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Radiation from a Beam of Modulated Electrons (The Schwarz-Hora Effect)

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The mechanism of radiation from a beam of modulated electrons (Schwarz-Hora effect) is discussed. It is shown that the effect should be quadratic (at a given velocity of the electrons) in the current arriving to the screen, being due to a coherent emission from the electrons in the beam; no emission linear in the current, specific to the modulation, is expected. In the conditions of the experiment, however, the calculated radiated power turns out to be at least 10^3 times smaller than the observed power.

I. INTRODUCTION

The purpose of this paper is to discuss the interesting effect recently observed by Schwarz and Hora[1,2]: When a beam of electrons (Fig. 1) passes through a thin crystal with a superimposed laser beam, it is observed (this is the Schwarz-Hora effect) that the electrons produce light of the same color of the laser light when they impinge on a *nonluminescent* screen.

This effect poses two problems: (a) to calculate the modulation of the wave function of each electron when it crosses the crystal in the presence of the laser light, and (b) to try to clarify the mechanism by which the modulated electrons produce light when they impinge on the nonluminescent screen.

Problem (a) is a standard though complicated problem in quantum mechanics[3]; although we have solved it quite generally (in the eikonal approximation), in this note we shall only report some schematic formulas – by now rather well known[2,4] – which are necessary for the discussion of problem (b). It is indeed to this problem that we shall focus our attention here. We can anticipate the results of this note as follows:

(1) The rate of radiation at the laser frequency $\bar{\omega}$ by an *individual electron* is substantially independent of whether the wave function of such an electron is or is not modulated at frequency $\bar{\omega}$; in other words, if we call "incoherent" the radiation emitted by the individual electrons impinging on the screen, the modulation of the wave function of the electrons does not produce an enhancement at frequency $\bar{\omega}$ above the ordinary transition or bremsstrahlung radiation in the incoherent radiation rate.

(2) There is the possibility of a preferential emission at frequency $\bar{\omega}$ only when the cooperative, or more precisely, coherent effect of all the modulated electrons is taken into account.

(3) If the radiation observed by Schwarz and Hora is due to the above coherent emission, the power emitted from the screen should be proportional (for a given electron velocity) to the square of the electron current; it becomes in our opinion very important to establish this point experimentally.

(4) The power emitted at frequency $\bar{\omega}$ from the screen by the above coherent mechanism is, however, according to our estimates, *at least* 10^3 times smaller than the power of 10^{-10} W observed by Schwarz and Hora.

II. THE MODULATED WAVE FUNCTION

For simplicity we consider here a one-dimensional schematization; the electron is assumed to

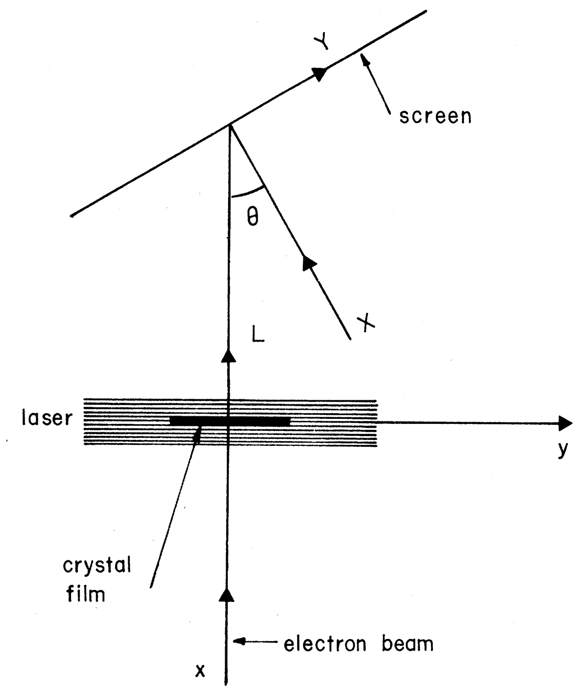

FIG. 1. A scheme of the experiment showing the notation used in the discussion.

travel along the x axis both before and after traversing the crystal. With reference to the real situation this amounts, essentially, to considering only the central Laue spot, neglecting the others.

As to the wave packet of each electron, this will be assumed to have a Gaussian shape, both in the direction of propagation (x axis) and in the transverse plane. Indicating with $G_{X,v}$ the wave packet of an electron centered at time t around the point $x = X + vt$, $y = z = 0$, and moving with velocity v, we shall therefore write

$$G_{X,v} = (\pi^{3/2}D^2a)^{-1/2}\exp\left\{-\left[\frac{(x-X-vt)^2}{2a^2}+\frac{y^2+z^2}{2D^2}\right]\right\}$$
$$= G^{\parallel}_{X,v}\,G_{\perp}\,, \tag{1}$$

where the last equality defines the longitudinal (that is, along the x axis) and transverse packets.[5]

In this notation the wave function of an electron which has traversed the crystal film with the superimposed electric field of the laser can be written

$$\varphi = a_0 G_{X,v_0} e^{i(p_0 x - E_0 t)} + a_+ G_{X,v_+}\exp i(p_+ x - E_+ t)$$
$$+ a_- G_{X,v_-}\exp i(p_- x - E_- t)\,, \tag{2}$$

where a_0 is the amplitude of the unperturbed wave and a_+, a_- are the amplitudes of the two main satellites which correspond to an increase or decrease $\hbar\bar{\omega}$ of the electron energy by the laser electric field. It is, indeed, in (2),

$$E_\pm = \frac{p_\pm^{\,2}}{2m} = E_0 \pm \hbar\bar{\omega}\,. \tag{3}$$

The numerical values of a_0, a_+, a_- depend on the characteristics of the crystal film and on the amplitude and polarization of the electric field of the laser.

Note that the three wave packets in (2) move with slightly different velocities, so that they should fail to overlap at a distance from the crystal larger than

$$L_{max} \approx 2\frac{m v_0^{\,2}}{\hbar\bar{\omega}}\,a\,. \tag{4}$$

At a distance larger than this no effect due to the superposition of the three terms in (2) should persist[6]; at smaller distances, to be accurate, the effects of the different velocities of the three packets should be taken into account, but we shall, for simplicity, put the following in (2): $G_{X,v_0} = G_{X,v_+} = G_{X,v_-}$. Thus the wave function (2) can be approximately rewritten

$$\varphi_X(\vec{x},t) = G_{X,v_0}\left[a_0\exp i\,(p_0x - E_0t) + a_+\exp i(p_+x - E_+t)\right.$$
$$\left. + a_-\exp i(p_-x - E_-t)\right]$$
$$\equiv \varphi_0 + \varphi_+ + \varphi_-\,, \tag{5}$$

where the last step is a definition of φ_0, φ_+, and φ_-; note also that in what follows the suffix in v_0 will be omitted.

Using the wave function (5), it is straightforward to construct the probability density

$$\rho_p(xyz,t) = G_{X,v}^{\,2}\left(A + B\cos\frac{2\pi x}{\Lambda}\cos\frac{\bar{\omega}}{v}(x - vt)\right) \tag{6}$$

and the density of current of probability

$$\vec{J}_p(xyz,t) \cong \vec{v}\rho_p(xyz,t)$$
$$= \vec{v}G_{X,v}^{\,2}\left(A + B\cos\frac{2\pi x}{\Lambda}\cos\frac{\bar{\omega}}{v}(x - vt)\right), \tag{7}$$

where $A = a_0^{\,2} + a_+^{\,2} + a_-^{\,2}$ and $B = 4a_0a_+ = 4a_0a_-$ and where the suffix p recalls the fact that the above expressions are probability densities. In deriving (6) and (7) from (5), we have assumed $a_+ = a_-$ real and put[2]

$$\Lambda = \frac{\hbar}{p}16\pi\left(\frac{E_0}{\hbar\bar{\omega}}\right)^2 = 1.64\text{ cm}.$$

The existence (noted by Schwarz[2]) of maxima and zeros in the modulating term in the expressions (6) and (7) is particularly interesting; it is quite remarkable that, corresponding to those maxima and zeros, maxima and minima are experimentally observed in the radiated power at frequency $\bar{\omega}$ from the screen when the position of the screen is continuously changed.[2]

III. INCOHERENT RADIATION

We prove now that an individual electron with a modulated wave function (5) when impinging on the screen cannot emit radiation at frequency $\bar{\omega}$ in an amount substantially different than it would do if its wave function were not modulated. The proof is extremely simple and quite general. The wave function (5) is the sum of three terms. Call T the evolution operator of this wave function under the action of the screen; T is a linear operator. The amplitude for emission of a photon at any frequency is therefore the sum of the amplitudes from each of the three addends in the wave function; no one of these amplitudes is enhanced at frequency $\bar{\omega}$. This proves our assertion.

Note that on the basis of the expressions (6) and (7), one might have been tempted to argue as follows: *Each* electron has a charge density $e\rho_p(xyz,t)$ and a current density $e\vec{J}_p(xyz,t)$; therefore it produces an electric field at the screen at frequency $\bar{\omega}$. The screen is polarized by this field and therefore each electron arriving at the screen contributes additively to the radiation emitted at frequency $\bar{\omega}$. This argument is wrong as is shown by the previous argument; indeed, ρ_p and $\vec{J}_p$ are probability densities and not charge current densities. In conclusion, aside from the ordinary bremsstrahlung or transition radiation, the rate of radiation from the screen at frequency $\bar{\omega}$ cannot be linear in the number of electrons arriving per second on the screen. We shall now show, instead, that a rate of emission at frequency $\bar{\omega}$ proportional to the square of the above number is expected.

IV. COHERENT RADIATION

The main result that we shall prove in this section will be the following: The rate of radiation at frequency $\bar{\omega}$ is the same rate that would be produced by a *classical* current density, i.e.,

$$\vec{J}(xyz,t)=e\frac{n}{L}\vec{v}G_\perp{}^2(yz)\times\left[A+B\cos\frac{2\pi(x+L)}{\Lambda}\cos\frac{\bar{\omega}}{v}(x-vt)\right], \tag{8}$$

impinging on the screen. [L is the distance between the crystal and the screen and n is the number of electrons contained in this space at any given instant.] In (8) and from now on, the origin of the x axis is chosen at the screen, while in the formulas given so far it was at the crystal.

Let us consider the current-density operator $\vec{\mathfrak{J}}(\vec{x})$; for what follows, the only important point is that $\vec{\mathfrak{J}}(\vec{x})$ is additive. It is

$$\vec{\mathfrak{J}}(\vec{x})=\sum_{i=1}^{n}\vec{\mathfrak{J}}_{\vec{x}_i}(\vec{x}), \tag{9}$$

where

$$\vec{\mathfrak{J}}_{\vec{x}_i}(\vec{x})=(\vec{P}_i/m)\delta(\vec{x}_i-\vec{x}). \tag{10}$$

The matrix element for radiation of a photon of momentum $\vec{k}$, energy $\bar{\omega}$ is given by

$$M_{IF}^{(X)}=\int_0^T\left\langle F\left|e^{-iH_0t}\frac{1}{\sqrt{2\omega}}\int d\vec{x}\,\mathfrak{J}_\epsilon(\vec{x})e^{i(\vec{k}\cdot\vec{x}-\omega t)}e^{iH_0t}\right|I\right\rangle dt. \tag{11}$$

Here H_0 is the free-electron Hamiltonian $[H_0=-(\hbar^2/2m)\sum_i\Delta_i]$, $\mathfrak{J}_\epsilon(\vec{x})$ is the projection of the current operator [given by (9) and (10)] along the polarization of the photon, and $|I\rangle$ and $\langle F|$ are the initial and final states of the electron beam at $t=0$. In terms of the wave function of an individual electron, given by (5), the wave function corresponding to the state I can be written as

$$\Psi_I=\prod_{i=1}^{n}\varphi_{X_i}(\vec{x}_i\,|\,t=0), \tag{12}$$

where the product is extended over all the electrons of the beam. In writing this wave function we have neglected the antisymmetrization because, in the actual situation, the wave packets of the individual electrons are sufficiently far apart with respect to their longitudinal size a.[7]

The matrix element (11) depends on the values of the X_i of each electron, a dependence which is noted in (11) by the superscript (X); to be more precise, $X\equiv(X_1,X_2,\ldots,X_n)$ characterizes the configuration of the centers of all the wave packets of the electrons in the beam at $t=0$.

Inserting now (9) in (11), we get

$$M_{IF}^{(X)}=\sum_i\langle F|R_i|I\rangle, \tag{13}$$

where we have put

$$R_i\equiv R_{x_i}=\frac{1}{\sqrt{2\omega}}\int_0^T dt\int d\vec{x}\,e^{-iH_0t}\vec{\epsilon}\cdot\vec{\mathfrak{J}}_{\vec{x}_i}(\vec{x})e^{i(\vec{k}\cdot\vec{x}-\omega t)}e^{iH_0t}. \tag{14}$$

Let us first consider the matrix element $M_{IF}^{(X)}$(13) to a state $F\neq I$. Due to the additive structure of (13) and to the product structure (12) of wave function Ψ_I only final states F where just one electron has changed its state have a nonvanishing matrix element. Therefore the power emitted in transitions to states $F\neq I$ is additive in the contributions from the individual electrons; but, as we have shown in Sec. III, there is no enhancement at the modulation frequency $\bar{\omega}$ in the power emitted individually by the electrons.

We can therefore confine ourselves to considering the matrix element (13) to the state $F=I$ and calculate the rate of radiation as

$$\frac{dP^{(X)}}{dt}=\frac{1}{T}\sum_{i\neq j}\langle I|R_i|I\rangle\langle I|R_j^\dagger|I\rangle\rho\,, \tag{15}$$

where we have omitted in the sum (15) terms with $i=j$ because again they constitute a contribution to the power emitted by the individual electrons.[8] In (15), ρ is the density of the photon final states.

We have still to perform on Eq. (15) the average over all the possible X configurations in the beam. In order to do this, we assume that at a given time the center of the wave packet of each electron has the same probability of being anywhere between $-L$ and 0, where $-L$ is the abscissa of the crystal and 0 that of the screen; by this assumption we neglect again any correlation due to the Pauli principle. The operation of averaging is thus represented by

$$\mathrm{Av}_X=\frac{1}{L^n}\int dX_1dX_2\cdots dX_n\,. \tag{16}$$

The rate of radiation is therefore

$$\begin{aligned}\frac{dP}{dt}&=\mathrm{Av}_X\frac{dP^{(X)}}{dt}\\&=\frac{1}{T}\frac{1}{L^n}\int_{-L}^{0}\prod_k dX_k\sum_{i\neq j}\langle I|R_i|I\rangle\langle I|R_j^\dagger|I\rangle\rho\\&=\frac{1}{T}\frac{n(n-1)}{L^2}\int_{-L}^{0}dX_1dX_2\langle I|R_1|I\rangle\langle I|R_2^\dagger|I\rangle\rho\\&=\frac{1}{T}\frac{n(n-1)}{L^2}\left|\int_{-L}^{0}\langle I|R_1|I\rangle dX_1\right|^2\rho\\&\cong\frac{1}{T}\frac{n^2}{L^2}\left|\int_{-L}^{0}\langle I|R_1|I\rangle dX_1\right|^2\rho\,.\end{aligned} \tag{17}$$

In (17) we have used the fact that $\langle I|R_i|I\rangle$ depends only on X_i and that the functional dependence on X_i is the same for all the values of i. Let us now define

$$\vec{\mathfrak{J}}(\vec{x},t)=\frac{n}{L}\int_{-L}^{0}dX\int d\vec{\xi}\,\varphi_X^*(\vec{\xi},t)\vec{\mathfrak{J}}_{\vec{\xi}}(\vec{x})\varphi_X(\vec{\xi},t), \tag{18}$$

where

$$\varphi_X(\vec{x}t)=e^{iH_0t}\varphi_X(\vec{x}),$$

and observe that $\vec{\mathfrak{J}}(\vec{x},t)$ so defined is precisely the classical current (8); the quantity

$$\frac{n}{L}\int_{-L}^{0}\langle I|R_1|I\rangle dX_1,$$

which appears in (17), can therefore be written, recalling the expression (14) of R, as

$$\frac{n}{L}\int_{-L}^{0}\langle I|R_1|I\rangle dX_1=\frac{1}{\sqrt{2\omega}}\int_0^T dt\int d\vec{x}\,\vec{\epsilon}\cdot\vec{\mathfrak{J}}(\vec{x}t)\times\exp i(\vec{k}\cdot\vec{x}-\omega t)\,. \tag{19}$$

We finally obtain from (19) the power emitted in the element of solid angle along $\vec{k}$ ($\hbar=1$),

$$\begin{aligned}dW_{\vec{k}}&=\omega\frac{dP}{dt}\\&=\frac{1}{2T}\left|\int_0^T dt\int d\vec{x}\,\vec{\epsilon}\cdot\vec{\mathfrak{J}}(\vec{x},t)\exp i(\vec{k}\cdot\vec{x}-\omega t)\right|^2\rho\,,\end{aligned} \tag{20}$$

which is precisely the power at frequency ω which would be produced by the classical current $\vec{\mathfrak{J}}(\vec{x},t)$; note that $\vec{\mathfrak{J}}(\vec{x},t)$ has a harmonic time dependence at frequency $\bar{\omega}$ which leads to a $\delta(\omega-\bar{\omega})$ from the time integral in (20); this proves the result stated at the beginning.

We end this section with two remarks.

(1) In the above calculation we have treated the screen simply as a boundary condition on the current (8); it is indeed this boundary condition that allows the radiation to be emitted. We have, however, neglected so far the fact that the incident electrons do produce an induced current in the screen which also radiates. In the case of a metallic screen this induced current is simply the mirror image of the incident current with respect to the surface of the screen. In Eq. (20) we shall therefore from now on *interpret* $\vec{\mathfrak{J}}(\vec{x},t)$ *as being given by Eq.* (8) *plus its mirror image*. Formally, this can be obtained replacing (10) by

$$\frac{\vec{P}_i}{m}\delta(\vec{x}-\vec{x}_i)-\frac{\vec{P}_{iM}}{m}\delta(\vec{x}-\vec{x}_{iM}),$$

where $\vec{x}_{iM}$, $\vec{P}_{iM}$ are the images of $\vec{x}_i$, $\vec{P}_i$ with respect to the screen. For a dielectric screen the situation is somewhat more complicated although we cannot foresee at the moment a difference in the order of magnitude between the rates of radiation against a metal or a dielectric screen.

(2) Equation (20) can be simplified by decomposing the current into positive- and negative-frequency parts:

$$\vec{\mathfrak{J}}(\vec{x},t)=\vec{\mathfrak{J}}^{(+)}(\vec{x})e^{i\bar{\omega}t}+\vec{\mathfrak{J}}^{(-)}(\vec{x})e^{-i\bar{\omega}t}\,. \tag{21}$$

We get

$$\begin{aligned}dW_k&=\pi\delta(\omega-\bar{\omega})\rho\left|\int d\vec{x}\,\vec{\epsilon}\cdot\vec{\mathfrak{J}}^{(+)}(\vec{x})e^{i\vec{k}\cdot\vec{x}}\right|^2\\&=\pi\delta(\omega-\bar{\omega})\rho\,|\vec{\epsilon}\cdot\vec{\mathfrak{J}}^{(+)}(\vec{k})|^2,\end{aligned} \tag{22}$$

where $\vec{\mathfrak{J}}^{(+)}(\vec{k})$ is the $\vec{k}$ component of the Fourier transform of $\vec{\mathfrak{J}}^{(+)}(\vec{x})$. This expression will be used in Sec. V for the calculation of the rate of radiation.

V. RATE OF RADIATION

We shall consider a conducting screen normal to the xy plane and at an angle θ with the incident beam of electrons (compare Fig. 1; $x\cos\theta+y\sin\theta=0$ defines the plane of the screen) and calculate using Eq. (22) the power emitted into photons of momentum $\vec{k}$.

Performing a rotation to new coordinates $\vec{R} \equiv X, Y, Z$ such that the plane of the screen becomes the new plane $X = 0$, we can rewrite such current as

$$\vec{J}^{(+)} = \tfrac{1}{2} e \vec{v} B \cos(2\pi/\Lambda)(L + \vec{n}\cdot\vec{R})\exp[-i(\bar{\omega}/v)\vec{n}\cdot\vec{R}]\, G_\perp([\vec{R} - \vec{n}(\vec{n}\cdot\vec{R})]^2)\Theta(-X)$$
$$- \tfrac{1}{2} e \vec{v}_M B \cos(2\pi/\Lambda)(L - \vec{n}'\cdot\vec{R})\exp[i(\bar{\omega}/v)\vec{n}'\cdot\vec{R}]\, G_\perp([\vec{R} - \vec{n}'(\vec{n}'\cdot\vec{R})]^2)\Theta(X), \tag{23}$$

where $\vec{n} = (\cos\theta, \sin\theta, 0)$ and $\vec{n}' = (\cos\theta, -\sin\theta, 0)$. The second term in (23) is the mirror current mentioned in Sec. IV. Inserting this expression, Eq. (23), of the current in (22) and performing the integrations, we obtain for the power radiated into photons of momentum between $\vec{k}$ and $\vec{k} + d\vec{k}$ the expression

$$\int \frac{dW_k}{d\omega} d\omega = \pi S_{\text{pol}} |\vec{\epsilon}\cdot\vec{J}^{(+)}(\vec{k})|^2 \frac{dk_x}{d\omega}\frac{dk_y\, dk_z}{(2\pi)^3}$$
$$\cong \frac{B^2}{2\pi}\frac{e^2}{4\pi}\frac{n^2}{L^2}\frac{v^2}{\bar{\omega}^2}\cos^2\left(\frac{2\pi L}{\Lambda}\right)\exp\left\{-\left[\tfrac{1}{4}k_z^2 D^2 + \frac{[k_y - (\bar{\omega}/v)\sin\theta]^2 D^2}{4\cos^2\theta}\right]\right\}\frac{\sin^2\theta}{\cos^2\theta}\frac{dk_y\, dk_z}{(1 - \sin^2\theta/v^2)^{1/2}}, \tag{24}$$

a formula which holds for

$$\bar{\lambda} \equiv \frac{c}{\bar{\omega}} \ll D \quad\text{and}\quad \sin\theta \lesssim \frac{v}{c} - \frac{\lambda}{D}$$

owing to the approximations performed in the course of the calculation.

From this equation the total power can be calculated by integrating over the directions of $\vec{k}$. Owing to the high directionality of the emitted radiation this integration is easy, and we finally get for the total power (in Heaviside units and having put $c = 1$)

$$W_r = B^2 \frac{e^2}{4\pi}\frac{n^2}{L^2} v^2 \frac{1}{D^2\bar{\omega}^2} \times \cos^2\left(\frac{2\pi L}{\Lambda}\right)\left(1 - \frac{\sin^2\theta}{v^2}\right)^{1/2}\frac{\sin^2\theta}{\cos\theta}. \tag{25}$$

Let us first discuss Eq. (24). The high directionality shown by (24) is expected. Indeed, the electric field accompanying the beam of electrons is directed predominantly along the direction of the beam[9]; the Fourier components of this field are reflected by the screen.

By the same argument it is easy to understand that the maximum angle for which we can have reflection is determined by

$$\sin\theta_{\max} \cong v/c, \tag{26}$$

a relation which can be read from (24). Putting there $k_z = 0$ and $k_y = (\bar{\omega}/v)\sin\theta$ [remember that the Gaussians in (24) are very narrow: $D \approx 10\bar{\lambda} = 10c/\bar{\omega}$; see below], the equation $k_x^2 + k_y^2 + k_z^2 = \bar{\omega}^2/c^2$ can be solved for real k_x only if $\sin\theta < v/c$.

The second noteworthy feature of Eqs. (24) and (25) is the presence of the factor $\cos^2(2\pi L/\Lambda)$; this factor gives the periodic variation of the power emitted with the distance L of the screen to the crystal, as already mentioned in Sec. II. Note that the simple factorization of the factor $\cos^2(2\pi L/\Lambda)$ in (24) or (25) is not exact, but is practically so; it only implies $\Lambda \gg \lambda$.

Before ending this section two points deserve some attention.

(1) If the mechanism of emission of light is not reflection as considered so far, but diffuse reflection (the surface of the screen has many facets oriented at random), the light will not be emitted in a definite direction, but will be diffused more or less isotropically. Still it can be understood easily that also in this case the order of magnitude of the total diffused power is given by (25), where of course the dependence on the angle of incidence θ is omitted; i.e.,

$$W_d \approx B^2 \frac{e^2}{4\pi}\frac{n^2}{L^2} v^2 \frac{1}{D^2\bar{\omega}^2}\cos^2\frac{2\pi L}{\Lambda} \quad (D \gg \bar{\lambda}). \tag{27}$$

(2) So far we have assumed in the calculations that $D \gg c/\bar{\omega} = \bar{\lambda}$. This assumption seems well satisfied, as we shall see in Sec. VI. It is, however, convenient to write the formula similar to (27) in the opposite extreme where $D \ll \bar{\lambda}$. In this case we obtain (in the limit $D \to 0$), for the order of magnitude of the diffused power,

$$W_d' \approx B^2 \frac{e^2}{4\pi}\frac{n^2}{L^2} v^2 \cos^2\left(\frac{2\pi L}{\Lambda}\right) \quad (D \ll \bar{\lambda}), \tag{28}$$

a formula which can be understood by simple dimensional arguments.

VI. NUMERICAL EVALUATION OF THE EMITTED POWER

We now evaluate the order of magnitude of the power radiated at the screen given by Eq. (27). In this equation the quantity $(n/L)v = dn/dt$ can be deduced immediately from the known current density of 0.5 μA as given by Schwarz and Hora:

$$\frac{n}{L}v = \frac{dn}{dt} = 0.5\frac{10^{-6}}{1.6\times 10^{-19}} \cong 3\times 10^{12}\ \text{sec}^{-1}, \quad (29)$$

where we have assumed that all the current goes into one spot, an assumption which, of course, goes in the direction of increasing the calculated power.

Let us now discuss the parameter D which intervenes in (27). Here a reliable estimate is quite difficult; indeed, D depends on several quantities in a complicated way: (1) the transverse size of a wave packet when emitted by the electron gun; (2) the focalization system of the electron beam before the electrons impinge on the crystal; (3) the details of the diffraction by the crystal and, in particular, the size of the microcrystals in the crystal giving rise to the diffraction; and (4) the distance of the screen from the crystal.

A schematic way to estimate D in (27) can be the following: If d is the transverse radius of the part of the wave function of an electron moving towards a given spot immediately after the crystal, the wave packet will then expand with an angle of divergence $\approx h/2pd$. If the screen is at a distance L from the crystal, the radius at the screen is $D = d + (hL/2pd)$, or, preferably,

$$D \approx \left(d^2 + \frac{h^2L^2}{4p^2d^2}\right)^{1/2}. \quad (30)$$

It appears from (30) that D has a minimum value given by $D_{\min} \approx (hL/p)^{1/2}$; taking as a typical value $L = 20\,\text{cm}$ and recalling that it is $p = 10^{-17}$ (c.g.s. units), it turns out that $D_{\min} \approx 10^{-4}$ cm. In the actual experimental conditions, d in (30) should be the smaller of the following two radii: that of a microcrystal in the diffracting film and that of the wave function of an electron incident on the crystal. The electron beam has, according to Ref. 2, a diameter of a few microns; probably we are not far from the value $D_{\min}$ given above. In any case, in the following estimates we shall insert in (27) a value of 10^{-4} cm for D, again in the direction of increasing most probably the calculated power.[10]

Finally, we leave B in (27) as a free parameter for the moment, but we recall that B^2 must be less than 2, by definition[11]; choosing a distance L such that $\cos^2(2\pi L/\Lambda) = 1$, we get

$$W_a \approx 0.5\times 10^{-6}B^2\ \text{erg/sec} = 0.5\times 10^{-13}B^2\ \text{W}. \quad (31)$$

It is seen that even for $B^2 = 2$ we obtain a power which is three orders of magnitude smaller than the power of $\approx 10^{-10}$ W observed by Schwarz. If we take into account that the numbers inserted in (27) have been chosen in the direction of increasing the calculated power, the discrepancy may well increase further; an increase in the discrepancy by two other orders of magnitude should not be considered improbable.

We do not see at the moment how to avoid this discrepancy. Note that, as (28) shows, in the limit $D\to 0$ we might gain two orders of magnitude, but, as appears from the previous discussion, we do not see at the moment any way to justify such limit.

VII. FINAL REMARKS

The list of results emerging from the above analysis has already been presented in Sec. I. While we have nothing to add to that list, to which we refer for the main conclusions, it is appropriate to mention briefly three further points.

1. *Dependence of the power W of the Schwarz-Hora radiation at several values of L on the polarization $\alpha = E_x/E$ of the laser electric field.* We are not able to explain why the intensity of the radiation changes with the polarization of the laser in a different way at different distances (Fig. 5 of Ref. 2). Indeed, we would expect that the polarization of the laser affects only the coefficients a_0, a_+, and a_- in (5) and therefore the quantity B in (8). This would imply that $dW/d\alpha$ should be independent of L, contrary to observation.

2. *Bremsstrahlung.* The following question arises, of course: What is the expected power radiated by normal bremsstrahlung in the visible region from the electrons impinging on the screen? This power is linear in the number of electrons hitting the screen per unit time and we have shown that it is essentially independent of the modulation of the electron's wave function. In the notation of Sec. IV, this power is proportional to

$$\sum_i \sum_F |\langle I|R_i|F\rangle|^2$$

summed over all the final states F including the state $F = I$.

Experimentally, this power looks negligible; indeed, it is only when the laser is in operation that radiation from the nonfluorescent screen is observed. If the laser is off, that is, if the electrons are not modulated, only a weak purple phosphorescence is observed on the nonfluorescent screen, with a luminosity becoming weaker and weaker as the vacuum in the system is improved.[2]

The question is now the following: If we try to calculate the power emitted incoherently in bremsstrahlung, do we get more or less than the coherent power calculated previously? We shall not answer this question here, but only confine ourselves to a few comments in order to indicate some problems which arise.

If we take a model in which the electrons stop instantaneously at the surface of the metal, the order of magnitude of the energy emitted in bremsstrahlung between ω and $\omega + d\omega$ by one electron is given, very roughly, by

$$\frac{1}{137\pi} \times \tfrac{2}{3}\beta^2 \hbar\, d\omega\,.$$

Therefore the power emitted by all the electrons integrated over all the frequencies in the visible turns out to be, in the experimental conditions, $\simeq 10^{-3}$ erg/sec, one thousand times larger than the power emitted coherently at the frequency $\bar{\omega}$ according to the previous calculation. But of course a model in which the velocity of an electron vanishes instantaneously is unrealistic; the formula given above constitutes an upper limit, and both the spectrum of the radiation and the total energy emitted depend critically on the damping time of the current associated to the electron when the electron enters the surface. It seems preferable to wait for more experimental data [particularly on point (3) listed in the Introduction] before discussing further this matter and the related question of how our previous calculation of the coherent emission might be affected also by the detailed behavior of the electrons near to or inside the surface.

3. *Further investigations.* Finally we mention here that it would be of interest to determine the maximum distance L for which the Schwarz-Hora effect is observed. As shown in Eq. (4) this depends on the longitudinal size of the wave packet of an electron. However, Coulomb effects, which we have neglected, should be taken into account in an accurate evaluation of this point.

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[1]H. Schwarz and H. Hora, Appl. Phys. Letters 15, 349 (1969).

[2]H. Schwarz, report, 1970 (unpublished).

[3]A classical treatment of the phenomenon (classical bunching) has been considered by (a) R. L. Harris and R. F. Smith, Nature 225, 502 (1970); (b) P. L. Rubin, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu 11, 356 (1970) [Soviet Physics JETP Letters 11, 239 (1970)]; (c) B. Oliver and L. Cutler, Phys. Rev. Letters 25, 273 (1970). A quantum-mechanical treatment seems however necessary as discussed, e.g., in Ref. 4(a) and as shown most directly by the maxima and minima in the intensity of the emitted radiation when the crystal screen distance is changed continuously.[2]

[4](a) A. Varshalovich and M. D'yakonov, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu 11, 596 (1970) [Soviet Physics JETP Letters 11, 411 (1970)]; (b) L. Favro, D. Fradkin, and P. Kuo, Phys. Rev. Letters 25, 202 (1970).

[5]It is assumed that all the packets have their centers along the x axis, and that they all have the same transverse radius D and longitudinal size a.

[6]Because the effect is observed for values of L up to 35 cm, it follows from (4) that a should be larger than $\approx 5\times 10^{-4}$ cm.

[7]The average distance between two electrons is $\approx 3 \times 10^{-3}$ cm.

[8]The sum $\sum_i \sum_F |\langle I|R_i|F\rangle|^2$, where the set F extends to all the final states F including I, constitutes what we have called ordinary bremsstrahlung. It would be of course incorrect to separate artificially in the above sum the term $F=I$ from the others. A few comments on the ordinary bremsstrahlung will be given in Sec. VII.

[9]We have assumed in writing (2) that the transverse structure of the current (8) is a smoothly decreasing function of y^2+z^2 (a Gaussian). It is possible that in the real situation the transverse structure of the current is more complicated; indeed, an analysis of the diffraction and modulation process at the crystal shows that the precise structure of the transverse part of the current is rather complicated. To simulate such complications, we have repeated the calculation described above introducing in the current an oscillatory behavior in the y direction, represented by a multiplicative factor of the form $\sin qy$. The angular distribution is changed (corresponding to the presence in the incident beam of the transverse Fourier component q) but, with reasonable values of q, the order of magnitude of the reflected intensity is not changed.

[10]Note, incidentally, that with such a small value of D as given above, the fact that the spots appear so large in the photographic plate with respect to their mutual distance should perhaps be attributed to a multiple reflection of the originally emitted photon by many microcrystals in the screen at distances from the original point of emission up to $\approx 10^{-3}$ cm. Note also that an accurate analysis of the apparent size of the spots in the photographic plate might perhaps give an experimental answer to the question of the linear versus quadratic dependence of the effect from the current. In fact the numbers of electrons arriving to the various spots are quite different and the ratios between the apparent magnitudes of the spots on the nonfluorescent screen should be different, if a quadratic dependence holds, from the corresponding ratios on the fluorescent screen where the dependence is linear.

[11]This follows immediately from the definition [after Eq. (7)] of A and B and from the normalization of the wave function (2).