PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOL. 4, NO. 10 15 NOVEMBER 1971

Taub-NUT (Newman, Unti, Tamburino) Metric and Incompatible Extensions*

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(Received 6 July 1971)

A Kruskal-like extension of the Taub-NUT (Newman, Unti, Tamburino) metric is given, clearly exhibiting the unusual properties of this metric. The new extension is incompatible with the extension of Misner and Taub, referred to as Taub-NUT space, and also with the further non-Hausdorff extension of the latter that has been proposed by various authors. The geodesic incompleteness of these various extensions is discussed.

The Taub-NUT (Newman, Unti, Tamburino) metric, represented here in Schwarzschild-like coordinates,

$$\begin{aligned} ds^2 &= (r^2+l^2)(d\theta^2+\sin^2\theta d\phi^2) \\ &\quad + U(dt+2l\cos\theta d\phi)^2 - U^{-1}dr^2\,, \qquad (1) \\ U &\equiv -1+2(mr+l^2)/(r^2+l^2)\,, \end{aligned}$$

has recently been the object of considerable interest because of its unusual properties.[1] Misner and Taub,[2,3] by imposing a certain identification on (1), extended the metric. In fact, they obtained two distinct Hausdorff extensions, isometric to each other and representative of what is called Taub-NUT space, which however cannot be adopted simultaneously without abandoning the Hausdorff property. (Some authors define manifold to include this property and others not to, but in any case it would be very awkward not to have it in the context of relativity.) The purpose of the present paper is to point out that metric (1) taken as is, without any identification, can also be extended analytically. A maximal such extension is presented. It is incompatible with the identification used by Misner and Taub in the sense that imposing the (analytically continued) identification on the extension destroys one of its most fundamental properties as a topological space (T_1) and thereby its character as a manifold.

Two types of singularity are apparent in metric (1). One type occurs at $U=0$ $[r=r_\pm \equiv m\pm(m^2+l^2)^{1/2}]$ and represents a Killing horizon[4]; the other occurs at $\theta=0,\ \pi$, where the determinant of the components of the metric vanishes. Misner[3] showed that, because of the second term on the right, for $l\neq 0$ the latter type of singularity is not the usual degeneracy associated with spherical coordinates on the 2-sphere. Both types of singularity are treated by Misner and Taub.[2,3] They impose on (1) the identification

$$(\phi, t) = (\phi+(n+m)2\pi,\ t+(n-m)4l\pi) \quad \text{for all integers } n, m. \qquad (2)$$

The two distinct isometric extensions of (1) arise from Eddington-Finkelstein-like transformations[5]

$$2l\psi = t \pm \int U^{-1}dr\,. \qquad (3)$$

From these transformations we obtain two metrics,

$$\begin{aligned} ds^2 &= (r^2+l^2)(d\theta^2+\sin^2\theta d\phi^2) + 4l^2U(d\psi+\cos\theta d\phi)^2 \\ &\quad \mp 4l(d\psi+\cos\theta d\phi)dr\,, \qquad (4) \end{aligned}$$

valid for all r. The identification (2) becomes

$$(\phi, \psi) = (\phi + (n+m)2\pi,\ \psi + (n-m)2\pi). \tag{5}$$

Depending upon the choice of sign in (3), different branches of the Killing horizon are covered. The singularity in the θ coordinate is due to the degeneracy of Euler angle coordinates on the 3-sphere. The transformation[1]

$$\begin{aligned} w &= e^{r/2}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi), \\ x &= e^{r/2}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi), \\ y &= e^{r/2}\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi), \\ z &= e^{r/2}\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \end{aligned} \tag{6}$$

for $0<\theta<\pi$, $0<\phi<2\pi$, $0<\psi<4\pi$, $-\infty<r<\infty$ takes the metrics (4) into

$$\begin{aligned} ds^2 &= (r^2+l^2)(\sigma_x^{\,2}+\sigma_y^{\,2}) + 4l^2U\sigma_z^{\,2} \pm 4l\sigma_z dr, \\ \sigma_x &\equiv 2e^{-r}(x\,dw - w\,dx - z\,dy + y\,dz), \\ \sigma_y &\equiv 2e^{-r}(y\,dw + z\,dx - w\,dy - x\,dz), \\ \sigma_z &\equiv 2e^{-r}(z\,dw - y\,dx + x\,dy - w\,dz), \\ dr &= 2e^{-r}(w\,dw + x\,dx + y\,dy + z\,dz), \end{aligned} \tag{7}$$

with $e^r = w^2+x^2+y^2+z^2$. The elegance of these coordinates is that they cover the manifold $M = \{(w,x,y,z)\in R^4 \mid w^2+x^2+y^2+z^2 \neq 0\}$, homeomorphic to $S^3\times R$, and hence define a one-map atlas[6] for the manifold. It should be emphasized that neither Eddington-Finkelstein–like transformation provides a complete treatment of the horizons, and this accounts for many of the strange properties of Taub-NUT space. For instance, geodesics approaching one horizon branch pass through normally, while those approaching the other branch appear to spiral indefinitely.[1,7] This is a natural consequence of Eddington-Finkelstein–like coordinates and the periodicity of the ψ coordinate.

Our extension of metric (1) without any identification is given by Kruskal-like transformations[8]

$$\begin{aligned} u_\pm &= \left(\frac{r-r_\pm}{r_\pm}\right)^{1/2}\left(\frac{r-r_\mp}{r_\pm}\right)^{r_\mp/2r_\pm} e^{r/2r_\pm}\cosh t/2r_\pm, \\ v_\pm &= \left(\frac{r-r_\pm}{r_\pm}\right)^{1/2}\left(\frac{r-r_\mp}{r_\pm}\right)^{r_\mp/2r_\pm} e^{r/2r_\pm}\sinh t/2r_\pm. \end{aligned} \tag{8}$$

The transformation with the upper sign throughout is one-to-one from the half-plane $r_+<r<\infty$, $-\infty<t<\infty$ onto the quadrant $|v_+|<u_+$, while the transformation with the lower sign is one-to-one from $-\infty<r<r_-$, $-\infty<t<\infty$ onto $|v_-|<u_-$. Note that for $l=0$ ($r_+=2m$, $r_-=0$) the transformation with the upper sign is the Kruskal transformation[8] for the Schwarzschild metric. Applying these transformations to (1), we obtain two metrics:

$$\begin{aligned} ds^2 &= (r^2+l^2)(d\theta^2+\sin^2\theta\, d\phi^2) \\ &\quad + f^2\left[du_\pm^{\,2} - dv_\pm^{\,2} - (2l/r_\pm)(u_\pm dv_\pm - v_\pm du_\pm)\cos\theta\, d\phi \right. \\ &\qquad \left. - (l/r_\pm)^2(u_\pm^{\,2}-v_\pm^{\,2})\cos^2\theta\, d\phi^2\right], \end{aligned} \tag{9}$$

where

$$f^2 \equiv 4r_\pm^{\,4}(r^2+l^2)^{-1}\left(\frac{r-r_\mp}{r_\pm}\right)^{1-r_\mp/r_\pm} e^{-r/r_\pm}$$

is an analytic function of $u_\pm^{\,2}-v_\pm^{\,2}$ by (8). These metrics immediately extend analytically to the entire $(u_\pm, v_\pm)$ plane, where $r_-<r<\infty$ for the upper sign and $-\infty<r<r_+$ for the lower sign. There are still singularities at $\theta=0,\pi$.

These metrics can be patched together as indicated in Fig. 1 to form a complete analytic extension of the (totally geodesic) (r,t) surface.[9] In the diagram, I and III label NUT regions and II labels Taub regions. Metric (9) with the upper sign covers the squares consisting of two blocks labeled I and two blocks labeled II, and the metric with the lower sign covers the squares consisting of two blocks labeled II and two blocks labeled III. One of the metrics (4) without any identification covers each strip of three adjoining blocks going up from left to right, whether I, II, III or III, II, I; the other metric covers downgoing strips, and the

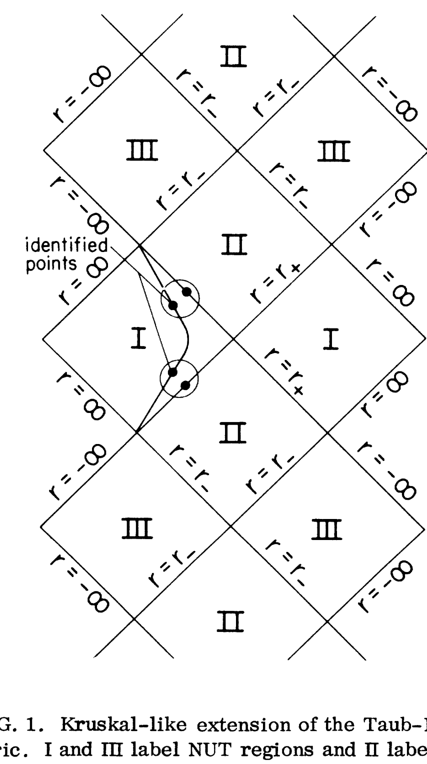

FIG. 1. Kruskal-like extension of the Taub-NUT metric. I and III label NUT regions and II labels Taub regions.

transformation between the metrics on the overlap regions is obtained from (3). Our extension takes account of the bifurcate nature of each horizon.[4] On the other hand, we shall prove that it will not allow identification (5). Thus, the procedure given earlier for covering the $\theta = 0, \pi$ singularities is not available, and our extension is geodesically incomplete. Geroch[10] and Hajicek[11] have suggested carrying out both Eddington-Finkelstein–like transformations simultaneously, maintaining identification (5), and accepting the resulting non-Hausdorff extension of Taub-NUT space.

We show next how this non-Hausdorff property arises. The identification (5) in the subspaces of constant ϕ is

$$(r, \psi, \theta, \phi) = (r, \psi + 4\pi, \theta, \phi) \tag{10}$$

and can be described as an identification on the orbits of the Killing vector $\xi \equiv \partial/\partial\psi = 2l(\partial/\partial t)$. The one-parameter group of isometries generated by the Killing vector ξ is given by

$$x^i \circ U_\lambda(p) \equiv U^i_\lambda(p) \equiv [\exp(\lambda\xi)x^i](p), \tag{11}$$

where the x^i are coordinate functions on a neighborhood of the point p. The identification (10) is equivalent to

$$U_\lambda(p) = U_{\lambda+4\pi}(p). \tag{12}$$

In $(u_\pm, v_\pm, \theta, \phi)$ coordinates,

$$\xi = (l/r_\pm)\left(v_\pm \frac{\partial}{\partial u_\pm} + u_\pm \frac{\partial}{\partial v_\pm}\right)$$

and

$$U^i_\lambda(u_\pm, v_\pm, \theta, \phi) = (u_\pm \cosh\lambda l/r_\pm + v_\pm \sinh\lambda l/r_\pm,\ v_\pm \cosh\lambda l/r_\pm + u_\pm \sinh\lambda l/r_\pm,\ \theta, \phi). \tag{13}$$

The orbits are hyperbolas in the $(u_\pm, v_\pm)$ plane and half-lines on the horizons, excluding the origin, which is a fixed point of ξ. Any two points on different branches of a horizon and in the same subspace of constant θ and ϕ violate the Hausdorff property when identification (5) is imposed. It can be shown that for an arbitrarily small neighborhood of each of the two points there is an orbit of the Killing vector ξ which intersects both neighborhoods and has a point in one neighborhood that is identified with a point in the other (see Fig. 1). Worse still, the point p: $(u_+, v_+, \theta, \phi) = (0, 0, \theta_0, \phi_0)$ cannot be a regular point of the manifold under the identification, since every neighborhood of it contains every point on the lines $u_+ = \pm v_+$, $\theta = \theta_0$, $\phi = \phi_0$, violating the T_1 property[12] of a manifold. Without the identification, p is a regular point with a spray of geodesics. This implies that the non-Hausdorff extension of Taub-NUT space obtained by deleting such points is still geodesically incomplete, contrary to Hajicek's[11] claim. Incidentally, every timelike geodesic in the totally geodesic subspace $\theta = \theta_0$, $\phi = \phi_0$ that goes through p also goes through its diagonally opposite corner point q: $(u_-, v_-, \theta, \phi) = (0, 0, \theta_0, \phi_0)$ of a block II in Fig. 1. Hence p is conjugate to q along these geodesics.

We are faced with a dilemma. From a topological point of view identification (5) is disastrous, creating a non-Hausdorff manifold and forcing the deletion of otherwise regular points. Yet the only way to cover the singularities at $\theta = 0, \pi$ is by means of this identification, which leads to the global topology $S^3 \times R$ suggested by the commutation relations of the Killing vectors and their action on the space.[3] We have to choose whether to adopt this identification, and either way the result is unpleasant.

Misner and Taub[2] have proved that Taub-NUT space is maximal, i.e., it has no Hausdorff extension. Our extension of the Taub-NUT metric is also maximal. The relationship of our extension to Taub-NUT space can be understood more clearly by examining transformation (6). The two-dimensional subspaces of constant θ and ϕ, including $\theta = 0, \pi$, are topologically cylinders, $S^1 \times R$, and are totally geodesic. Taub-NUT space is simply connected, since it is topologically $S^3 \times R$, but if we remove the two disjoint cylinders $\theta = 0$ and $\theta = \pi$, the space is no longer simply connected and we can consider the universal covering space. We arrive at metric (4) without any identification, and we can then carry out its Kruskal-like extension. Actually, there is no reason to delete both cylinders. If we delete just one, say the $\theta = \pi$ cylinder, and then consider the universal covering space, a Kruskal-like extension is still possible, the extension being given by transformations (8) applied to the Taub-NUT metric in Eq. (2) of Ref. 3. The ϕ coordinate in that metric is periodic with period 2π, and the polar coordinate singularity at $\theta = 0$ is to be covered by another patch. Incidentally, there is nothing special about the cylinders $\theta = 0$ and $\theta = \pi$, which appear distinguished merely because of a particular choice of Euler angle coordinates on S^3. We can remove any one of the totally geodesic cylinders of Taub-NUT space and obtain a maximal extension of its universal covering space. The particular cylinder chosen can never be reattached to the manifold afterwards. These extensions are maximal in the same way that the universal covering space of the punctured plane is maximal.[13] Therefore, these extensions also have incomplete and inextensible geodesics. The remaining totally geodesic cylinders are "unwrapped" and their complete extensions are represented by

Fig. 1.

Incompatible extensions can be exhibited most simply in the two-dimensional definite metric $ds^2 = dx^2 + f(x)^2 d\alpha^2$, where f is an odd periodic analytic function with numerically different slopes at neighboring zeros, say $f(x) = \sin(x + a\sin x)$, $|a| < 1$. Near $x = 0$, $f \approx (1+a)x$ and, as with polar coordinates, the identification of α modulo $2\pi/(1+a)$ permits $x = 0$ to be included as a regular point. The same holds for $x = \pi$ with $1+a$ replaced by $1-a$; but the two identifications are incompatible.

Incompatible extensions have arisen previously in relativity. Silberstein[14] has presented the metric

$$\begin{aligned} ds^2 &= e^{2\nu}dx_4^{\,2} - e^{-2\nu}[e^{2\lambda}(dx_1^{\,2} + dx_2^{\,2}) + x_1^{\,2}dx_3^{\,2}]\,, \\ \nu &\equiv -L_1/r_1 - L_2/r_2\,, \\ \lambda &\equiv -\frac{x_1^{\,2}}{2}\left(\frac{L_1^{\,2}}{r_1^{\,4}} + \frac{L_2^{\,2}}{r_2^{\,4}}\right) \\ &\quad + \frac{L_1 L_2}{a}\left(\frac{x_1^{\,2} + x_2^{\,2} - a^2}{[(x_1^{\,2} + x_2^{\,2} - a^2)^2 + 4a^2 x_1^{\,2}]^{1/2}} - 1\right)\,, \\ r_1^{\,2} &\equiv x_1^{\,2} + (x_2 + a)^2, \quad r_2^{\,2} \equiv x_1^{\,2} + (x_2 - a)^2, \end{aligned} \tag{14}$$

where L_1, L_2, and a are nonzero constants, $a > 0$. The degeneracy at $x_1 = 0$, as pointed out by Einstein and Rosen[15] (after correcting a mistake in the original expression for λ), is the cause of the trouble. Either the degeneracy at $x_1 = 0$, $|x_2| > a$ can be covered by identifying x_3 modulo 2π, or the degeneracy at $x_1 = 0$, $|x_2| < a$ by identifying x_3 modulo $2\pi e^{-2L_1L_2/a}$, but not both. This metric has been dismissed because neither identification gives anything physically reasonable. However, for the Taub-NUT metric with identification (5) Taub space is interpreted as an empty homogeneous but anisotropic cosmological model, a mixmaster universe[16] with an additional symmetry, namely the Killing vector ξ, and an attempt has also been made to interpret the NUT metric without the identification.[17] Our extension (Fig. 1) illustrates very clearly the unusual properties of the Taub-NUT metric and the problems that can be encountered in the large in analytically extending a metric.

We wish to thank B. DeWitt, K. Kuchař, W. D. Markiewicz, C. W. Misner, and J. A. Wheeler for interesting discussions.

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*Work supported in part by the National Science Foundation under Grant No. GP 7669.

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