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[1] R. E. Morganstern, Phys. Rev. D 3, 2946 (1971).

[2] R. E. Morganstern, Phys. Rev. D 3, 616 (1971).

[3] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[4] The rather odd-looking transformation on the differential $d\bar{t}$ in Eq. (4) results from the fact that we insist on a "universal time" $\bar{t}$ as our fourth coordinate so that it can no longer be dimensionless, but must carry the units of time. This may be remedied by introducing the new dimensionless parameter η according to $d\bar{t} = \bar{a}_l(\eta)d\eta$ and $dt = ad\eta$ so that the line elements are then related by the second of Eqs. (4). However, we still obtain the same relation between the two universal times t and $\bar{t}$ if we wish to reintroduce them.

[5] This statement is obviously true because it was just the UT of Eq. (4) which generated the other formalisms and their new cosmic times.

[6] In fact, the gravitational constant may be considered as either increasing or decreasing, depending upon the units of measure. Since $\varphi_{(\alpha)} = \varphi_{(0)}\lambda^{\alpha}$, and $G\sim\varphi_{(\alpha)}^{-1}$, it is evident that for $\alpha = 1$, the gravitational constant is decreasing, while for $\alpha = -1$, it is increasing.

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Exact Solutions to Radiation-Filled Brans-Dicke Cosmologies*

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Using the Robertson-Walker metric, exact general solutions to the Brans-Dicke cosmologies for $p = \frac{1}{3}\rho$ are found. The two first integrals which exist for this case reduce the problem to quadratures. In the case of flat space the quadratures may be integrated explicitly in terms of elementary functions, while in the curved spaces ($k = \pm 1$), the solutions may be expressed in terms of elliptic integrals. The limit as a approaches ∞ (or $a_{\max}$ for a closed universe) may be evaluated for positive, negative, and zero curvature, and leads to the rather surprising result that the scalar field λ approaches a constant.

I. INTRODUCTION

In a previous paper[1] we have written down the cosmological equations for the Brans-Dicke (BD) theory in arbitrary units under the usual assumptions of a Robertson-Walker metric and an isotropic matter tensor in comoving coordinates. Here we find exact solutions for flat ($k=0$) and curved spaces ($k=\pm 1$) in the case of a radiation-filled universe. We initially restrict ourselves to the gravitational unit formalism[2] ($\alpha = 0$ of Ref. 1), and find the limiting behavior of the solutions for $t\to 0$ and $t\to\infty$. The rather unexpected result appears that the scalar field λ asymptotically approaches a constant in a radiation-filled universe. Finally, we outline a general procedure for obtaining the solutions in other units, i.e., in terms of their appropriate cosmological time.[3] In general, this procedure cannot be carried out analytically except in the limiting behavior as $t\to 0, \infty$. In the limit as $t\to 0$ the scalar field $\lambda[t_{(\alpha)}]$, the expansion parameter $a_{(\alpha)}[t_{(\alpha)}]$, and the density $\rho_{(\alpha)}[t_{(\alpha)}]$ display somewhat different time dependences for the three natural unit systems,[4] while in the limit as $t\to\infty$ there is no difference, since $\lambda\to$ constant.

II. FIELD EQUATIONS AND SOLUTIONS

For $\alpha = 0$ and $\epsilon = \frac{1}{3}$ the field equations of Ref. 1 reduce to

$$(\dot{a}/a)^2 = \tfrac{1}{3}\varphi_0^{-1}\rho + \tfrac{1}{12}(2\omega+3)\dot{\lambda}^2 - ka^{-2}, \tag{1a}$$

$$a^3\dot{\lambda} = B \equiv a_0^{3}\dot{\lambda}_0, \tag{1b}$$

$$\rho a^4 = q \equiv \rho_0 a_0^{4}, \tag{1c}$$

where we have suppressed bracketed subscripts (0) referring to the units, and where a dot denotes differentiation with respect to the cosmic time $t_{(0)}$. We have also absorbed a factor of 8π into the definition of ρ and have set $c^2 = 1$. Substituting the two first integrals (1b) and (1c) into (1a), we obtain

$$\frac{dt}{da} = (\tfrac{1}{3}\beta B^2 + \tfrac{1}{3}\varphi_0^{-1}qa^2 - ka^4)^{-1/2}a^2, \tag{2}$$

where $\beta = \frac{1}{4}(2\omega+3)$. Equation (3) immediately yields the quadrature

$$t(a) = \int_{u=0}^{u=a} (\tfrac{1}{3}\beta B^2 + \tfrac{1}{3}\varphi_0^{-1}qu^2 - ku^4)^{-1/2}u^2du, \tag{3}$$

where we have set the integration constant equal to zero so that $a=0$ when $t=0$. Equation (3) gives a implicitly as a function of t and, therefore, Eq. (1c) immediately gives ρ implicitly as a function of t. Equation (1b) may now be solved for $\Lambda(a)$ by using the expression for dt/da in Eq. (2). We obtain[5]

$$\Lambda-\Lambda_0=B\int_{u=a_0}^{u=a}[\tfrac{1}{3}\beta B^2+\tfrac{1}{3}\varphi_0^{-1}qu^2-ku^4]^{-1/2}u^{-1}du. \tag{4}$$

We have thus reduced the problem of the BD radiation cosmology to the integration of quadratures (3) and (4).

In the case of flat space ($k=0$), Eqs. (3) and (4) may be integrated immediately to yield

(i) for $k=0$,

$$\begin{aligned}&t=(\tfrac{1}{3}\beta B^2)^{-1/2}\mu^{-3/2}\,\tfrac{1}{2}\{\xi(1+\xi^2)^{1/2}-\ln[\xi+(1+\xi^2)^{1/2}]\},\\&\lambda=\lambda_0\{\xi^{-1}[(1+\xi^2)^{1/2}-1]\}^{(3/\beta)^{1/2}}\{\xi_0^{-1}[(1+\xi_0^2)^{1/2}-1]\}^{-(3/\beta)^{1/2}},\\&\rho=q\mu^2\xi^{-4},\end{aligned} \tag{5}$$

where

$$\xi=\mu^{1/2}a,\quad \mu=\varphi_0^{-1}q/\beta B^2.$$

For the curved-space cases, we can write down the solutions in terms of elliptic integrals of the first and second kind, viz.,

(ii) for $k=-1$,

$$\begin{aligned}&t(a)=a[(l_-^2+a^2)/(m_-^2+a^2)]^{1/2}-l_-E(\alpha,p),\\&\lambda=\lambda_0\{[(l_-^2+a^2)^{1/2}(m_-^2+a^2)^{-1/2}-l_-/m_-][(l_-^2+a^2)^{1/2}(m_-^2+a^2)^{-1/2}+l_-/m_-]^{-1}\}^{\frac{1}{2}(3/\beta)^{1/2}}\{a=a_0\}^{-\frac{1}{2}(3/\beta)^{1/2}};\end{aligned} \tag{6}$$

(iii) for $k=+1$,

$$\begin{aligned}&t(a)=(l_+^2+m_+^2)^{1/2}E(\gamma,r)-\frac{m_+^2}{(l_+^2+m_+^2)^{1/2}}F(\gamma,r)-a\left(\frac{l_+^2-a^2}{m_+^2+a^2}\right)^{1/2},\\&\lambda=\lambda_0\{[(m_+^2+a^2)^{1/2}(l_+^2-a^2)^{-1/2}-m_+/l_+][(m_+^2+a^2)^{1/2}(l_+^2-a^2)^{-1/2}+m_+/l_+]^{-1}\}^{\frac{1}{2}(3/\beta)^{1/2}}\{a=a_0\}^{-\frac{1}{2}(3/\beta)^{1/2}},\end{aligned} \tag{7}$$

where $\{a=a_0\}$ is the expression in braces above evaluated at $a=a_0$, $\lambda_0=\lambda(a_0)$, and

$$\begin{aligned}&E(\alpha,p)=\int_0^{\sin\alpha}(1-p^2y^2)^{1/2}(1-y^2)^{-1/2}dy,\\&F(\gamma,r)=\int_0^{\sin\gamma}(1-y^2)^{-1/2}(1-r^2y^2)^{-1/2}dy,\\&\alpha=\arctan(a/m_-),\quad p=(l_-^2-m_-^2)^{1/2}/l_-,\\&\gamma=\arcsin\left[(a/l_+)\left(\frac{l_+^2+m_+^2}{m_+^2+a^2}\right)^{1/2}\right],\\&p=(l_-^2-m_-^2)^{1/2}/l_-,\\&r=l_+/(l_+^2+m_+^2)^{1/2},\\&l_-^2=\tfrac{1}{2}(\tfrac{1}{3}\varphi_0^{-1}q)[1+(1-S)^{1/2}],\\&m_-^2=\tfrac{1}{2}(\tfrac{1}{3}\varphi_0^{-1}q)[1-(1-S)^{1/2}],\\&l_+^2=a_{\max}^2=\tfrac{1}{2}(\tfrac{1}{3}\varphi_0^{-1}q)[(1+S)^{1/2}+1],\\&m_+^2=\tfrac{1}{2}(\tfrac{1}{3}\varphi_0^{-1}q)[(1+S)^{1/2}-1],\\&S\equiv 12\beta B^2/(\varphi_0^{-1}q)^2.\end{aligned} \tag{8}$$

III. LIMITING BEHAVIOR OF SOLUTIONS

Although the analytic solutions given in Sec. II completely solve the problem of the BD radiation-filled universe, they do not make their behavior very apparent in the two important regions near the initial singularity ($a\to 0$) and for late epochs $a\to\infty$ ($k=-1,0$) or $a\to a_{\max}$ ($k=+1$). In addition, we shall find in Sec. IV that although the general problem of transforming the present solutions to other unit-formalisms cannot be solved analytically, it can be achieved in the two limiting regions. This might give some further insight as to the possible significance of units in the BD theory.

For these reasons, we now turn to the problem of finding the limiting behavior of the solutions of the last section. There are two methods which may be used to find the behavior of these solutions. The first simply involves expanding the exact solutions in the appropriate regions; the second involves asymptotic expansion of the quadratures given in Eqs. (3) and (4). The second method can be used even when the explicit solutions are not known, and moreover it provides a convenient check on the asymptotic behaviors of the exact solutions to within an additive constant[6] (see Appendix). We have computed the limiting behavior of the solution using both methods, and of course we find that the results agree. We summarize them as follows:

(i) For $k=0$,

$$a\to 0\begin{cases} t=\frac{1}{3}(\frac{1}{3}\beta B^2)^{-1/2}a^3\,, \\ \lambda=\lambda_0 e^{-J_0(a_0)}(\frac{1}{2}\mu^{1/2}a)^{(3/\beta)^{1/2}}\,, \end{cases}$$

$$a\to\infty\begin{cases} t=\frac{1}{2}(\frac{1}{3}\varphi_0{}^{-1}q)^{-1/2}a^2\,, \\ \lambda=\lambda_0\exp[-J_0(a_0)][1-B(\frac{1}{3}\varphi_0{}^{-1}q)^{-1/2}a^{-1}]; \end{cases} \tag{9}$$

(ii) for $k=-1$,

$$a\to 0\begin{cases} t=\frac{1}{3}(\frac{1}{3}\beta B^2)^{-1/2}a^3\,, \\ \lambda=\lambda_0\exp[-J_-(a_0)](1-S)^{(3/\beta)^{1/2}/4}(\frac{1}{2}\mu^{1/2}a)^{(3/\beta)^{1/2}}\,, \end{cases}$$

$$a\to\infty\begin{cases} t=a-2^{-1/2}(\frac{1}{3}\varphi_0{}^{-1}q)^{1/2}[1+(1-S)^{1/2}]^{1/2}E(\frac{1}{2}\pi,p), \\ \lambda=\lambda_0\exp[-J_-(a_0)][(1-S)/(1+2S^{1/2})^2]^{(3/\beta)^{1/2}/4}(1-\frac{1}{2}Ba^{-2}); \end{cases} \tag{10}$$

(iii) for $k=1$,

$$a\to 0\begin{cases} t=\frac{1}{3}(\frac{1}{3}\beta B^2)^{-1/2}a^3\,, \\ \lambda=\lambda_0\exp[-J_+(a_0)](1+S)^{(3/\beta)^{1/2}/4}(\frac{1}{2}\mu^{1/2}a)^{(3/\beta)^{1/2}}\,, \end{cases}$$

$$a\to a_{\max}\begin{cases} t=t_{\max}-(\frac{1}{3}\varphi_0{}^{-1}q)^{1/2}\times\frac{1}{2}[(1+S)^{1/2}+1][1+S]^{-1/4}[1-(a/a_{\max})^2]^{1/2}, \\ \lambda=\lambda_0\exp[-J_+(a_0)]\{1-(3/\beta)^{1/2}2^{-1/2}[(1+S)^{1/2}-1][1+S]^{-1/4}[1-(a/a_{\max})^2]^{1/2}\}, \end{cases} \tag{11}$$

where the quantity $J_k(a_0)$ ($k=0,\pm1$) is the expression for the integral Λ (i.e., drop Λ_0) evaluated at a_0 [see Eqs. (5)–(8)] and $t_{\max}=t(a_{\max})$. From the above solutions (9)–(11), we see that as $a\to 0$ the expansion parameter has the same $t^{1/3}$ behavior for all k and the common time scale $\tau=\frac{1}{3}(3/\beta B^2)^{1/2}$, which is determined by the initial conditions on the scalar field by means of the constant $B=\dot{\Lambda}_0 a_0{}^3$ [cf. Eq. (1b)]. For $a_0\sim10^{28}$ cm and $\dot{\Lambda}_0\sim10^{-19}$ (the value obtained by equating the scalar energy density to the radiation energy density), we find $B\sim10^{65}$, so that the time scale is extremely small near the singularity. It is also evident that for all k the scalar field λ approaches zero as $t^{(3/\beta)^{1/2}/3}$.

In the limit as a approaches ∞ (or $a_{\max}$), we find that the scalar field approaches a constant. The time scales and time dependences involved in Eqs. (9)–(11) are quite different from one another. The $k=0$ time scale $\tau=\frac{1}{2}(3/\varphi_0{}^{-1}q)^{1/2}$ depends only upon the initial condition of the matter field. On the other hand, the $k=-1$ time scale is independent of both the matter and scalar field initial conditions. The $k=+1$ time scale depends upon both the matter and scalar-field initial conditions.

IV. UNIT-TRANSFORMED COSMOLOGIES

Up to now we have considered the $\alpha=0$ BD formalism and have obtained quadratures and limiting forms for ρ, a, and λ in a radiation-filled universe. In order to find the solutions in arbitrary units (α), we may (at least in principle) simply apply the following units transformation[7]:

$$a_{(\alpha)}=\lambda^{-\frac{1}{2}\alpha}a_{(0)}, \tag{12a}$$

$$dt_{(\alpha)}=\lambda^{-\frac{1}{2}\alpha}dt_{(0)}, \tag{12b}$$

$$\rho_{(\alpha)}=\lambda^{2\alpha}\rho_{(0)}, \tag{12c}$$

where the bracketed subscripts refer to the units in which quantities are expressed. However, the solutions for $a_{(0)}$, $\rho_{(0)}$, and λ, which appear in Eqs. (12), have been obtained as functions of $t_{(0)}$ not $t_{(\alpha)}$, viz.,

$$t_{(0)}=t_{(0)}[a_{(0)}]\,,\quad \lambda=\lambda[a_{(0)}]\,,\quad \rho=\rho_{(0)}[a_{(0)}]\,, \tag{13}$$

where the square brackets denote functional dependence. In order to express Eqs. (12a) and (12c) as functions of $t_{(\alpha)}$, we first must integrate Eq. (12b) with the help of Eqs. (13) to obtain

$$t_{(\alpha)}[a_{(0)}]=\int\lambda^{-\frac{1}{2}\alpha}[a_{(0)}]t'_0[a_{(0)}]da_{(0)}\,. \tag{14}$$

Equation (14) may be inverted (at least numerically) to give $a_{(0)}=a_{(0)}[t_{(\alpha)}]$, and then, using this expression together with Eqs. (13) in Eqs. (12a) and (12c), we obtain the desired solution

$$a_{(\alpha)}[t_{(\alpha)}]=\lambda^{-\frac{1}{2}\alpha}[a_{(0)}[t_{(\alpha)}]]a_{(0)}[t_{(\alpha)}], \tag{15a}$$

$$\rho_{(\alpha)}[t_{(\alpha)}] = \lambda^{2\alpha}[a_{(0)}[t_{(\alpha)}]]\,\rho_{(0)}[a_{(0)}[t_{(\alpha)}]]\,, \qquad (15b)$$

where the relation between the two cosmic times is given functionally as

$$t_{(0)} = t_{(0)}[a_{(0)}[t_{(\alpha)}]]\,. \qquad (15c)$$

Although the procedure outlined above cannot, in general, be carried out analytically, it can be performed analytically in the limiting regions considered in Sec. III. We find the following results for $k=0, \pm1$ as $a\to 0$:

$$\begin{aligned} a_{(\alpha)} &= B_k^{-\frac{1}{2}\alpha}(A_k t_{(\alpha)})^{[1-\frac{1}{2}\alpha(3/\beta)^{1/2}]/[3-\frac{1}{2}\alpha(3/\beta)^{1/2}]}, \\ \rho_{(\alpha)} &= q a_{(\alpha)}^{-4}, \\ \lambda &= B_k(A_k t_{(\alpha)})^{(3/\beta)^{1/2}/[3-\frac{1}{2}\alpha(3/\beta)^{1/2}]}, \\ t_{(0)} &= (\tfrac{1}{3}\beta B^2)^{-1/2}(A_k t_{(\alpha)})^{3/[3-\frac{1}{2}\alpha(3/\beta)^{1/2}]}, \end{aligned} \qquad (16)$$

where

$$B_k = \lambda_0 \exp[-J_k(a_{(0)0})](1+kS)^{(3/\beta)^{1/2}/4}(\tfrac{1}{2}\mu^{1/2})^{(3/\beta)^{1/2}},$$
$$A_k = B_k^{\frac{1}{2}\alpha}[3-\tfrac{1}{2}\alpha(3/\beta)^{1/2}].$$

In Table I we display the dependence of $a_{(\alpha)}[t_{(\alpha)}]$, $\lambda[t_{(\alpha)}]$, and $\rho_{(\alpha)}[t_{(\alpha)}]$ on the units specified by $\alpha=0, \pm1$ as obtained from Eq. (16) for the choice $\omega=5$. Although the time dependence does not show any drastic changes as we change units (and therefore cosmic times) it does seem to make a significant difference in the behavior of the density where the exponent changes from -0.84 ($\alpha=1$) to -1.68 ($\alpha=-1$) compared with the exponent of -2 for GR. As ω becomes larger the differences from unit to unit become smaller.

In the limit of large a, λ approaches a constant for the radiation cosmologies and, therefore, $a_{(\alpha)}[t_{(\alpha)}]$ has the same functional form as $a_{(0)}[t_{(0)}]$ and similarly for $\rho_{(\alpha)}$. Thus we may conclude that the effects of the choice of units on the time dependence of ρ, a, and λ cannot be distinguished at the present epoch.

V. SUMMARY AND CONCLUSIONS

We have reduced the problem of the BD radiation cosmology to that of integrating quadratures and have found explicit solutions in the cases of flat and curved spaces ($k=0, \pm1$). If we take our initial condition on the scalar field to be $\dot\Lambda_0=0$ ($B=0$) the solutions reduce to those of GR. On the other hand, $\dot\Lambda_0\neq0$ gives the solutions (5)–(8), in which the sourceless scalar field varies extremely slowly from zero at the initial singularity to a finite value as $a\to\infty$ (or $a_{\max}$).

A units transformation of the $\alpha=0$ cosmology to more general units (α) has been made in the two asymptotic regions, and it is found that the dependence on units is significant only near the initial singularity. Since λ approaches a constant at the present epoch, it does not affect the unit transformed solutions there (except for over-all scale factors).

Finally, we note that according to Eqs. (1b) and (1c), the ratio of radiation to scalar energy density is

$$\rho c^2[(2\omega+4)\dot\Lambda^2c^2/32\pi G_0]^{-1} = G_0\rho_0 a_0^{\,4}[(2\omega+4)/32\pi]^{-1}B^{-2}a^2.$$

From the above expression, it is seen that the radiation energy density must eventually dominate as a becomes large for open universes. For a closed universe, this dominance of radiation over scalar energy density will depend critically upon the value of $\dot\Lambda_0$, the ratio taking on the value unity for $\dot\Lambda_0 \approx 10^{-19}$ sec^{-1}.

In any event, the scalar-field energy density dominates as we approach the initial singularity of the cosmology, and the scalar field must surely play an important role there.

TABLE I. Effect of units on time dependence of radiation cosmology for $\omega=5$.

λ	$\lambda[t_{(\alpha)}]$	$a_{(\alpha)}[t_{(\alpha)}]$	$\rho_{(\alpha)}[t_{(\alpha)}]$
1	$t_{(1)}^{0.38}$	$t_{(1)}^{0.21}$	$t_{(1)}^{-0.84}$
0	$t_{(0)}^{0.32}$	$t_{(0)}^{0.33}$	$t_{(0)}^{-1.33}$
−1	$t_{(-1)}^{0.28}$	$t_{(-1)}^{0.42}$	$t_{(-1)}^{-1.68}$
GR	const	$t^{0.50}$	$t^{-2.00}$

APPENDIX

In this appendix we shall describe a second method for evaluating the limiting behavior of the cosmological solutions. Let $I_k(a)$ and $J_k(a)$ denote the value of the integrals in Eqs. (3) and (4) at the point a, and let $\tilde I_k(a)$ and $\tilde J_k(a)$ denote the integrals of the dominant terms (for the appropriate limit) in the integrands evaluated at the point a. Further, let α be the appropriate limit 0, ∞, or $a_{\max}$ and let M be a finite number in the neighborhood of these limits. Then, by using an intermediate point, say a^*, to split up the integrals, we may write

$$t(a) = \lim_{a^*\to\delta;\,a\to M}\left[\int_0^{a^*} + \int_{a^*}^{a}\right]$$
$$= \lim_{a^*\to\delta}\left[\int_0^{a^*} + \int_{a^*}^{a}\lim_{a^*,\,a\to M}\right],$$
$$t(a) = \left\{\lim_{a^*\to\delta}\left[I_k(a^*) - \tilde I_k(a^*)\right]\right\} + \tilde I_k(a). \qquad (17)$$

And similarly,

$$\Lambda(a)-\Lambda_0=\left\{\lim_{a^*\to\delta}\left[J_k(a^*)-\tilde{J}_k(a^*)\right]-J_k(a_0)\right\}+\tilde{J}_k(a). \quad (18)$$

In Eqs. (17) and (18) the bracketed terms assure us that the limiting solution matches properly to the exact solution. It is to be noted that the limit $a^*\to\delta$ is to be taken after the exact integration is performed. Thus, the exact solutions are required, in general, to evaluate these limits.

Using this procedure, we can immediately write down the asymptotic dependence of the solutions for $a\to 0$, viz.,

$$t(a)=\lim_{a^*\to 0}\left[I_k(a^*)-(\tfrac{1}{3}\beta B^2)^{-1/2}\tfrac{1}{3}a^{*3}\right]+(\tfrac{1}{3}\beta B^2)^{-1/2}(\tfrac{1}{3}a^3).$$

Since a^* must drop out of the solution and, moreover, since we have chosen $a=0$ at $t=0$ the limit term above must vanish. Similarly, as $a\to 0$ we have for λ

$$\lambda=\lambda_0\lim_{a^*\to 0}\{\exp[J_k(a^*)](a^*)^{-(3/\beta)^{1/2}}\}\exp[-J_k(a_0)]a^{(3/\beta)^{1/2}}.$$

Here the limit term must be determined from the exact solution as given in the text. The other limits are computed similarly and agree in form with those of the text.

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[1]R. E. Morganstern, preceding paper, Phys. Rev. D 4, 278 (1971).

[2]R. H. Dicke, Phys. Rev. 125, 2163 (1962).

[3]It is important to emphasize that the scalar field, expansion parameter, and density are here expressed as functions of the cosmic time associated with the particular units (α) under consideration. The reason for this is that only the cosmic time t_α corresponds to a Robertson-Walker metric (see Ref. 1). Therefore, our results must be expressed as functions of the cosmic time if they are to be meaningful.

[4]$\alpha=0$, 1, -1 correspond, respectively, to units for which G, m, or Gm is constant (see Ref. 1).

[5]Note that we do not choose the lower limit a_0 to be zero in Eq. (4), since as $a\to 0$ the integral is logarithmically divergent and the resulting logarithmic ratio $\ln(a/a_0)$ would become undefined as both a and $a_0\to 0$. The net result is that the limiting behavior of λ/λ_0 becomes uncertain to within a multiplicative constant. This circumstance is avoided by our choice of $a_0\neq 0$.

[6]The constant cannot be determined by this method, but only from the exact solutions.

[7]This UT differs from Eqs. (4) of Ref. 1 because here we are transforming from the $\alpha=0$ formalism to the general α formalism, whereas there we transformed from $\alpha=1$ (unbarred) to the general α (barred formalism). It might also be in order to add the rather obvious comment that the UT does not affect the equation of state $p=\epsilon\rho$, since ϵ is dimensionless and p and ρ have essentially the same dimensions.

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Effect of Gravitational Light Deflection on the Proposed Gyroscope Test of the Lense-Thirring Effect

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From a previous analysis of the proposed gyroscope test of the Lense-Thirring effect, it is known that all perturbations that contribute more than 10^{-3} sec/yr to the precession of the gyroscope must be taken into account. A frame of reference is obtained by rigidly attaching the gyro housing to a telescope which is trained on some reference star. Here we point out that the deflection of the light from the reference star due to the sun's gravitational field can give rise to an apparent precession of the gyroscope equal to $(4.1\times10^{-3}\cot\frac{1}{2}\theta)$ sec, where θ is the angle between the earth-sun direction and the earth-star direction ($\pi\geq\theta>0$). Thus, over a six-month period, this could amount to as much as 1.75 sec, depending on the angle which the sun-star line makes with the ecliptic.

In a recent analysis of the proposed gyroscope test of the Lense-Thirring effect, Barker and O'Connell[1] pointed out that all perturbations that contribute more than 10^{-3} sec/yr to the precession of the gyroscope must be taken into account. In particular, with regard to the Lense-Thirring contribution to the precession of a gyroscope in a satellite in a circular polar orbit 300 miles above