S-Wave $\pi\pi$ Scattering in the Effective-Potential Approximation*

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The $I=0$ and $I=2$ $\pi\pi$ S-wave phase shifts are calculated, using the effective-potential model with ρ and f^0 exchange which has yielded an earlier successful ρ-f^0 double bootstrap. Scattering lengths and a broad S-wave resonance in fair qualitative agreement with current experimental indications are obtained. However, the $I=0$ phase shift becomes negative between about 350 and 700 MeV, in disagreement with the dispersion-theory analysis of Morgan and Shaw and with current-algebra predictions on the energy derivative of the amplitude.

For some time there has been intensive effort devoted to the difficult problem of obtaining information on the S-wave $\pi\pi$ system. As a result of additional experimental data of various kinds, plus improved procedures for obtaining information from data on $\pi+N\to 2\pi+N$, a somewhat clearer picture seems to be emerging. For example, a recent paper by Morgan and Shaw[1] (hereafter referred to as MS) reports having obtained an essentially unique set of S-wave phase shifts up to 850 MeV through the combination of available experimental information with the use of forward-scattering dispersion relations. The scattering lengths obtained in MS are in agreement with Weinberg's theoretical predictions,[2] based on current algebra. It thus seems worthwhile to see whether the bootstrap calculations, which now seem to be reasonably successful in giving an understanding of the higher partial waves in the resonance region,[3-5] can also give at least a partial understanding of the behavior of the S waves. The bootstrap calculations are, of course, likely to be much less reliable for the S wave, because of the well-known fact that the absence of a centripetal barrier in the S wave makes it more sensitive to short-range forces or distant singularities, and also because of the possibility that the S wave might be especially sensitive to neglected inelastic channels. Nonetheless, it would seem to be of interest to see how far one can go in understanding the S waves by the same methods which appear to be fairly successful for other aspects of low-energy $\pi\pi$ scattering.

Johnson and Collins have reported their results for the S-wave scattering lengths and phase shifts in the third paper in Ref. 4. Their results for the scattering lingths are in excellent agreement with the current-algebra predictions[2] and with the analysis of MS. The phase shifts are also in reasonable agreement with those of MS, though tending perhaps to fall somewhat lower. In view of the possible theoretical difficulties in S-wave calculations, however, it would appear to be valuable to compare the results obtained from different calculational procedures. We report here on the results obtained for S-wave scattering by the methods of I. To summarize briefly the method employed there, one obtains the $\pi\pi$ partial-wave scattering amplitudes in the direct (s) channel from the solution of the nonrelativistic Schrödinger equation with an energy-dependent effective potential, constructed from the t-channel absorptive part using the method devised by Balázs.[6] The t-channel absorptive part is taken to be dominated by the ρ and f^0 contributions.

Here we calculate the low-energy S-wave scattering, in both the $I=0$ and $I=2$ states, by this method, using the $\rho+f^0$ exchange potential, with the parameters chosen to have the self-consistent values found in I. The values obtained for the isospin-0 and isospin-2 S-wave scattering lengths, a_0 and a_2, are $a_0=0.72$ and $a_2=-0.28$. The values obtained by MS, consistent with the current-algebra results,[2] are $a_0=0.16\pm0.04$ and $a_2=-0.05\pm0.01$. Hence there is only rough qualitative agreement between the effective-potential scattering lengths and those obtained by Morgan and Shaw from the analysis of experimental data; the effective-potential results do have the correct sign, but while they are fairly small in magnitude, they are still larger than the MS values by factors of 4 or 5. However, the values obtained for the scattering lengths are very sensitive to small variations in the parameters of the input potential. If one takes the potential to correspond to the present best physical values of the ρ and f^0 masses and widths,[7] one obtains $a_0=0.083$ and $a_2=-0.046$, values essentially in agreement with MS, although in this case the self-consistency is much worse, as one obtains output resonance widths differing by more than 50% from the input values. Since this is an approximate calculation, there must, of necessity, be some uncertainty in the values of the input parameters which should be used in calculating the scattering lengths. In particular, there is no *a priori* reason why these should be the values which maximize the self-consistency between the input and output ρ and

f^0 parameters, since the effect of the approximations is presumably different in the S waves than in the P and D waves; of course, if the method is to have any meaning or usefulness, the values of the parameters to be used in calculating the scattering lengths ought to be reasonably close to the "bootstrap" or self-consistent values. However, if the results turn out to be highly sensitive to moderate variations in the input parameters, this must be reflected in a large uncertainty in the predicted values of the scattering lengths. If one takes, as seems perhaps reasonable, the difference between the self-consistent values of the ρ and f^0 parameters obtained in I and their physical values as an indication of the uncertainty in the potential parameters which should be used in calculating the scattering lengths, then the corresponding range of values for the scattering lengths is, as we have seen, $a_0 = 0.40 \pm 0.32$ and $a_2 = -0.16 \pm 0.12$, so that the results for the scattering lengths seem to be consistent with, but by no means imply, the values obtained by MS.

In Fig. 1 we give the values of the S-wave phase shifts as a function of the c.m. energy, obtained with the potential determined by the parameters of I. For comparison, we also show the upper and lower bounds of the range of values for the phase shifts obtained by MS in their analysis. (Note: The value of the $I=2$ phase shift at the mass of the ρ is taken as an input parameter, whose general order of magnitude is taken from other work,[8] by MS and is not determined by their analysis.) We also show the lower of the two sets of phase shifts obtained by Scharenguivel *et al.*[9] from extrapolation to the pion pole of the forward-backward asymmetry in $\pi^- p \to \pi^- \pi^+ n$. A second set of phase shifts, lying higher and involving a fairly narrow resonance and related to those shown in Fig. 1 by the up-down ambiguity,[10] is also obtained by Scharenguivel *et al.*, and, in fact, preferred by them on the basis of considerations related to the absorption model. That set is in disagreement with the analysis of MS, as well as with the results here; the latter fact is not surprising in view of the well-known difficulty of generating sharp S-wave resonances. For the sake of clarity this second set of phase shifts from Ref. 9 has been omitted from Fig. 1.

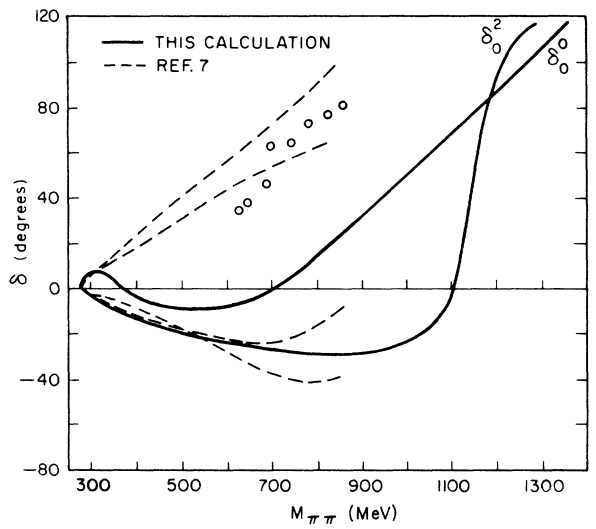

FIG. 1. The solid lines give the effective-potential results for the $I=0$ and 2 S-wave phase shifts, δ_0^0 and δ_0^2, as obtained in this calculation. The upper and lower pairs of dashed lines indicate the range of values for δ_0^0 and δ_0^2, respectively, obtained by Morgan and Shaw in Ref. 1. The experimental points, indicated by the circles, are from Ref. 9. The error bars are omitted for the sake of clarity; the errors in the experimental points are of the order of $\pm 10°$.

There is fair qualitative agreement at high energy between the $I=0$ S-wave phase shifts obtained from the effective-potential model and those found in Refs. 1 and 9. However, the experimental value of the phase shift appears to reach 90° in the vicinity of 1 BeV, while the phase shift yielded by the effective-potential calculation reaches 90° at an energy about 200 MeV higher. Also, the slope of the theoretical phase shift as a function of energy in the region where it goes through 90° is somewhat less than twice that suggested by the experimental results, although the resonance predicted by the effective-potential results is still very broad, with a full width of about 500 MeV. In the energy region below the ρ mass, however, the effective-potential-model phase shifts become negative, whereas the MS phase shifts, obtained by the use of forward-scattering dispersion relations in the energy region below that at which there are experimental results, remain positive throughout the whole region down to threshold. As we discuss below, the negative dip in the $I=0$ phase shift reflects a departure of the effective-potential amplitude from the predictions of current algebra. Finally, we may remark that, unlike the scattering lengths, the behavior of the phase shifts is rather insensitive to moderate variations in the parameters of the potential.

The general behavior of the $I=2$ phase shifts obtained from the effective-potential calculation is very reasonable up to an energy of about 1100 MeV. At that point one finds highly anomalous behavior, namely a narrow $I=2$ S-wave resonance. This effect occurs as a result of the fact that at these energies, because of its more rapid energy dependence, the f^0-exchange potential, which is attractive in all isospin states, comes to dominate the ρ-exchange force, which is repulsive in states with $I=2$. Despite the drastic difference between this

behavior and that found in the real world, we are not inclined to view this result as a cause for great concern. This is because of the fact that the theoretical $I=2$ scattering is being determined by the (relatively) small difference in the magnitudes of the large ρ- and f^0-exchange potentials at these energies, in contrast to the situation in the $I=0$ state where these potentials enter with the same sign. Hence the $I=2$ results are likely to be particularly sensitive to short-range forces, and hence unreliable. We might note, for example, that the presence of an additional short-range force due to the exchange of an $I=1$ particle, e.g., the $j=3$ recurrence of the ρ trajectory, could well have the effect of suppressing the undesired $I=2$ resonance and could at the same time, by providing additional attraction, cause the $I=0$ phase shift to rise somewhat more rapidly as a function of energy. Hopefully, because of its shorter range, such a force would have comparatively little effect on the results of I.

We now compare the results of this calculation with the predictions of current algebra. We denote by $A(s,t,u)$ the invariant amplitude for $I=0$ (equal to the scattering amplitude multiplied by $\sqrt{s}$), where s, t, and u are the usual Mandelstam variables and we suppress the superscript which denotes isospin. Then current algebra, coupled with the assumption that A can be represented as a first-degree polynomial in s, t, u, and the external masses when all the variables are of order one (we take, of course, $m_\pi=1$, as we have throughout) yields two predictions for A on the mass shell, so that $s+t+u=4$, which may be summarized as follows[2]:

$$A(1,t,u)\approx 0.02, \tag{1a}$$

$$\partial A(1,t,u)/\partial s\approx 0.1, \tag{1b}$$

provided t and u lie within the range of validity of the expansion. One then assumes that the expansion remains valid, and that it remains sufficient to consider only first-order terms, all the way up to the physical threshold, $s=4$. A necessary condition for the expansion to bear any relation to the actual amplitude at threshold is that the discontinuity across the physical cut near threshold must be small. As far as the necessity for only first-order terms is concerned, Khuri has argued that at least the quadratic terms should be small.[11] With these assumptions one then derives the well-known result for the scattering length, $a_0\approx 0.16$. This agrees with MS, and hence, as discussed in the above, agrees roughly with the effective-potential calculation; as we shall see, this is probably fortuitous. If this linear extrapolation is valid, then

$$\partial A/\partial s\approx 0.1 \tag{2}$$

in the vicinity of threshold. [Eq. (2) will not hold right at threshold, since the low-energy part of the cut, even if weak, makes an infinite contribution to $\partial A/\partial s$ at the branch point. Again, however, if $\mathrm{Im}A$ is small at low energy, Eq. (2) can reasonably be expected to hold in the low-energy region as long as one stays a little bit above the threshold branch point. Equation (2) should also hold for $\mathrm{Re}A$, even at threshold, provided $\mathrm{Im}A$ is small.] We assume, naturally, that in the region near threshold, the only important contribution to $\partial A/\partial s$ is from the S wave. One then finds that Eq. (2) is badly violated by the effective-potential amplitude. To see this, we note that the value of A at threshold is given by (we suppress the t and u arguments of A, since we assume the mass-shell amplitude depends only on s in the region of interest) $A(4)=2a_0$ while, because the phase shift goes through 0 and dips into the negative region at about $s=7$, we have $A(7)\approx 0$. Hence we can estimate $\partial A/\partial s$ in the low-energy region by $\partial A/\partial s\approx -2a_0/(7-4)=-(\frac{2}{3})a_0$, so that $\partial A/\partial s\approx -0.5$. In fact, the disagreement is even larger, since when one calculates numerically the value of $\mathrm{Re}(\partial A/\partial s)$ at threshold (rather than the average of $\partial A/\partial s$ over the range $4\leq s\leq 7$) one obtains $\partial\mathrm{Re}A/\partial s\approx -1.0$. This also indicates that A is quite nonlinear in the effective-potential model, i.e., $\partial^2 A/\partial s^2$ is rather large at threshold. (A numerical evaluation shows the third and fourth derivatives are also large.) Consequently, there would be no reason to expect an amplitude which resembled that predicted by the effective-potential model to obey the current-algebra predictions at threshold, even if they were to hold in the unphysical region where they are derived. Finally, we note that predictions of the effective-potential calculation concerning the results of extrapolating the amplitude below threshold are suspect, owing to the known failure, emphasized by Finkelstein,[12] of the effective-potential method near $s=0$. This results from the presence of a factor of $s^{-1/2}$ in the approximate form of the potential used in calculations.

The general qualitative features which we have been discussing, namely, the negative value of $\partial A/\partial s$, the nonlinearity at threshold, and the negative dip in the phase shift, appear to be inherent in the model and are unchanged by reasonable changes in the potential. For example, these results are essentially unchanged when the potential is taken to correspond to the physical, rather than the self-consistent values of the resonance parameters. The fact that negative values are obtained for the $I=0$ phase shift in the energy range between about 350 and 700 MeV is unfortunate, in that the method should be more reliable there than at the

higher energies where the resonance appears.

It is, perhaps, of interest to compare these results with those obtained from the effective-potential model with ρ exchange alone, with ρ parameters being determined by requiring the ρ to bootstrap itself. Such results are given by Finkelstein[12] for the $I=0$ case. He obtains a negative value of the scattering length, a_0, and an $I=0$ S-wave phase shift which becomes *rapidly large and negative,* passing through $-90°$ for c.m. energy in the vicinity of 400 MeV. Hence the inclusion of f^0 exchange, which yields bootstrap values for the ρ and f^0 parameters in much better agreement with experiment, also produces S-wave scattering which, qualitatively at least, is much closer to what appear to be the experimental results.

In conclusion, the model discussed in I predicts scattering lengths and a broad S-wave resonance in rough agreement with experiment. However, the $I=0$ phase shift becomes negative in the region between threshold and the vicinity of the ρ mass, in substantial disagreement with the results of Morgan and Shaw. The introduction of f^0 exchange, in addition to ρ exchange, which led to a fairly successful double-bootstrap calculation in I, does improve the S waves, at least qualitatively, as compared with the result obtained from the use of a self-consistent ρ-exchange potential alone.

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