Time-Dependent Solutions of Einstein's Equations

Clement Leibovitz
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada
(Received 19 February 1971)

Spherically symmetric time-dependent solutions are presented. One class of solutions could represent the interior of an incompressible sphere undergoing at its surface a process of condensation or evaporation. A large class of solutions of the equation $T_2^2=0$ is also obtained. A generalization of the Oppenheimer-Snyder solution is found. Two solutions obeying an equation of state are described.

We shall suppose that the reader is aware of the importance of time-dependent solutions.[1] We shall therefore not elaborate on this subject and shall present directly our results.

I. CONDENSATION AND EVAPORATION AT THE SURFACE OF AN INCOMPRESSIBLE BODY

Let us consider the element

$$ds^2=-A(r)dr^2-r^2(d\theta^2+\sin^2\theta d\psi^2)+C(r)dt^2, \quad (1)$$

which will be written alternatively as

$$ds^2=-e^{\omega}dr^2-r^2d\Omega^2+e^{\sigma}dt^2. \quad (2)$$

The equation $T_1^1=T_2^2$ reduces to[2]

$$-\frac{A'}{A^2}\left(\frac{C'}{4C^2}+\frac{1}{2r}\right)+\frac{1}{A}\left(\frac{C''}{2C}-\frac{C'}{2rC}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0. \quad (3)$$

It can also be written

$$2\sigma''+\sigma'^2-\sigma'\left(\frac{2}{r}+\frac{A'}{A}\right)=\frac{-2A'}{2A}+\frac{4}{r^2}-\frac{4A}{r^2}. \quad (4)$$

This equation was solved giving A in terms of an arbitrary function σ. However, if we consider Eq. (4) as a differential in σ (A being an arbitrary function), it becomes a Riccati differential equation that can be solved only if we know a particular solution σ_0. Taking the particular solution as a parameter, we can solve Eq. (4) and obtain

$$C=e^{\sigma}=e^{\sigma_0}\left(h\int r\sqrt{A}\,e^{-\sigma_0}dr+g\right)^2 \quad (5)$$

(h and g are arbitrary constants). A is given by[2]

$$A=\frac{e^{\sigma_0}(2+r\sigma_0')^2\exp\left(-4\int\frac{\sigma_0'dr}{2+r\sigma_0'}\right)}{-4r^2\int\left[\frac{e^{\sigma_0}}{r^3}(2+r\sigma_0')\exp\left(-4\int\frac{\sigma_0'dr}{2+r\sigma_0'}\right)\right]dr+\text{const}\times r^2}. \quad (6)$$

The interesting thing is that Einstein's time-dependent field equations remain satisfied if we consider h and g as arbitrary functions of time instead of being constants. The relevant equations are[3]

$$-8\pi T_1^1=e^{-\omega}\left(\frac{\sigma'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \quad (7)$$

$$-8\pi T_2^2=e^{-\omega}\left(\frac{\sigma''}{2}-\frac{\omega'\sigma'}{4}+\frac{\sigma'^2}{4}+\frac{\sigma'-\omega'}{2r}\right)-e^{-\sigma}\left(\frac{\ddot{\omega}}{2}+\frac{\dot{\omega}^2}{4}-\frac{\dot{\omega}\dot{\sigma}}{4}\right), \quad (8)$$

$$8\pi T_4^4=e^{-\omega}\left(\frac{\omega'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \quad (9)$$

$$8\pi T_4^1=-e^{-\omega}(\dot{\omega}/r). \quad (10)$$

A look at the field equations shows that the equations $T_1^1=T_2^2$ and $T_4^1=0$ remain satisfied if any arbitrary constant occurring in C *and not occurring in A* is replaced by an arbitrary function of time. Therefore, the equation

$$C=e^{\sigma}=e^{\sigma_0}\left[h(t)\int r\sqrt{A}\,e^{-\sigma_0}dr+g(t)\right]^2, \quad (5')$$

together with Eq. (6), represents a comoving time-dependent solution for a perfect fluid.

More specifically, every static solution for $A=e^{\omega}$ and e^{σ_0} of Eqs. (7)–(10) generates a family of time-dependent solutions given by A and e^{σ}, the latter being defined by (5′).

An interesting feature of these solutions is that they cannot be joined to the vacuum solution. It is shown in Appendix A that the junction conditions reduce in our case to

$$\frac{dR}{dt}=0 \text{ and } p=0,$$

in which $r=R(t)$ is the equation of motion of the junction surface. The junction surface must therefore be comoving and, on it, the pressure of the interior solution must be zero. It should therefore be possible to find a constant value R for r for which the pressure is zero at all times. However, we would have on one hand from (7)

$$\sigma' = \frac{e^{\omega(R)}}{R} - \frac{1}{R},$$

which implies that, for the same value R of r, σ' is time-independent. However, we have from (5′)

$$\sigma' = \frac{\sigma_0' + 2h(t) r\sqrt{A}\, e^{-\sigma_0}}{h(t)\int r\sqrt{A}\, e^{-\sigma_0} dr + g(t)}.$$

This last expression can be time-independent only if $h(t)/g(t)$ is a constant. In this case $C=e^{\sigma}$ reduces to a function of r multiplied by a function of t and therefore corresponds to a static solution.

Equation (9) shows that $\rho = T_4^4$ is time-independent, while Eqs. (7) and (8) show that $p = -T_1^1 = -T_2^2$ is time-dependent. The proper mass within a constant coordinate radius is time-independent (since ρ and the 3-geometry are time-independent).

This can be seen also by writing down the mass function[4] giving the total energy enclosed in the sphere of coordinate radius r. In the case of our metric this function $m(r,t)$ is given simply by

$$2m(r,t) = r(1-e^{-\omega}),$$

which is time-independent.

The sphere must be made of incompressible matter (since density is not affected by changes in pressure); moreover, the incompressible matter must be inhomogeneous since the density varies along the radius.

We could choose a constant arbitrary value for the radius R of the sphere and consider the solution as describing the interior of a sphere of constant mass (and incompressible matter) when its surface is undergoing time-dependent pressures given by Eq. (7).

We may also consider the case of the sphere being surrounded by a gas exerting a constant pressure p_0 on the surface of the sphere. The radius of the sphere would then be given by solving *for* r the equation [deduced from Eq. (7)]

$$8\pi p_0 = e^{-\omega}\left(\frac{\sigma'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2},$$

in which σ' is the time-dependent function deduced from Eq. (5′). We shall therefore obtain a time-dependent solution $r=R(t)$. If r is a time-increasing function we could say that some of the external gas has condensed on the surface of the sphere. Similarly, if $r=R(t)$ is a decreasing function of time we could say that the surface of the sphere is undergoing a process of evaporation.

It may be objected that, in the case of condensation, the condensed matter would have different densities according to the coordinate radius at which condensation occurs. This objection cannot be held in case the interior solution is of constant density.

Such will be the case if we start with the Einstein static solution with $A=(1-r^2/a^2)^{-1}$ and $e^{\sigma_0}=1$. We obtain through Eq. (5′)

$$C = e^{\sigma} = [h(t)(1-r^2/a^2)^{1/2} + g(t)]^2.$$

This solution is the Schwarzschild interior solution with two of the arbitrary constants replaced by functions of time. For such a solution the condensation at any coordinate radius will always give us matter with the same constant density.

II. SOLUTIONS WITH $T_2^2=0$

Einstein[5] has studied the case $T_1^1=0$, $T_2^2\neq 0$. The inverse case, $T_2^2=0$ and $T_1^1\neq 0$, is also worthy of interest. Let us consider, for instance, a spherical body of variable mass. Such a body cannot be imbedded in vacuum. However, the junction conditions[6] with an appropriate exterior solution do not involve T_2^2 so that we may in principle join a perfect fluid for which $T_2^2=-p$ at its surface with an exterior solution for which $T_2^2=0$ provided some conditions involving the discontinuity of T_1^1 are satisfied. Another instance of interest is that of gravitational collapse; an interior solution with $T_1^1\neq 0$ and $T_2^2=0$ would be an improvement to Oppenheimer and Snyder's solution with zero pressure. The Oppenheimer-Snyder solution does not allow the study of the effect of pressure on the gravitational collapse. However, a solution introducing a radial component of pressure would permit the study of its effect on the evolution of a collapsing body. This effect, though different from the one produced by the isotropic pressure of a perfect fluid, would nevertheless give good indications, since the radial component of pressure is the relevant one in radial motions [the two other components, $p_2=p_3$, are absent in the expression of the time derivative of the mass function[4] $m_4 = r^2(T_1^1 r_4 - T_4^1 r_1)4\pi$].

We will consider the following metric:

$$ds^2 = -A dr^2 - B(d\theta^2 + \sin^2\theta d\psi^2) + C dt^2,$$

or (11)

$$ds^2 = -e^{\omega} dr^2 - e^{\mu} d\Omega^2 + e^{\sigma} dt^2,$$

for which the equation $T_2^2=0$ may be written[7]

$$e^{-\omega}(2\sigma'' + \sigma'^2 + 2\mu'' + \mu'^2 - \mu'\omega' - \sigma'\omega' + \mu'\sigma') + e^{-\sigma}(\dot\omega\dot\sigma + \dot\mu\dot\sigma - \dot\omega\dot\mu - 2\ddot\omega - \dot\omega^2 - 2\ddot\mu + \dot\mu^2) = 0. \tag{12}$$

This is a nonlinear differential equation that seems to be quite intractable; we have been able, however, to obtain two large classes of solutions, the derivation of which is given in Appendix B. The first class of solutions is given by

$$A=\frac{uf'^2}{B}\,\frac{\exp\int\frac{B'^2f\,dr}{2f'B^2}}{\int\left(\frac{f'}{f}\exp\int\frac{B'^2f\,dr}{2f'B^2}\right)dr},$$
$$C=\frac{u\dot f^2}{B}\,\frac{\exp\int\frac{\dot B^2f\,dt}{2\dot fB^2}}{\int\left(\frac{\dot f}{f}\exp\int\frac{\dot B^2f\,dt}{2\dot fB^2}\right)dt},\tag{13}$$

while the second class is given by

$$A=\frac{vf'^2}{B}\exp\int\frac{B'^2f\,dr}{2B^2f'},$$
$$C=\frac{w\dot f^2}{B}\exp\int\frac{\dot B^2f\,dt}{2B^2\dot f}.\tag{14}$$

In both cases B and f are arbitrary products of functions of r and t, while u, v, and w are arbitrary constants.

If in Eq. (14) we take, for instance, $B=t^{4/3}r^2$ and $f=t^{4/3}(r^2-2mr)^{1/2}$, we obtain a particular solution the line element of which is

$$ds^2=-at^{4/3}\left(\frac{dr^2}{1-2m/r}+r^2d\Omega^2\right)+(1-2m/r)dt^2.\tag{15}$$

The components of the energy-momentum tensor are in this case

$$T_1^1=T_2^2=T_3^3=0,$$
$$8\pi T_4^4=\frac{4}{3t^2(1-2m/r)}=\frac{4}{3t^2}+\frac{8m}{3t^2(r-2m)},$$
$$8\pi T_4^1=\frac{-4m}{3ar^2t^{7/3}},\quad 8\pi T_1^4=\frac{4m}{3(r-2m)^2t}.$$

If at a given constant time t_0 we make the transformation

$$\bar r=\sqrt{a}\ t_0^{2/3}r,$$

we obtain for the 3-geometry at $t=t_0$

$$\frac{d\bar r^2}{1-2m\sqrt{a}\ t_0^{2/3}/r}+r^2d\Omega^2.\tag{16}$$

The results suggest that the singularity is that of a variable mass given by $m\sqrt{a}\,t^{2/3}$. This is confirmed by computing the mass function; we thus obtain

$$m(r,t)=m\sqrt{a}\,t^{2/3}+\frac{2a^{3/2}}{9}\,\frac{r^3}{1-2m/r}.$$

The mass function is a sum of a function of time and a function of r. The first term gives the time-dependent mass singularity already computed, while the second term gives the nonsingular mass within the coordinate radius r; this nonsingular mass being time-independent shows that any increase in mass due to the matter flow within a given coordinate radius is equal to the amount of mass collected by the singularity.

The expression for T_4^1 shows that the flux of momentum is zero for $m=0$, though we then have $8\pi T_4^4=4/3t^2$. In this case the quantity of matter within a comoving sphere of constant coordinate r is time-independent, since the volume element is $a^{3/2}t^2r^2\sin\theta d\theta dr d\psi$.

For $m\neq 0$ there are two contributions to ρ; we can write $\rho=\rho_1+\rho_2$ with

$$8\pi\rho_1=\frac{4}{3t^2}\quad\text{and}\quad 8\pi\rho_2=\frac{8m}{3t^2(r-2m)}.\tag{17}$$

The second contribution ρ_2 may be considered as arising totally from the singularity; as the mass of the singularity increases, a current of matter must come from infinity towards the singularity, adding its contribution to the background density. ρ_1 gives essentially the uniform background density. The "perturbing" density ρ_2 tends to zero at great distance from the origin.

The density of matter current as represented by T_4^1 is proportional to r^{-2} so that at constant time we have equal currents of matter through all spheres centered at the origin. The solution represents the effect of time-increasing inhomogeneity in an otherwise homogeneous universe of flat three-geometry (in our particular coordinates).

The metric is valid for $2m<r$ and represents possibly an exterior solution for an interior one representing a variable mass.

III. A GENERALIZATION OF THE OPPENHEIMER - SNYDER SOLUTION

The line element of the Oppenheimer-Snyder (OS) solution[7] is given by

$$ds^2=-Adr^2-Bd\Omega^2+Cdt^2,$$
$$A=B'^2/4B,\quad B=(Ft+G)^{4/3},\quad C=1,\tag{18}$$

F and G being arbitrary functions of r.

The solution may be characterized as that representing spherically symmetric dust matter with a flat 3-geometry in comoving coordinates $[T_4^1=0,\ A=(d\sqrt{B}/dr)^2,\ p=0]$. The calculated density is

$$8\pi\rho=\tfrac{4}{3}(t+G/F)^{-1}(t+G'/F')^{-1}.\tag{19}$$

We propose the following generalization of the OS solution (the derivation of which is given in Appendix C).

$$A = B'^2/4B\,,\qquad B = \dot\tau(t)^{-2/3}[F(r)\tau(t)+G(r)]^{4/3},\quad C=1, \tag{20}$$

in which F, G, and τ are arbitrary functions of their argument. We have in this case

$$T_4^1 = 0,\quad -8\pi(T_1^1 = T_2^2 = T_3^3) = -(\ddot\tau/\dot\tau)^2 + \tfrac{2}{3}\dddot\tau/\dot\tau\,. \tag{21}$$

The pressure is uniform through space, though time-dependent. Expression (21) for the pressure is invariant for any homographic transformation of τ with constant coefficients

$$\overline{\tau} = \frac{a\tau+b}{c\tau+d}\,. \tag{22}$$

For $\tau = t$ the solution (20) reduces to the OS solution; therefore, the OS solution can be obtained for the more general choice of

$$\tau = \frac{at+b}{cf+d}\,, \tag{23}$$

where a, b, c, and d are constants. The calculated density for the solution (20) is

$$8\pi\rho = \frac{[2\ddot\tau^2 - \dddot\tau(\tau+G/F)][2\ddot\tau^2 - \dddot\tau(\tau+G'/F')]}{3\dot\tau^2(\tau+G/F)(\tau+G'/F')}\,. \tag{24}$$

A. The Solution as a Generalization for the Case $\lambda \neq 0$

If we take for τ the expression

$$\tau = \frac{a\tan(ut)+b}{c\tan(ut)+d} \tag{25}$$

(a, b, c, and d being constants), the calculated pressure becomes

$$8\pi p = \tfrac{4}{3}u^2\,. \tag{26}$$

Therefore, if we use the cosmological constant and give to it the value $\lambda = -\frac{4}{3}u^2$, we obtain a solution with zero pressure which generalizes the OS solution for the case $\lambda \neq 0$. Replacing $\tan(ut)$ by $\tanh(ut)$ in (25), the pressure becomes $8\pi p = -\frac{4}{3}u^2$, and λ must then have the value $\lambda' = \frac{4}{3}u^2$ so that the generalization can be made for positive and negative values of the cosmological constant.

B. The Solution as a Collection of Different Friedmann Regions

It is proved in Appendix D that a perfect fluid, in comoving coordinates with cosmological time ($\partial C/\partial r = 0$) and for which B [as defined by Eq. (11)] is a product of a function of time by a function of r, represents necessarily a Friedmannian metric. Therefore, writing

$$B = F^{4/3}\dot\tau^{-2/3}(\tau + G/F)^{4/3}\,, \tag{27}$$

we see that wherever in space $G/F = \text{constant}$, the metric will be Friedmannian. Therefore, if we take for G/F a continuous function which is constant over a number of nonoverlapping domains of r, we will have joined several Friedmannian regions in the most easy way. We may take for definiteness $F(r) = r^{3/2}$ (which fixes the meaning of the radial coordinate) and write $P = G/F = G r^{-3/2}$ so that

$$B = r^2\dot\tau^{-2/3}[\tau(t)+P(r)]^{4/3},\quad A = B'^2/4B\,, \tag{28}$$

where

$$\begin{aligned} &P \text{ is a constant } P_1 \text{ for } r_1<r<r_2,\\ &P \text{ is a function of } r \text{ for } r_2<r<r_3,\\ &P \text{ is a constant } P_2 \text{ for } r_3<r<r_4\,. \end{aligned} \tag{29}$$

The continuous function P in the interval $r_2<r<r_3$ is to be such that[8]

$$P(r_2)=P_1,\quad P(r_3)=P_2,\quad P'(r_2)=P'(r_3)=0\,. \tag{30}$$

The metric defined by (28) and (29) describes two Friedmannian regions (the first and last regions) joined smoothly by the non-Friedmannian region of the second interval.

It should be remarked that all regions have a flat 3-geometry so that the Friedmann regions that can be joined are those of zero 3-curvature.

IV. TIME-DEPENDENT SOLUTIONS WITH AN EQUATION OF STATE

We are proposing two such solutions. The first one is given by the following line element:

$$ds^2 = -(b/r^2)dr^2 - r^2(t-c)(h-t)d\Omega^2 + r^2dt^2 \tag{31}$$

(b, c, and h are constant). Calculations give

$$\begin{aligned} -8\pi(T_1^1 = T_2^2 = T_3^3) &= 8\pi p\\ &= \frac{3}{b} + \frac{1}{4r^2}\frac{(c-h)^2}{(t-c)^2(h-t)^2} + \lambda,\\ 8\pi T_4^1 &= 0,\\ 8\pi\rho &= -\frac{3}{b} + \frac{1}{4r^2}\frac{(c-h)^2}{(t-c)^2(h-t)^2} - \lambda, \end{aligned} \tag{32}$$

so that we have for the equation of state

$$\rho = p - 2\lambda - 6/b\,. \tag{33}$$

For $\lambda = -3/b$ the solution is cosmological with $\rho = p > 0$ at all points in space. The element is valid for $c<t<h$ (if we suppose $h<c$). It corresponds to a universe starting at time $t=c$ and ending at time $t=h$. For $\lambda < -3/b$ we have $\rho > p$. ρ is then positive, but p may assume negative values. The solution, however, cannot be joined with Schwarzschild's exterior solution because the equation $p=0$ has no solution for r independent of time (which is a prerequisite in comoving coordinates; see Appendix A). The second solution corresponds to the following line element:

$$ds^2 = \frac{-2a}{1-r^2u}dr^2 - r^2\{a + C_1 \sinh[(2c/a)^{1/2}t] + C_2 \cosh[(2c/a)^{1/2}t]\}d\Omega^2 + crdt^2 , \tag{34}$$

from which we calculate

$$\begin{aligned} &T_1^4 = 0 , \\ &8\pi p = \frac{1}{r^2}\left(\frac{a^2 + C_2{}^2 - C_1{}^2}{2a\{a + C_1 \sinh[(2c/a)^{1/2}t] + C_2 \cosh[(2c/a)^{1/2}t]\}^2}\right) - \frac{3u}{2} , \\ &8\pi \rho = \frac{1}{r^2}\left(\frac{a^2 + C_2{}^2 - C_1{}^2}{2a\{a + C_1 \sinh[(2c/a)^{1/2}t] + C_2 \cosh[(2c/a)^{1/2}t]\}^2}\right) + \frac{3u}{2} , \end{aligned} \tag{35}$$

so that

$$\rho = p + 3u/8\pi . \tag{36}$$

For $C_1 = C_2 = 0$ the solution is static and can be smoothly joined to Schwarzschild's exterior solution with a coordinate radius

$$R = (3u)^{-1/2}$$

and a total mass

$$m = \tfrac{1}{3}(3u)^{-1/2} . \tag{37}$$

For $C_2 = -C_1$ the solution is time-dependent and tends asymptotically to the static solution as t tends to infinity. For $C_2 = C_1$ the solution is time-dependent and starts asymptotically from the static solution.

In order to avoid a singularity in the metric, as well as in the density and in the pressure at a finite time, we must have

$$C_2 > |C_1| . \tag{38}$$

The time-dependent solutions do not oscillate about the equilibrium position. However, it cannot be deduced from this that the static solution is unstable; in fact, none of the time-dependent solutions can be imbedded in a vacuum ($p = 0$ has no time-dependent solution for r; see Appendix A). The stability of the static solution must be judged from the behavior of perturbations compatible with the embedding in vacuum.

For $a^2 + C_2{}^2 - C_1{}^2 = 0$ we have a solution characterized by homogeneous pressure and density and by the relation $\rho + p = 0$. The solution, however, is different from De Sitter's solution, which has one more characteristic, namely, that of being isotropic at every point in comoving coordinates.

The solution (34) becomes oscillatory if we replace a by $-a$. However, in this case we must also have $r^2u > 1$. If we consider the shell of matter within the coordinate radii r_1 and r_2 such that $r_2 > r_1 > u^{-1/2}$, we then have a shell of matter oscillating under the action of forces applied at the boundary surfaces $r = r_1$ and $r = r_2$. The value of the forces may be deduced from the expression of p for the corresponding values of r.

The two solutions of this paragraph have with Friedmann's solution the following characteristic: They are the only solutions representing a perfect fluid which, in comoving coordinates, correspond to expressions for A, B, and C which are products of a function of r by a function of t. Friedmann's solution has one more characteristic, namely, $dC/dr = 0$.

APPENDIX A: THE JUNCTION OF A COMOVING SOLUTION WITH A SCHWARZSCHILD EXTERIOR SOLUTION

The junction conditions are[6]

$$\Delta T_1^1 - \frac{dR}{dt}\Delta T_1^4 = 0 , \tag{A1}$$

$$\Delta T_4^1 - \frac{dR}{dt}\Delta T_4^4 = 0 , \tag{A2}$$

where $r = R(t)$ is the equation of motion of the junction surface. For a vacuum $T_v^u = 0$, while for a comoving perfect fluid $T_1^1 = -p$ and $T_4^4 = \rho$ so that (A1) and (A2) may be written in our case

$$-p = 0, \tag{A3}$$

$$\frac{dR}{dt}\rho = 0 . \tag{A4}$$

Equations (A3) and (A4) can be satisfied in two ways:

(1) $p[R(t), t] \equiv 0$, $\rho[R(t), t] \equiv 0$, i.e., by finding a function $r = R(t)$ for which p and ρ become zero.

(2) By finding a constant r_0 such that for $R = r_0$ we have $p(R, t) \equiv 0$.

The first way is impossible for most equations of state.

APPENDIX B: DERIVATION OF SOLUTIONS OF THE EQUATION $T_2^2=0$

The relevant equation is[9]

$$e^{-\omega}(2\sigma''+\sigma'^2+2\mu''+\mu'^2-\mu'\omega'-\sigma'\omega'+\mu'\sigma') + e^{-\sigma}(\dot\omega\dot\sigma+\dot\mu\dot\sigma-\dot\omega\dot\mu-2\ddot\omega-\dot\omega^2-2\ddot\mu+\dot\mu^2)=0\,. \tag{B1}$$

As a first step we look for the class of solutions in which each of $A=e^{\omega}$, $B=e^{\mu}$, and $C=e^{\sigma}$ is a product of a function of time by a function of r. In this case μ', ω', σ' are functions of r only while $\dot\mu$, $\dot\omega$, $\dot\sigma$ are functions of t only.

Writing with the evident notation $A=R_AT_A$, $B=R_BT_B$, and $C=R_CT_C$, Eq. (B1) becomes equivalent to two ordinary differential equations:

$$\frac{R_C}{R_A}(2\sigma''+\sigma'^2+2\mu''+\mu'^2-\mu'\omega'-\sigma'\omega'+\mu'\sigma')=a \tag{B2}$$

and

$$\frac{T_A}{T_C}(\dot\omega\dot\sigma+\dot\mu\dot\sigma-\dot\omega\dot\mu-2\ddot\omega-\dot\omega^2-2\ddot\mu-\dot\mu^2)=-a\,, \tag{B3}$$

where a is an arbitrary constant.

Two classes of solutions are to be considered according to a being equal to zero or not. The two equations (B2) and (B3) are still very complicated. However, the liberty of choosing the system of coordinates allows us to impose a condition on σ, ω, and μ. In order that the condition be a restriction on the choice of the coordinates only without restricting the generality of the solution, we must be sure that the condition *is not* covariant.

A very convenient coordinate condition for equation (B2) is

$$2(\sigma''+\mu'')+(\sigma'+\mu')^2=0, \tag{B4}$$

the solution of which is

$$e^{\sigma}=T_C\frac{(r+b)^2}{R_B}\,. \tag{B5}$$

Equation (B2) becomes

$$\frac{R_C}{R_A}(-\mu'\omega'-\sigma'\omega'-\mu'\sigma')=a, \tag{B6}$$

from which ω may be calculated in terms of μ [σ being given by (B5)]. The solution once obtained may be generalized, and we get rid of condition (B4) by performing coordinate transformations.

The solution obtained for the case $a=0$ looks as follows:

$$A=\frac{uh'^2g^2}{R_BT_B}\exp\int\frac{R_B'^2h\,dr}{2R^2h'}\,,$$
$$B=R_BT_B\,, \tag{B7}$$
$$C=\frac{vg^2h^2}{R_BT_B}\exp\int\frac{\dot T_B^2g\,dt}{2T^2\dot g}\,,$$

in which h and R_B are arbitrary functions of r while g and T_B are arbitrary functions of t (u and v being arbitrary constants).

If we write $f=g(t)h(r)$, we may write the solution as follows:

$$A=\frac{uf'^2}{B}\exp\int\frac{B'^2f\,dr}{2B^2f'}\,,$$
$$B=RT\,, \tag{B8}$$
$$C=\frac{v\dot f^2}{B}\exp\int\frac{\dot B^2f\,dt}{2B^2\dot f}\,.$$

In (B8) B and f are restricted to be each one a product of a function of r by a function of t.

APPENDIX C: DERIVATION OF THE GENERALIZATION OF THE OS SOLUTION

The OS line element is given by

$$ds^2=-A\,dr^2-B(d\theta^2+\sin^2\theta\,d\psi^2)+dt^2\,, \tag{C1}$$

$$A=\frac{B'^2}{4B}=\left(\frac{d\sqrt{B}}{dr}\right)^2, \tag{C2}$$

$$B=[F(r)t+G(r)]^{4/3}\,. \tag{C3}$$

Equation (C2) is characteristic of a flat 3-geometry and is a particular solution of the equation $T_4^1=0$ for the metric (C1). By adhering to Eq. (C2), we are restricting ourselves to solutions having a flat 3-geometry in comoving coordinates.

Taking into account that $T_4^1=0$, the equation $T^{\mu}_{1;\mu}=0$ takes the form

$$\frac{\partial T_1^1}{\partial r}+\frac{B'}{B}(T_1^1-T_2^2)=0\,. \tag{C4}$$

Therefore, in order to have $T_1^1=T_2^2$, it is enough to have

$$\frac{\partial T_1^1}{\partial r}=0 \quad\text{or}\quad T_1^1=f(t)\,. \tag{C5}$$

Calculating T_1^1 for the line element (C1), and taking (C2) into consideration, Eq. (C5) leads to

$$-\ddot B+\dot B/4B=Bf(t)\,. \tag{C6}$$

Writing $\dot B=uB$, we have

$$-\dot u-\tfrac{3}{2}u^2=f(t), \tag{C7}$$

which is a Riccati equation that can be solved only if we know a particular solution. Taking advantage of the fact that $f(t)$ is arbitrary, we write

$$-\dot u-\tfrac{3}{2}u^2=-\dot g(t)-\tfrac{3}{2}g^2(t)\,, \tag{C8}$$

with $g(t)$ an arbitrary function of time. We now have a particular solution $u=g(t)$. Performing the integration and changing the expression of the arbitrary function we obtain

$$B=[\dot{\tau}(t)]^{-2/3}[F(r)\tau(t)+G(r)]^{4/3}. \tag{C9}$$

APPENDIX D: CHARACTERIZING FRIEDMANN'S SOLUTION

We intend to show that the following statement is true: A spherically symmetric metric with cosmological time representing a comoving perfect fluid is a Friedmannian metric if $B=R(r)T(t)$; R and T being functions of their arguments and B being the metric coefficient occurring in

$$ds^2=-Adr^2-B(d\theta^2+\sin^2\theta d\psi^2)+dt^2. \tag{D1}$$

The comoving condition as given by Tolman[10] is

$$A=(B'^2/4B)f(r) \tag{D2}$$

and may be written with $B=R(r)T(t)$,

$$A=Tg(r). \tag{D3}$$

The equation $T^1_1=T^2_2$ gives, in terms of T, R, and their derivatives,

$$-\frac{1}{RT}+\frac{1}{2Tg}\left(\frac{R'}{R}\right)^2=\frac{1}{2Tg}\frac{R''}{R}+\frac{1}{4Tg}\frac{g'R'}{gR}, \tag{D4}$$

the solution of which is

$$g=\frac{R'^2}{4R+aR^2}\quad (a=\text{constant}). \tag{D5}$$

Now, $R(r)$ may be chosen arbitrarily, if necessary, by a transformation $r=r(\bar{r})$, $t=\bar{t}$ which does not alter the comoving character of the line element; moreover, Eq. (D5) is covariant relative to such a transformation. Let us therefore take $R=r^2$; we obtain

$$g=\frac{1}{1+\frac{1}{4}ar^2}, \tag{D6}$$

so that we can write

$$ds^2=-T\left[\frac{dr^2}{1+\frac{1}{4}ar^2}+r^2(d\theta^2+\sin^2\theta d\psi^2)\right]+dt^2, \tag{D7}$$

which is one of the forms of Friedmann's solution.

We may also state: A spherically symmetric element with cosmological time representing a perfect fluid is a Friedmannian metric if $B=R_1T$ and $A=R_2T$ in an obvious notation.

It is easy to check that in this case $T^1_4=T^4_1=0$, and therefore, this second statement reduces to the preceding one.

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[1]Works dealing with exact time-dependent solutions: G. C. McVittie, Astrophys. J. 140, 401 (1964); 143, 682 (1966); Ann. Inst. Henri Poincaré A6, 1 (1967); A7, 10: (1967); H. Bondi, Proc. Roy. Soc. (London) A281, 39 (1964); Nature 215, 838 (1967); Monthly Notices Roy. Astron. Soc. 107, 410 (1947); I. N. Thompson and G. J. Whitrow, *ibid.* 136, 207 (1967); H. Nariai, Progr. Theoret. Phys. (Kyoto) 38, 92 (1967); W. B. Bonnor and M. C. Foulkes, Monthly Notices Roy. Astron. Soc. 137, 239 (1967); R. C. Folman, Proc. Natl. Acad. Sci. U. S. 20, 169 (1934); B. Datt, Z. Physik 108, 314 (1938); J. Pachner, Bull. Astron. Inst. Czech. 17, 105 (1966); A. Banerjee, Proc. Phys. Soc. (London) 91, 747 (1967).

[2]C. Leibovitz Phys. Rev. 185, 1664 (1969).

[3]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, Oxford, England, 1934), p. 251.

[4]This function describes the total energy within coordinate r as a function of time. It has been introduced by Bardeen, Misner, and Sharp and has been used by Cahill and McVittie. We refer here more specifically to M. E. Cahill and G. C. McVittie, J. Math. Phys. 11, 1382 (1970).

[5]A. Einstein, Ann. Math. 40, 922 (1939).

[6]W. Israel, Proc. Roy. Soc. (London) 248A, 404 (1958).

[7]J. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).

[8]The continuity of $P'(r)$ is necessary if we demand that A be continuous at a junction surface; however it has been shown (see Ref. 11) that A may be discontinuous in comoving coordinates provided we enforce the continuity of B'^2/A. In our case this last quantity is equal to $4B$, and its continuity is obtained by demanding the continuity of $P(r)$ only. The continuity of $P'(r)$ is therefore not necessary.

[9]L. Landau and L. Lifshitz, *Classical Theory of Fields*, (Addison-Wesley, Reading, Mass., 1951), p. 311.

[10]R. Tolman, private communication to J. Oppenheimer and H. Snyder (see Ref. 7).

[11]C. Leibovitz, Nuovo Cimento 60B, 254 (1969).