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Radiative Corrections to the Muon Polarization in $K_{\mu 3}^{\pm}$ Decays*

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Radiative corrections to the muon polarization vector in $K_{\mu 3}^{\pm}$ decays have been calculated. The model, previously used, is based on a phenomenological weak K-π vertex and perturbation theory. The answer depends logarithmically on a cutoff, although the corrections are not nearly as sensitive to variations in the cutoff as those for unpolarized decays. All terms which contribute to order α have been retained, with no approximations concerning the smallness of the muon mass or the "real" inner bremsstrahlung. The corrections to the three components of the polarization vector (longitudinal, transverse, and perpendicular) have been evaluated numerically at points in the Dalitz plot, for various complex values of ξ. In addition, the integrated correction to the degree of polarization as a function of muon energy has also been computed. It is found that these corrections are small, generally less than or equal to 1%, except in kinematically queer regions, where their relative size can be appreciable.

I. INTRODUCTON

It is generally recognized that a measurement of the polarization of the muons from $K_{\mu 3}$ decays provides a sensitive and independent means of determining the parameter $\xi(q^2)$.[1] In fact, at each point in the Dalitz plot, the muon is 100% polarized along a direction which depends on the value of $\xi(q^2)$ at that point. This method has been emphasized by Cabibbo and Maksymowicz.[2] The presence of electromagnetic interactions, however, alters this conclusion in that the muon polarization is no longer 100% in any direction, but is somewhat less. Therefore, an estimate of the radiative corrections to the polarization of muons from $K_{\mu 3}$ decays is necessary in order to apply this method for studying $\xi(q^2)$.

In this paper, we extend our previous calculations of radiative corrections to $K_{\mu 3}$ decays[3-6] to cover the case of polarized muons from $K_{\mu 3}^{\pm}$ decays. Two other attempts at estimating these corrections have been published.[7,8] In the first, Benfatto, Nicolo, and Rossi employ a shortcut based on estimating the change in a coefficient $b=\frac{1}{2}m_\mu(\xi-1)$ due to the contribution of the vertex diagrams in the radiative correction. This procedure is faulty in that the inner-bremsstrahlung contribution is neglected. However, it does give a rough impression of the size of the corrections. The status of the second calculation is less certain. Becherrawy's[8] results for the radiative corrections to unpolarized $K_{l3}^{\pm}$ decays do not agree with ours, and there are some obvious errors in his calculation of the inner-bremsstrahlung contribution to the muon polarization.[9] It appears that a closer look at these radiative corrections is justified.

The model which we have used assumes the usual phenomenological Lagrangian, in which the weak K-π vertex is described by two form factors, $f_+(q^2)$ and $f_-(q^2)$, where $q^2=(p_K-p_\pi)^2$. Electromagnetic interactions are added via the minimal gauge-invariant coupling which is implied by the substitution $p_\alpha \to p_\alpha - eA_\alpha$ for the charged particles present. The radiative corrections are computed using the standard perturbation theory to first order in α, with the assumption that the form factors are constant.

This model, which has been described elsewhere,[3-6] suffers from a number of limitations, principally the logarithmic dependence of the corrections on an ultraviolet cutoff Λ. Several authors have suggested the possibility of eliminating the ultraviolet divergence by including the contribution of the axial-vector part of the weak hadronic current and/or by introducing vector mesons which mediate the weak interaction.[7,8,10-16] The net effect of these types of models is to replace the conventional cutoff parameter Λ with a parameter characteristic of the particular model, typically on the order of a few BeV. Therefore, the numerical results for the radiative corrections will not be very different from the simple cutoff-dependent model, which we elect to use below. Of course, if the numerical estimates are sensitive to the value of the cutoff, they are useful only as an indication of the general order of magnitude of the corrections. Our model also neglects the momentum dependence of the form factors in calculating the radiative corrections. Experimentally, the momentum dependence of the form factors is weak, i.e., the parameters $\lambda_\pm$ in the expansion $f_\pm(0)(1+\lambda_\pm q^2/m_\pi^2)$ are small. If the form factors are assumed to be smoothly varying functions of q^2, it is reasonable to hope that their momentum dependence also produces an effect of order $\lambda_\pm$ on the radiative corrections.

II. CALCULATION

The starting point for the calculation of the radiative corrections to the polarization of muons from $K^+_{\mu 3}$ decays[17] is the matrix elements given previously,[18] and we will adhere to the notation of those papers with some minor exceptions. The only difference is, since we are interested in polarized muons, we must insert a spin projection operator[19] $H_+ = \frac{1}{2}(1 + i\gamma_5 \not{n})$ in front of the muon Dirac spinor, where $p_\mu \cdot n = 0$ and $n^2 = -1$. In the muon rest system, where $n_\alpha = (0, \hat{n}_R)$, the operator H_+ projects onto a spin direction parallel to the unit vector $\hat{n}_R$. After the insertion of $H_+ v_\mu$ in place of v_μ in the matrix elements, the transition rate for the decay into polarized muons can be calculated using the standard covariant trace techniques.

The terms in the decay rate which depend on the four-vector, n_α, can be expressed in terms of the unit vector in the muon rest frame, $\hat{n}_R$, by means of a Lorentz transformation. If a_α is an arbitrary four-vector, then

$$a\cdot n = \hat{n}_R \cdot \left[\frac{\vec{p}_\mu}{m_\mu}\left(a_0 - \frac{\vec{a}\cdot\vec{p}_\mu}{E_\mu + m_\mu}\right) - \vec{a}\right], \qquad (1)$$

where $(a_0, \vec{a})$ are the components of a_α in the system where the muon has energy-momentum $(E_\mu, \vec{p}_\mu)$. The decay rate for muons polarized along the direction $\hat{n}_R$ in their rest frame can then be written in the form

$$\Gamma(E_\mu, E_\pi; \hat{n}_R) = \tfrac{1}{2}[\Gamma(E_\mu, E_\pi) + \hat{n}_R \cdot \vec{A}(E_\mu, E_\pi)], \qquad (2)$$

where $\Gamma(E_\mu, E_\pi)$ is the decay rate for unpolarized muons.[6] The "polarization vector" of the muons is conventionally defined by

$$\vec{\mathcal{P}}(E_\mu, E_\pi) = \vec{A}(E_\mu, E_\pi)/\Gamma(E_\mu, E_\pi), \qquad (3)$$

so that the decay rate for polarized muons can also be written

$$\Gamma(E_\mu, E_\pi; \hat{n}_R) = \tfrac{1}{2}\Gamma(E_\mu, E_\pi)[1 + \hat{n}_R \cdot \vec{\mathcal{P}}(E_\mu, E_\pi)]. \qquad (4)$$

The component of the polarization vector in the direction of $\hat{n}_R$ is called the "degree of polarization along $\hat{n}_R$," which is simply the ratio of the difference and sum of the transition rates for muons polarized parallel to, or antiparallel to, $\hat{n}_R$, that is,

$$\hat{n}_R \cdot \vec{\mathcal{P}}(E_\mu, E_\pi) = \frac{\Gamma(\text{up}) - \Gamma(\text{down})}{\Gamma(\text{up}) + \Gamma(\text{down})} = \frac{\Gamma(E_\mu, E_\pi; \hat{n}_R) - \Gamma(E_\mu, E_\pi; -\hat{n}_R)}{\Gamma(E_\mu, E_\pi)}. \qquad (5)$$

For each of the quantities $\Gamma(E_\mu, E_\pi)$, $\Gamma(E_\mu, E_\pi; \hat{n}_R)$, $\vec{A}(E_\mu, E_\pi)$, and $\vec{\mathcal{P}}(E_\mu, E_\pi)$ we shall designate the contribution of zero order in α, and the radiative corrections to first order in α, by the subscripts 0 and RC, respectively. For example

$$\vec{\mathcal{P}}(E_\mu, E_\pi) = \vec{\mathcal{P}}_0(E_\mu, E_\pi) + \vec{\mathcal{P}}_{RC}(E_\mu, E_\pi), \qquad (6)$$

where

$$\vec{\mathcal{P}}_{RC}(E_\mu, E_\pi) = [\vec{A}_{RC}(E_\mu, E_\pi) - \Gamma_{RC}(E_\mu, E_\pi)\vec{\mathcal{P}}_0(E_\mu, E_\pi)]/\Gamma(E_\mu, E_\pi). \qquad (7)$$

Equation (7) shows that one should expect a radiative correction to the muon polarization vector of the same order of magnitude as the correction to the Dalitz-plot transition rate, even if $\vec{A}_{RC}(E_\mu, E_\pi)$ should accidentally vanish.

The radiative corrections to the above quantities originate from two sources: the virtual corrections (designated by the subscript V) and the inner bremsstrahlung. The latter can be conveniently split into two parts; the first is infrared divergent (designated by the subscript IR) and the second is the so-called real inner bremsstrahlung (designated by the subscript RIB). For example, we may write

$$\vec{A}_{RC}(E_\mu, E_\pi) = \vec{A}_V(E_\mu, E_\pi) + \vec{A}_{IR}(E_\mu, E_\pi) + \vec{A}_{RIB}(E_\mu, E_\pi). \qquad (8)$$

In the expressions below, however, we shall omit writing the infrared divergent terms, which cancel out exactly when the various contributions to the radiative corrections are combined. The distinction between the infrared and real inner bremsstrahlung is somewhat arbitrary, and no physical or kinematic constraint (such as a minimum photon energy or an energy resolution) is implied. The experimental conditions for which the present radiative corrections are calculated are that all photons, which are emitted in events for which the observed momenta of the muon and pion fit the three-body kinematics for $K^+_{\mu 3}$ decay, remain undetected.

Equivalent expressions for the zero-order muon polarization vector have been obtained by several authors.[2,20-22] In the center-of-mass system of the decaying particles, i.e., the kaon rest frame (to which all noncovariant expressions in this paper refer), we can express the muon polarization vector in terms of its components along the three mutually perpendicular unit vectors defined by

$$\hat{\epsilon}_L = \vec{p}_\mu / p_\mu,$$
$$\hat{\epsilon}_\perp = \vec{p}_\pi \times \vec{p}_\mu / |\vec{p}_\pi \times \vec{p}_\mu| \qquad (9)$$
$$\hat{\epsilon}_T = \hat{\epsilon}_L \times \hat{\epsilon}_\perp .$$

The longitudinal unit vector $\hat{\epsilon}_L$ is parallel to the muon momentum, $\hat{\epsilon}_T$ is transverse to $\hat{\epsilon}_L$ in the plane of the decay, while $\hat{\epsilon}_\perp$ is perpendicular to the decay plane. The zero-order muon polarization vector is

$$\vec{\mathcal{P}}_0(E_\mu, E_\pi) = \vec{A}_0(E_\mu, E_\pi)/\Gamma_0(E_\mu, E_\pi), \tag{10}$$

where

$$\begin{aligned}(2\pi)^3|f_+|^{-2}\vec{A}_0(E_\mu,E_\pi) = {}& [2E_\nu - (W_\pi - E_\pi)\,\mathrm{Re}(1-\xi)]\,m_K p_\mu \hat{\epsilon}_L \\ & - [m_K - E_\mu\,\mathrm{Re}(1-\xi) + \tfrac{1}{4}m_\mu^{\,2}|1-\xi|^2/m_K]\{[p_\mu(m_K - E_\pi) + p_\pi E_\mu \cos\Psi_0]\hat{\epsilon}_L + m_\mu p_\pi \sin\Psi_0 \hat{\epsilon}_T\} \\ & + \mathrm{Im}(\xi) m_\mu p_\mu p_\pi \sin\Psi_0 \hat{\epsilon}_\perp,\end{aligned} \tag{11}$$

and $\Gamma_0(E_\mu, E_\pi)$ is given by Eq. (1) of Ref. 6. The angle between the directions of the muon and the pion, when no inner bremsstrahlung is present, is Ψ_0, and

$$\cos\Psi_0 = (x_{\max} - 2p_\mu p_\pi)(2p_\mu p_\pi)^{-1}, \tag{12}$$

$$\sin\Psi_0 = [x_{\max}(4p_\mu p_\pi - x_{\max})]^{1/2}(2p_\mu p_\pi)^{-1}. \tag{13}$$

The result emphasized by Cabibbo and Maksymowicz[2] is that $|\vec{A}_0(E_\mu, E_\pi)| = \Gamma_0(E_\mu, E_\pi)$ or $|\vec{\mathcal{P}}_0(E_\mu, E_\pi)| = 1$, as can be verified explicitly from Eq. (11).

The virtual corrections to the vector $\vec{A}(E_\mu, E_\pi)$ are given by

$$\vec{A}_V(E_\mu, E_\pi) = A_{V,L}(E_\mu, E_\pi)\hat{\epsilon}_L + A_{V,T}(E_\mu, E_\pi)\hat{\epsilon}_T + A_{V,\perp}(E_\mu, E_\pi)\hat{\epsilon}_\perp, \tag{14}$$

where

$$\begin{aligned}(2\pi)^3|f_+|^{-2}(\pi/\alpha)A_{V,L}(E_\mu,E_\pi) = {}& m_K p_\mu\{2(W_\mu - E_\mu)\,\mathrm{Re}A + \tfrac{1}{2}(W_\pi - E_\pi)\,\mathrm{Re}[(1+\xi)(A^* + B)]\} \\ & - \{H_\mu^{\,2}\,\mathrm{Re}A + \tfrac{1}{2}(m_k E_\mu - m_\mu^{\,2})\,\mathrm{Re}[(1+\xi)(A^*+B)] + \tfrac{1}{4}m_\mu^{\,2}|1+\xi|^2\,\mathrm{Re}B\} \\ & \times m_K^{\,-1}[p_\mu(m_K - E_\pi) + p_\pi E_\mu \cos\Psi_0],\end{aligned} \tag{15}$$

$$\begin{aligned}(2\pi)^3|f_+|^{-2}A_{V,T}(E_\mu,E_\pi) = {}& -(\alpha/\pi)m_K^{\,-1}m_\mu p_\pi \sin\Psi_0 \\ & \times\{H_\mu^{\,2}\,\mathrm{Re}A + \tfrac{1}{2}(m_K E_\mu - m_\mu^{\,2})\,\mathrm{Re}[(1+\xi)(A^* + B)] + \tfrac{1}{4}m_\mu^{\,2}|1+\xi|^2\,\mathrm{Re}B\},\end{aligned} \tag{16}$$

$$(2\pi)^3|f_+|^{-2}A_{V,\perp}(E_\mu,E_\pi) = \tfrac{1}{2}(\alpha/\pi)m_\mu p_\mu p_\pi \sin\Psi_0\,\mathrm{Im}[(1+\xi)(A^*+B)]. \tag{17}$$

In the above expressions for the components of $\vec{A}_V(E_\mu, E_\pi)$, A and B are given by Eqs. (4) and (5) of Ref. 6.

The infrared contribution to the radiative correction to the vector $\vec{A}(E_\mu, E_\pi)$ may be written as

$$\vec{A}_{IR}(E_\mu, E_\pi) = (\alpha/\pi)I_0(E_\mu, E_\pi)\vec{A}_0(E_\mu, E_\pi) + \vec{A}'_{IR}(E_\mu, E_\pi), \tag{18}$$

where $I_0(E_\mu, E_\pi)$ is given by Eq. (9) of Ref. 6, and $\vec{A}'_{IR}(E_\mu, E_\pi)$ is evaluated below. In order to extract the infrared divergent terms as defined in Eq. (18), it is necessary to take explicit account of the fact that the angle Ψ between the muon and pion directions [and hence the direction of the unit vectors defined in Eq. (9)] depends on the invariant mass x of the undetected particles (the neutrino-photon combination). When inner bremsstrahlung is present, in place of Eqs. (12) and (13), one must use the relations

$$\cos\Psi = (2p_\mu p_\pi)^{-1}(x_{\max} - x - 2p_\mu p_\pi) \tag{19}$$

and

$$\sin\Psi = (2p_\mu p_\pi)^{-1}[(x_{\max} - x)(4p_\mu p_\pi - x_{\max} + x)]^{1/2}. \tag{20}$$

The evaluation of the infrared contribution involves limits of integrals of the general form

$$\lim_{\lambda\to 0}\int_{\lambda^2}^{x_{\max}} dx\, f(x,\lambda)\sin\Psi, \tag{21}$$

in addition to the more usual terms which contribute to $I_0(E_\mu, E_\pi)$. In Eq. (21), λ is the "fictitious" photon mass, and $f(x,\lambda)$ is a function which, for $\lambda = 0$, diverges like x^{-1} as x approaches zero. The correct evaluation of expressions such as Eq. (21), including terms of order zero in λ as well as the dominant $\ln\lambda$ terms, is facilitated by means of a standard Euler substitution.[23] The limit of the expression in Eq. (21) can be transformed into

$$\lim_{\lambda\to 0} \sin\Psi_0 \int_{\lambda^2}^{x_{\max}} dx\, f(x,\lambda) + O(\lambda) + \int_{t_{\min}}^{t_{\max}} dt\; 2t(1+t^2)^{-2}[(t^2-1)\sin\Psi_0 - 2t\cos\Psi_0]\bar{x}f(\bar{x},0), \tag{22}$$

where

$$\bar{x} = 4p_\mu p_\pi(\cos\Psi_0 - t\sin\Psi_0)(1+t^2)^{-1} \tag{23}$$

and

$$t_{\max} = -\tan\tfrac{1}{2}\Psi_0, \quad t_{\min} = \cot\Psi_0. \tag{24}$$

It turns out that for the integrals involved in the radiative corrections to the muon polarization in $K^{\pm}_{\mu 3}$ decays, the product $\bar{x}f(\bar{x},0)$ in Eq. (22) is independent of $\bar{x}$ (and, therefore, of t also). In this case, the second integral in Eq. (22) is easily found to be

$$\begin{aligned}(2p_\mu p_\pi)^{-1} I_t(E_\mu, E_\pi) &\equiv \int_{t_{\min}}^{t_{\max}} dt\, 2t(1+t^2)^{-2}[(t^2-1)\sin\Psi_0 - 2t\cos\Psi_0] \\ &= \{[-1 + 2\ln(2\sin\tfrac{1}{2}\Psi_0)]\sin\Psi_0 + (\pi - \Psi_0)\cos\Psi_0\}.\end{aligned} \tag{25}$$

We can now give the expression for the term $\vec{A}'_{IR}(E_\mu, E_\pi)$ defined in Eq. (18):

$$\begin{aligned}\vec{A}'_{IR}(E_\mu,E_\pi) = {}& (\alpha/\pi)(2\pi)^{-3}|f_+|^2\{-1 + (E_\mu/p_\mu)\ln[(E_\mu + p_\mu/m_\mu]\} \\ &\times\{-x_{\max}p_\mu \mathrm{Re}(1+\xi)\hat{\epsilon}_L + [m_K - E_\mu \mathrm{Re}(1-\xi) + \tfrac{1}{4}m_\mu{}^2|1-\xi|^2/m_K]\, p_\mu{}^{-1}[E_\mu x_{\max}\hat{\epsilon}_L - m_\mu I_t(E_\mu,E_\pi)\hat{\epsilon}_T] \\ &+ m_\mu \mathrm{Im}\,\xi I_t(E_\mu,E_\pi)\hat{\epsilon}_\perp\}.\end{aligned} \tag{26}$$

It can be seen from the logarithm in the first pair of brackets in Eq. (26) that the $\vec{A}'_{IR}(E_\mu,E_\pi)$ term is comparable in magnitude to similar parts of the $I_0(E_\mu,E_\pi)$ terms in Eq. (18).[24]

The real inner bremsstrahlung part of the radiative corrections to the muon polarization in $K^{\pm}_{\mu 3}$ decays can be expressed as a sum of integrals over x. Because the integrands involve products of rational functions of x, square roots, logarithms, and $\sin\Psi$, it was decided to do the integrations numerically on a computer. The integration subroutine, based on a Gaussian quadrature formula, was tested on some sample terms which can be evaluated analytically and was found to be as accurate as the computer library functions. Nevertheless, in a few relatively infrequent integrals, the convergence was so poor that the accuracy dropped to a few percent. However, these rare integrals were typically one to two orders of magnitude smaller than the other integrals which contributed. In view of the inherent uncertainty in the present model due to the cutoff, an accuracy of a few percent (better than $\frac{1}{2}\%$ in the vast majority of terms) in some parts of the real inner bremsstrahlung contribution was considered tolerable.

In order to write down the real inner bremsstrahlung contribution to the vector $\vec{A}_{RC}(E_\mu,E_\pi)$ in a manner consistent with the coordinate system implied by the unit vectors in Eq. (9), we define the components of $\vec{A}_{RIB}(E_\mu,E_\pi)$ by

$$\vec{A}_{RIB}(E_\mu,E_\pi) = A_{RIB,L}(E_\mu,E_\pi)\hat{\epsilon}_L + A_{RIB,T}(E_\mu,E_\pi)\hat{\epsilon}_T + A_{RIB,\perp}(E_\mu,E_\pi)\hat{\epsilon}_\perp. \tag{27}$$

The expressions for the components in Eq. (27) are admittedly more cumbersome than the covariant expression for the corresponding parts of $\Gamma(E_\mu, E_\pi; \hat{n}_R)$, but Eq. (27) is more directly related to experimental measurements. If we let the subscript i denote a longitudinal or transverse component (i.e., $i = L$ or T) then

$$A_{RIB,i}(E_\mu,E_\pi) = \frac{\alpha}{4\pi}\frac{|f_+|^2}{(2\pi)^3}\frac{m_\mu}{m_K}\int_0^{x_{\max}} dx\{T_K(p_K\cdot n)_i + T_\pi(p_\pi\cdot n)_i + \sum_{m,n,a,b} T(m,n;a,b)[n_\alpha I^\alpha_{m,n}(p_a,p_b)]_i\}, \tag{28}$$

and

$$A_{RIB,\perp}(E_\mu,E_\pi) = \frac{\alpha}{4\pi}\frac{|f_+|^2}{(2\pi)^3}\frac{m_\mu}{m_K}\mathrm{Im}(\xi)\int_0^{x_{\max}} dx\, T_\perp, \tag{29}$$

where the summation indices m, n serve to identify the invariant integrals and a, b refer to the particles μ, π, or K.

The notation $(n_\alpha a^\alpha)_i$, where a is an arbitrary four-vector, means the ith component of the vector $\vec{a}_R$ in the rest system of the muon, expressed in terms of the components $(a_0, \vec{a})$ in the center-of-

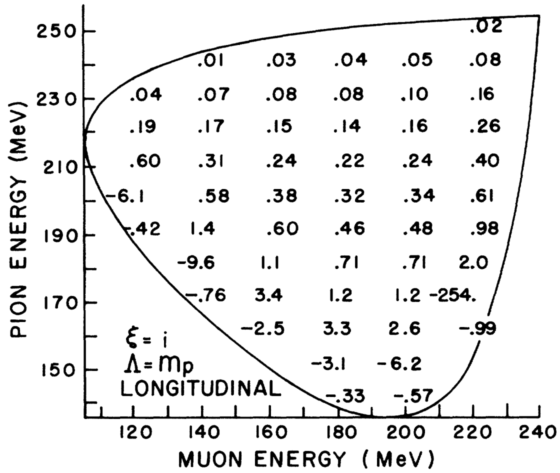

FIG. 1. Percent radiative corrections to the longitudinal component of the muon polarization at various points in the $K^+_{\mu 3}$ Dalitz plot (indicated by the decimal points).

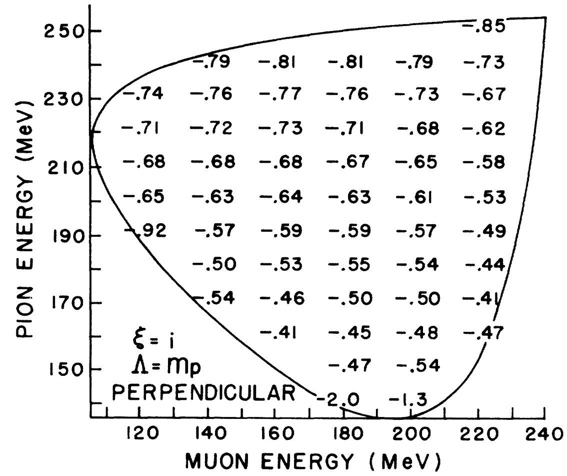

FIG. 3. Percent radiative corrections to the perpendicular component of the muon polarization at various points in the $K^+_{\mu 3}$ Dalitz plot (indicated by the decimal points).

mass system. From Eq. (1)

$$(n_\alpha a^\alpha)_i = \left[\frac{\vec{p}_\mu}{m_\mu}\left(a_0 - \frac{\vec{a}\cdot\vec{p}_\mu}{E_\mu + m_\mu}\right) - \vec{a}\right]\cdot\hat{\epsilon}_i\,. \tag{30}$$

For the two vectors which appear explicitly in Eq. (28)

$$\begin{aligned}
&(p_K\cdot n)_L = m_K p_\mu / m_\mu,\\
&(p_K\cdot n)_T = (p_K\cdot n)_\perp = 0,\\
&(p_\pi\cdot n)_L = (p_\mu E_\pi - p_\pi E_\mu \cos\Psi)/m_\mu,\\
&(p_\pi\cdot n)_T = -p_\pi \sin\Psi, \quad (p_\pi\cdot n)_\perp = 0.
\end{aligned} \tag{31}$$

Finally, the invariant integrals $I^\alpha_{m,n}(p_a, p_b)$ are defined in Appendix A, and the expressions for the coefficients T_K, T_π, $T_\perp$, and $T(m,n;a,b)$ are given in Appendix B.

III. RESULTS

The sum of the contributions indicated in Eqs. (14), (18), and (27) constitute the radiative corrections to the vector $\vec{A}(E_\mu, E_\pi)$ as indicated in Eq. (8). From this value of $\vec{A}_{RC}(E_\mu, E_\pi)$, together with the results of Sec. II in Ref. 6, the radiative corrections to the muon polarization vector $\vec{\mathcal{P}}_{RC}(E_\mu, E_\pi)$ as a function of position in the Dalitz plot can be found by the use of Eq. (7). We have evaluated $\vec{\mathcal{P}}_{RC}(E_\mu, E_\pi)$ on a computer, in 10-MeV bins throughout the Dalitz plot, for several complex values of ξ, and for cutoffs of one- and two-proton masses. A sampling of these results is shown in Figs. 1–6.

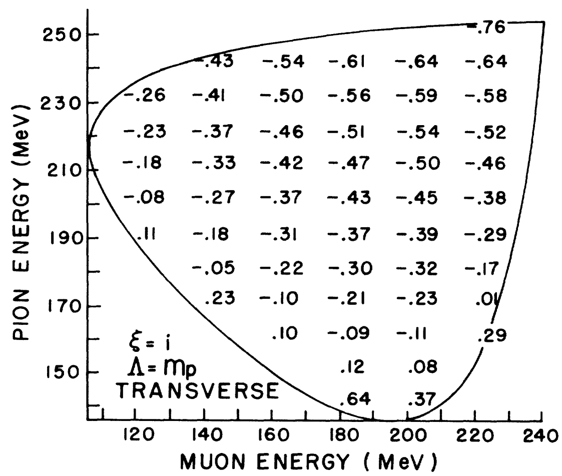

FIG. 2. Percent radiative corrections to the transverse component of the muon polarization at various points in the $K^+_{\mu 3}$ Dalitz plot (indicated by the decimal points).

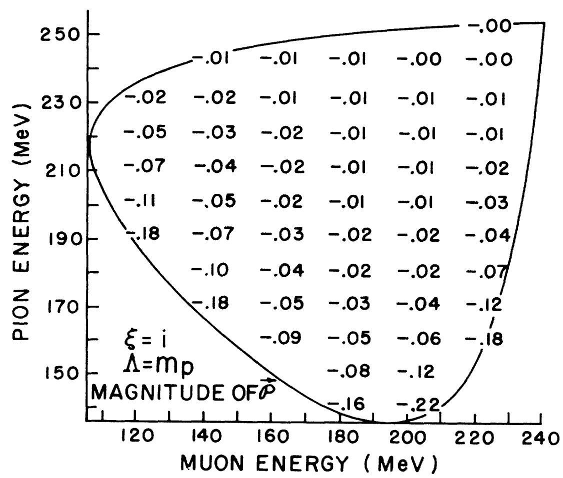

FIG. 4. Percent radiative corrections to the magnitude of the muon polarization at various points in the $K^+_{\mu 3}$ Dalitz plot (indicated by the decimal points).

Figs. 1–3 indicate the percent correction to each of the three components of the muon polarization vector, namely, $100\times \mathcal{P}_{\mathrm{RC},i}(E_\mu,E_\pi)$, where i stands for a longitudinal, transverse, or perpendicular component, respectively. In Fig. 4, we plot the percent correction to the magnitude of the polarization vector, namely, $100\times[|\vec{\mathcal{P}}(E_\mu,E_\pi)|-1]$. We have chosen a complex value of ξ, equal to one imaginary unit, so that all three components of the muon polarization would be present. Of course, the radiative correction to the magnitude of the polarization vector must be negative, as required by unitarity. This restriction was satisfied for all values of ξ and Λ, and at all points in the Dalitz plot, for which we calculated radiative corrections, which gives us additional confidence in the accuracy of our analysis. In Figs. 5 and 6 we show the actual values of the longitudinal and transverse polarizations and their radiative corrections, for two real values of ξ, namely ±1, as a function of pion energy and for a fixed muon energy of 190 MeV. The marked dependence on ξ, of the zero-order polarization, was noted in Ref. 2 and is also illustrated, at points throughout the Dalitz plot, in the literature.[25]

The size of the radiative corrections to the components of the muon polarization vector is small, generally less than or equal to 1% of the uncorrected values. An exception to this occurs in the lower portion of the Dalitz plot, for the percent radiative correction to the longitudinal component of the polarization vector, which is quite large (−254% at one point in Fig. 1). The reason for this is that

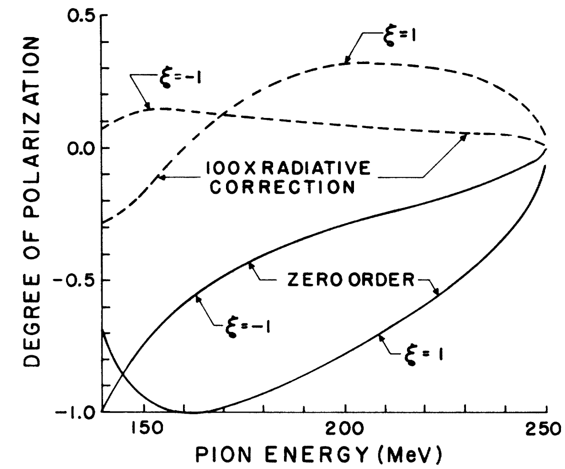

FIG. 6. Transverse component of the muon polarization in $K^+_{\mu 3}$ to zero order, and the first-order radiative corrections, for $E_\mu = 190$ MeV, $\xi=\pm1$, and $\Lambda = m_p$.

the zero-order expression for the longitudinal polarization may vanish at certain points in the Dalitz plot, while the radiative corrections do not have any such behavior. It follows immediately from Eq. (11) that $A_{0,L}(E_\mu,E_\pi)$ is a linear function of E_π; however, we refrain from giving the easily obtainable expression because the positions of the zeros in the Dalitz plot depend on ξ and do not appear to correspond to any particular kinematic configuration for the decay products. It may be that future measurements of the longitudinal polar-

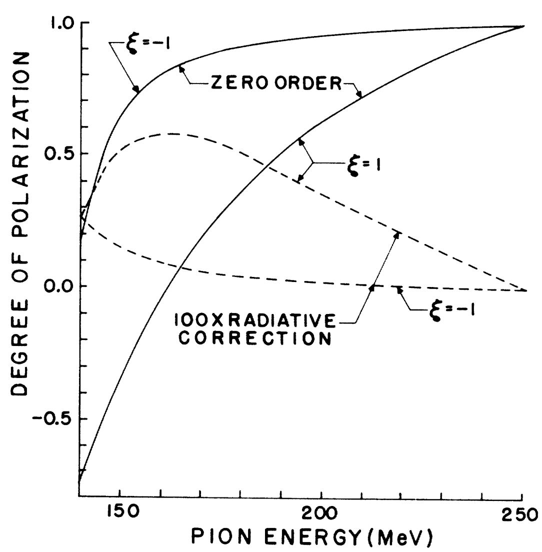

FIG. 5. Longitudinal component of the muon polarization in $K^+_{\mu 3}$ to zero order, and the first-order radiative corrections, for $E_\mu = 190$ MeV, $\xi=\pm1$, and $\Lambda = m_p$.

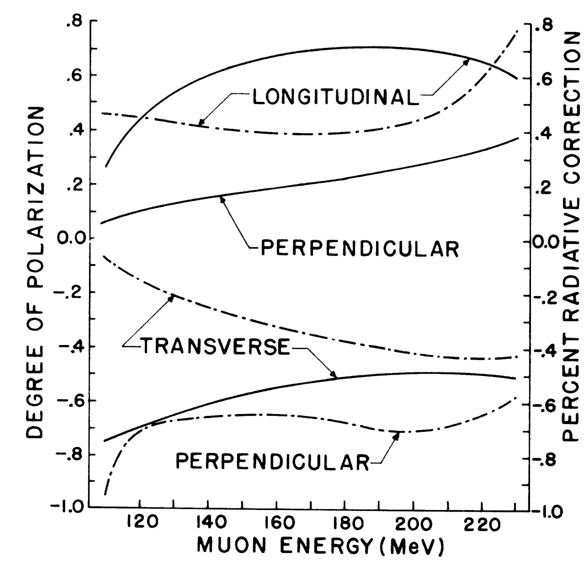

FIG. 7. Components of the polarization vector for the muon energy spectrum in $K^+_{\mu 3}$ decay. The solid lines are the zero-order contribution; the dashed-dot lines are the percent radiative corrections for $\xi = i$ and $\Lambda = m_p$.

ization, in Dalitz plot bins a few MeV wide, will be sensitive to this effect. It can readily be seen from Fig. 4 that the radiative corrections to the magnitude of the polarization vector are an order of magnitude smaller than the corrections to the components of the polarization vector. In other words, the radiative corrections change the direction of the polarization vector much more than its length.

In spite of their small size, the radiative corrections to the muon polarization are not especially sensitive to variations in the cutoff parameter, Λ. When Λ is doubled, from m_p to $2m_p$, the corrections are increased in size by between 0 and 25% depending on position in the Dalitz plot. This may be contrasted with the cutoff dependence of the unpolarized radiative corrections,[6] which were much more sensitive to a similar variation in Λ. For this reason, the numerical results of the present calculation probably give a more reliable indication of the radiative corrections than is the case with unpolarized $K^{\pm}_{\mu 3}$ events.

Finally, we have obtained the radiative corrections to the energy spectrum of polarized muons by numerical integration of the above corrections over pion energies. A sample of these results is given in Fig. 7, which shows the three components of the polarization vector and the percent radiative corrections as a function of muon energy, for $\xi = i$ and $\Lambda = m_p$. The size of the corrections in Fig. 7 is commensurate with the radiative corrections to the muon polarization throughout the Dalitz plot, indicating no strong cancellations. It should be noted that the magnitude of the muon polarization vector averaged over pion energies is no longer 100%, and, therefore, the radiative corrections to the magnitude can be positive and still be consistent with unitarity.

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APPENDIX A

The scalar invariant integrals are defined by

$$I_{m,n}(p_i,p_j)=\frac{1}{2\pi}\int\frac{d^3k}{k_0}\frac{d^3k'}{k'_0}\frac{\delta^{(4)}(P-k-k')}{(p_i\cdot k)^m(p_j\cdot k)^n} \tag{A1}$$

for $k^2=k'^2=0$. Those which contribute to the radiative corrections in this paper have been evaluated previously (in the Appendix to Ref. 4) with the exception of

$$\begin{aligned}4\beta_1{}^6I_{2,-3}(p_1,p_2)={}&x^2\{3(p_1p_2{:}P)(p_2P:p_1)^2+\tfrac{3}{2}\Delta_{12}[2\alpha_1(p_2P:p_1)-m_1{}^2(p_1p_2:P)]\}I_{1,0}(p_1,p_2)\\&+2m_1{}^{-2}x^2(p_2P:p_1)^3+\alpha_1(p_1p_2:P)(2\beta_1{}^2\beta_2{}^2+x\Delta_{12})+6x(p_2P:p_1)(\beta_1{}^2\beta_2{}^2-2x\Delta_{12}),\end{aligned} \tag{A2}$$

$$\begin{aligned}&P^2=x,\\&\alpha_i=p_i\cdot P,\\&\beta_i=(\alpha_i{}^2-m_1{}^2x)^{1/2},\\&\gamma_{ij}=[(p_i\cdot p_j)^2-m_i{}^2m_j{}^2]^{1/2},\\&\Delta_{ij}=\alpha_i(p_iP:p_j)+\alpha_j(p_jP:p_i)-x\gamma_{ij}{}^2=x^{-1}[\beta_i{}^2\beta_j{}^2-(p_ip_j:P)^2],\\&(ab:c)=(a\cdot c)(b\cdot c)-c^2(a\cdot b).\end{aligned} \tag{A3}$$

The vector invariant integrals are defined by

$$I^{\alpha}_{m,n}(p_1,p_2)=\frac{1}{2\pi}\int\frac{d^3k}{k_0}\frac{d^3k'}{k'_0}\frac{k^{\alpha}\delta^{(4)}(P-k-k')}{(p_i\cdot k)^m(p_j\cdot k)^n} \tag{A4}$$

also for $k^2=k'^2=0$. On invariance grounds $I^{\alpha}_{m,n}(p_1,p_2)$ must be a linear combination of p_1^{α}, p_2^{α}, and P^{α}, and since the invariant product of $I^{\alpha}_{m,n}(p_1,p_2)$ with each of these three vectors is proportional to a scalar invariant integral, the coefficients of the linear combination can easily be determined. One finds that[26]:

$$\begin{aligned}\Delta_{12}I^{\alpha}_{m,n}(p_1,p_2)={}&\tfrac{1}{2}x[p_1^{\alpha}(p_1P:p_2)+p_2^{\alpha}(p_2P:p_1)-P^{\alpha}\gamma_{12}{}^2]I_{m,n}(p_1,p_2)+[-p_1^{\alpha}\beta_2{}^2+p_2^{\alpha}(p_1p_2:P)+P^{\alpha}(p_1P:p_2)]I_{m-1,n}(p_1,p_2)\\&+[p_1^{\alpha}(p_1p_2:P)-p_1^{\alpha}\beta_1{}^2+P^{\alpha}(p_2P:p_1)]I_{m,n-1}(p_1,p_2).\end{aligned} \tag{A5}$$

APPENDIX B

The coefficients appearing in Eqs. (28) and (29) are

$$T_K \equiv (H_\mu^2 - m_\pi^2 + x)[I_{1,0}(p_\mu, p_K) - I_{0,1}(p_\mu, p_K)] + 2[2 - I_{1,-1}(p_\mu, p_K) - I_{-1,1}(p_\mu, p_K)]$$
$$+ \mathrm{Re}(1+\xi)[m_\mu^2 I_{2,-1}(p_\mu, p_\pi) + I_{-1,1}(p_\mu, p_K) - 1 + \tfrac{1}{2}x m_\mu^2 I_{1,1}(p_\mu, p_K) - (m_K^2 + m_\mu^2 - H_\mu^2 - H_\pi^2 + x) I_{1,0}(p_\mu, p_K)$$
$$+ (m_K^2 - m_K E_\nu - \tfrac{1}{2}H_\pi^2 + \tfrac{1}{2}m_\pi^2 + x) I_{0,1}(p_\mu, p_K)] - \tfrac{1}{2}x[H_\mu^2 + \tfrac{1}{4}m_\mu^2 |1+\xi|^2] I_{1,1}(p_\mu, p_K), \quad \text{(B1)}$$

$$T_\pi \equiv \mathrm{Re}(1-\xi)\{2 m_K E_\mu [I_{0,1}(p_\mu, p_K) - I_{1,0}(p_\mu, p_K)]$$
$$+ m_\mu^2 [I_{2,-1}(p_\mu, p_K) - I_{1,0}(p_\mu, p_K)] + m_K^2 [I_{0,1}(p_\mu, p_K) - I_{-1,2}(p_\mu, p_K)]\}, \quad \text{(B2)}$$

$$T(1,0;\mu,K) = -T(0,1;\mu,K)$$
$$\equiv m_K[2E_\pi + E_\mu \mathrm{Re}(1+\xi)], \quad \text{(B3)}$$

$$T(1,1;\mu,K) \equiv m_K(2E_\mu + E_\nu)(m_\pi^2 + \tfrac{1}{4}m_\mu^2|1+\xi|^2) - m_K E_\pi (H_\pi^2 - m_\mu^2)$$
$$+ \tfrac{1}{2}m_K \mathrm{Re}(1+\xi)[m_K x + (2E_\mu + E_\nu)(m_K^2 - H_\pi^2 - H_\mu^2) - E_\pi(H_\pi^2 - m_\mu^2)], \quad \text{(B4)}$$

$$T(2,0;\mu,K) \equiv -\tfrac{1}{2}(H_\pi^2 + m_\mu^2)(m_\pi^2 + \tfrac{1}{4}m_\mu^2|1+\xi|^2) + \tfrac{1}{2}(m_K^2 - H_\pi^2 - H_\mu^2 + x)[H_\mu^2 - m_\pi^2 - x - m_\mu^2 \mathrm{Re}(1+\xi)], \quad \text{(B5)}$$

$$T(0,2;\mu,K) \equiv -m_K^2[m_\pi^2 + \tfrac{1}{4}m_\mu^2|1+\xi|^2 + \tfrac{1}{2}(m_K^2 - H_\pi^2 - H_\mu^2 + x)\mathrm{Re}(1+\xi)], \quad \text{(B6)}$$

$$T(2,-1;\mu,\pi) = 2m_K(E_\nu - E_\mu) - 2x + m_\mu^2 \mathrm{Re}(1-\xi), \quad \text{(B7)}$$

$$T(2,-2;\mu,\pi) \equiv -2, \quad \text{(B8)}$$

$$T_\perp \equiv m_K p_\mu p_\pi [I_{1,0}(p_\mu, p_K) - I_{0,1}(p_\mu, p_K)] \sin\Psi$$
$$+ \{(p_\mu P : p_K)[I_{-1,1}(p_\mu, p_K) - 3] - (p_K P : p_\mu)[I_{1,-1}(p_\mu, p_K) - 3]$$
$$+ m_\mu^2[(p_K P : p_\mu) - (p_\mu p_K : P)] I_{2,-1}(p_\mu, p_K) - 4m_K^2 p_\mu^2 + m_K^2[(p_\mu P : p_K) - (p_\mu p_K : P)] I_{-1,2}(p_\mu, p_K)$$
$$+ [m_\mu^2(p_\mu P : p_K) + m_K E_\mu (p_\mu p_K : P) - (2m_K E_\mu - \tfrac{1}{2}x)(p_K P : p_\mu)$$
$$+ m_K E_\mu \beta_\mu^2 + m_K m_\mu^2 (E_\nu^2 - x) + \tfrac{1}{2}m_K^2 p_\mu^2 x] I_{1,0}(p_\mu, p_K)$$
$$+ [m_K^2(p_K P : p_\mu) - m_K E_\mu (p_\mu p_K : P) - (2m_K E_\mu - \tfrac{1}{2}x)(p_\mu P : p_K)$$
$$- m_\mu^2 m_K E_\mu (E_\nu^2 - x) - \tfrac{1}{2}x(m_K^2 p_\mu^2 + \beta_\mu^2)] I_{0,1}(p_\mu, p_K)$$
$$+ \tfrac{1}{2}x(m_K E_\mu[(p_\mu P : p_K) - (p_K P : p_\mu) + 2m_K^2 p_\mu^2] - \tfrac{1}{2}x m_K^2 p_\mu^2) I_{1,1}(p_\mu, p_K)\} (m_K p_\mu p_\pi \sin\Psi)^{-1}, \quad \text{(B9)}$$

where the notation used is the same as in Appendix A, and $P_\alpha \equiv (p_K - p_\mu - p_\pi)_\alpha$.

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Scaling and the Behavior of Nucleon Resonances in Inelastic Electron-Nucleon Scattering*

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The behavior of elastic scattering and of the electroproduction of nucleon resonances is shown to be closely related to the behavior of deep-inelastic electron-nucleon scattering. This relation is discussed in the context of duality ideas taken from strong-interaction processes. These ideas suggest that a substantial part of the observed behavior of inelastic electron-nucleon scattering is due to a nondiffractive component of virtual photon-nucleon scattering. Through finite-energy sum rules, quantitative relations between the elastic and resonance electroproduction form factors and the deep-inelastic scattering are derived and the behavior of inelastic scattering near threshold is calculated.

I. INTRODUCTION

High-energy inelastic electron-nucleon scattering is a unique probe of the charge distribution inside the nucleon and provides a method for searching for a possible substructure. Since experiments have revealed a large cross section for inelastic electron-proton scattering, there have been many different attempts to understand the physical origin of the observed regularities of the scattering, particularly the deep-inelastic scattering at high energies and large momentum transfers. In this paper we will show that the behavior of the deep-inelastic scattering is related in a striking way to the behavior of elastic scattering and of nucleon-resonance electroproduction. The relation between resonance electroproduction and deep-inelastic scattering is tied up closely with theoretical ideas, particularly about duality, which arise from the behavior of purely hadronic scattering processes. This leads us to a discussion of sum rules, and finally to quantitative relations between the elastic and resonance form factors and the inelastic structure functions. While we have dealt with these questions in a previous short paper,[1] we present here an extended discussion of the theoretical ideas as well as their consequences in quantitative detail.

We focus our attention on the process of inelastic electron-nucleon scattering where an electron of known energy (E) is scattered by a nucleon through a measured angle (θ) to a smaller final energy (E') due to the exchange of a single photon.[2] In general, the nucleon breaks up due to the scattering, and if only the final electron is observed, then the double differential cross section can be written as

$$\frac{d^2\sigma}{d\Omega' dE'}=\frac{4\alpha^2 E'^2}{q^4}\left[2W_1(\nu,q^2)\sin^2(\tfrac{1}{2}\theta)+W_2(\nu,q^2)\cos^2(\tfrac{1}{2}\theta)\right]. \tag{1}$$

The results of the scattering are thus summarized in the structure functions W_1 and W_2 which depend on the exchanged photon's laboratory energy, $\nu = E - E'$, minus the invariant mass squared, $q^2 = 4EE'\sin^2(\frac{1}{2}\theta)$. Knowing ν and q^2 from measuring the incident and scattered electron, the invariant mass W of the final hadrons is fixed by

$$s = W^2 = 2M_N\nu + M_N{}^2 - q^2 . \tag{2}$$

We can also consider inelastic electron scattering as a collision between the exchanged virtual photon and the target nucleon. One is then simply studying the total cross section of the process "γ" $+ p \to$ hadrons, where the hadrons have an invariant mass W, and we are able to vary the energy, mass, and polarization of the incident photon beam. This leads one to define total virtual photon-nucleon cross sections for transversely and longitudinally polarized photons, $\sigma_T(\nu,q^2)$ and $\sigma_S(\nu,q^2)$, which are related to W_1 and W_2 by[2]