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Brans-Dicke Cosmologies in Arbitrary Units: Solutions in Flat Friedmann Universes*

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Assuming a Robertson-Walker metric and an equation of state $p=\epsilon\rho$ $(0\le\epsilon\le 1)$, the Brans-Dicke (BD) field equations are found to yield a first integral of the form $a^{3(1+\epsilon)}\lambda^{\frac{1}{2}(1-3\epsilon)(1-\alpha)}$ $=$const, where α is the parameter denoting the particular units in which the field equations are expressed. This first integral is employed in the remaining field equations to yield polynomial solutions for a, ρ, and λ. For $\epsilon=0,\frac{1}{3}$, these solutions properly reduce to previously obtained pressure-free and radiation-filled universes, respectively.

I. INTRODUCTION

In a previous paper[1] we considered a general units transformation (UT) in which length, time, and reciprocal mass scaled as $\lambda^{\frac{1}{2}(1-\alpha)}$, and found that in terms of the parameter α, the Brans-Dicke (BD) field equations could be written down in the following form:

$$\begin{aligned}\overline{G}_{ij}&=(8\pi\varphi_0^{-1}/c^4)\lambda^{-\alpha}\overline{T}_{ij}+\tfrac{1}{2}(2\omega+3-\alpha^2)\Lambda_{,i}\Lambda_{,j}\\&\quad-\tfrac{1}{4}(2\omega+3+\alpha^2)\overline{g}_{ij}\Lambda_{,k}\Lambda^{,\bar{k}}+\alpha(\Lambda_{,i;\bar{j}}-\overline{g}_{ij}\overline{\Box}\Lambda),\\ \overline{\Box}\Lambda+\alpha\Lambda_{,k}\Lambda^{,\bar{k}}&=\frac{(8\pi\varphi_0^{-1}/c^4)\lambda^{-\alpha}\overline{T}}{2\omega+3},\end{aligned}\tag{1}$$

where $\Lambda\equiv\ln\lambda$, and the barred notation is kept in order to remind us that, in contrast with the original unbarred version of the BD theory, $\overline{T}^{ij}$ is not, in general, divergence-free, and in addition to remind us that the equations of motion are nongeodesic. In Ref. 1 we obtained the local Schwarzschild solution and found that the same predictions result for the three tests in all units. It has also been shown[2] that the BD theory predicts the same effects for the Schiff gyroscope in all units. These results are in accord with the units interpretation of these various formalisms and the dimensionless character of the measured quantities.

For local phenomena, units of measure are unimportant, but it appears that they may play a significant role in cosmological considerations. Although it is of course true that the various unit-transformed BD field equations are but different realizations of the same basic theory, the physical input data (mass and pressure distributions) must be expressed in units appropriate to the formalism we intend to use. It is not entirely clear how one can guarantee that this will be the case.

Since observationally one records only one numerical value for a given quantity (e.g., $\rho_0\sim 10^{-29}$–10^{-31} g/cm^3), one must consider carefully how this number was obtained, so as to determine relative to the *constancy* of which units it is being expressed.

Logically, it would seem that the units determined by the nature of observation are those associated with the dominant interaction. Thus, for example, physical measurements made in the laboratory (where gravitation is weak) would naturally be phrased in constant atomic units, and on the grander scale of the universe, measurements would be phrased in constant gravitational units. In principle, this criterion should certainly be correct, but in practice, astrophysical measurements are so very indirect that their interpretation in these terms is unclear.

It is not our purpose to attempt such an analysis here, but only to point out that if the actual process of measurement cannot be conceived of as being taken relative to all units being constant, then, assuming that an actual relative variation between units exists (as in BD theory), it follows that the number we record must be readjusted in such a manner that it may be considered as having been taken relative to one set of constant units (with respect to which other sets vary).

To illustrate these ideas more explicitly, let us consider the following idealized situation: Assume that all distance measurements are taken by parallax with the Earth-Sun orbit (assumed circular) as base. Then, taking note of the relation between the astronomical unit (AU) and the Schwarzschild radius of the sun R_s, viz., 1 AU $=\frac{1}{2}R_s(v/c)^{-2}$, all distances will be dimensionless *numbers* which represent a multiple of the AU (or equivalently R_s) taken as a *constant unit.* This number may be used directly in a theory for which R_s is constant. If, however, one wishes to use this number in a theory for which R_s varies (e.g., constant atomic units), then the number actually observed is an overestimate, since the unit we are now employing (varying R_s) was larger in the past (G is decreasing with time). Hence, we must use a number which is *smaller* than that which is actually observed if we enter this datum into the atomic-unit formalism. If now we assume, for example, that the *mass* is actually measured relative to *constant* atomic units, then the net result will be to increase the number we enter for the density of matter. It is therefore conceivable that the universe could actually have a density sufficient for closure even though the "observed" matter density is too small.

Thus we see that the assumption of an expanding universe leads to some interesting possibilities regarding the applicability of one BD formalism over another in the cosmological domain. For this reason we would like to investigate the BD cosmologies in arbitrary units.

II. COSMOLOGICAL EQUATIONS AND A CONSERVED QUANTITY

We take the usual form of the line element in comoving coordinates, viz.,

$$d\bar{s}^2 = -d\bar{t}^{\,2} + \bar{a}^2(\bar{t})[(1-kr^2)^{-1}dr^2 + r^2 d\Omega^2], \tag{2}$$

where $k = 1$, 0, and -1 correspond to closed, open Euclidean, and open hyperbolic spaces, respectively. It should be remarked at this point that the comoving coordinates in the original units of Brans and Dicke[3] are given by the line element

$$ds^2 = -dt^2 + a^2(t)[(1-kr^2)^{-1}dr^2 + r^2 d\Omega^2], \tag{3}$$

and we must identify

$$d\bar{t} = \lambda^{\frac{1}{2}(1-\alpha)}dt, \qquad \bar{a} = \lambda^{\frac{1}{2}(1-\alpha)}a. \tag{4}$$

This is in accord[4] with the general units transformation described in Ref. 1.

We take the matter tensor in comoving coordinates to be in the standard isotropic form

$$\bar{T}^k{}_l = \mathrm{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p}). \tag{5}$$

Using the line element (2) and the matter tensor (5) in the field equations (1) and suppressing the barred notation, we obtain for the $G^0{}_0$, $G^1{}_1$, and scalar field equations the following expressions:

$$3[(\dot{a}/a)^2 + ka^{-2}] = \varphi_0{}^{-1}\lambda^{-\alpha}\rho + \tfrac{1}{4}[2\omega + 3(1-\alpha^2)]\dot{\Lambda}^2 - 3\alpha(\dot{a}/a)\dot{\Lambda}, \tag{6a}$$

$$2(\ddot{a}/a) + (\dot{a}/a)^2 + ka^{-2} = -\varphi_0{}^{-1}\lambda^{-\alpha}p - \tfrac{1}{4}[2\omega + 3(1-\alpha^2)]\dot{\Lambda}^2 - 2\alpha\dot{\Lambda}(\dot{a}/a) - \alpha^2\dot{\Lambda}^2 - \alpha\ddot{\Lambda}, \tag{6b}$$

$$3(\dot{a}/a)\dot{\Lambda} + \ddot{\Lambda} + \alpha\dot{\Lambda}^2 = \varphi_0{}^{-1}\lambda^{-\alpha}(\rho - 3p)/(2\omega + 3), \tag{6c}$$

where a dot denotes differentiation with respect to t and we have taken $c^2 = 1$ and absorbed a factor of 8π into ρ and p. By taking the divergence of the right-hand side of Eq. (1), or alternatively, by appropriately combining Eqs. (6a) and (6b), we obtain the conservation identity

$$0 = \frac{d}{dt}\left\{a^3\left(\varphi_0{}^{-1}\lambda^{-\alpha}\rho + \tfrac{1}{4}[2\omega + 3(1-\alpha^2)]\dot{\Lambda}^2 - 3\alpha\frac{\dot{a}}{a}\dot{\Lambda}\right)\right\} + \frac{da^3}{dt}\left\{\varphi_0{}^{-1}\lambda^{-\alpha}p + \tfrac{1}{4}[2\omega + 3(1-\alpha^2)]\dot{\Lambda}^2 + 2\alpha\frac{\dot{a}}{a}\dot{\Lambda} + \alpha^2\dot{\Lambda}^2 + \alpha\ddot{\Lambda}\right\}. \tag{7}$$

The field equations (6a)–(6c) may be used to cast Eq. (7) in a more natural form (see Appendix A), viz.,

$$0 = \frac{d}{dt}(\rho a^3) + p\frac{d}{dt}(a^3) + \tfrac{1}{2}(1-\alpha)a^3\dot{\Lambda}(\rho - 3p). \tag{8}$$

If we now assume p and ρ are related by $p = \epsilon\rho$, Eq. (8) yields the first integral

$$\rho a^{3(1+\epsilon)}\lambda^{\frac{1}{2}(1-3\epsilon)(1-\alpha)} = \mathrm{const} \equiv q. \tag{9}$$

We note that for the choice $\alpha = 1$, which corresponds to the original BD theory, we recover the first integral familiar from general relativity since, for this choice, the matter tensor is conserved separately. We also see that for $\epsilon = \frac{1}{3}$ we again obtain a decoupling of the scalar and matter fields for all values of α.

Together with the first integral (9), we may take (6a) and (6c) as a complete set of equations for ρ, a, and λ. Equation (6b) is redundant and therefore unnecessary. In Sec. III below we find power-law solutions to this set of equations.

III. POWER-LAW SOLUTIONS IN FLAT SPACE

Let us consider flat-space cosmologies ($k = 0$) for which the solution has the form

$$a = a_0(t/t_0)^r, \quad \lambda = (t/t_0)^l, \quad \rho = \rho_0(t/t_0)^n. \tag{10}$$

Substituting Eqs. (10) into Eqs. (6a), (6c), and (9) we find, respectively,

$$r^2/t^2 = \tfrac{1}{3}\varphi_0{}^{-1}\rho_0(t/t_0)^{n-\alpha l} + [2\omega + 3(1-\alpha^2)][l^2/(12t^2)] - \alpha r l/t^2, \tag{11}$$

$$3rl/t^2 + l(\alpha l - 1)/t^2 = [(1-3\epsilon)/(2\omega+3)]\varphi_0{}^{-1}\rho_0(t/t_0)^{n-\alpha l}, \tag{12}$$

$$\rho_0 a_0{}^{3(1+\epsilon)}(t/t_0)^{n+3r(1+\epsilon)+\frac{1}{2}l(1-\alpha)(1-3\epsilon)} = \mathrm{const} = q. \tag{13}$$

First of all, we see that for solutions to exist we must have

$$n = \alpha l - 2. \tag{14}$$

Also, the constancy of Eq. (13) implies that the exponent must vanish, and this fact together with the relation (14) yields an expression relating r and l, viz.,

$$r = \frac{-l(1-\alpha)(1-3\epsilon) - 2\alpha l + 4}{6(1+\epsilon)}. \tag{15}$$

Combining Eqs. (11) and (12) so as to eliminate ρ_0 and using Eqs. (14) and (15) to eliminate n and r, respectively, we find, after some algebra, the following quadratic equation for l:

$$l^2[(1-3\epsilon)(1-3\epsilon-2\alpha)^2 + (6\omega+9)(1-\epsilon^2)(1-3\epsilon-2\alpha)] + l[-8(1-3\epsilon)(1-3\epsilon-2\alpha) - 4(1-\epsilon^2)(6\omega+9)] + 16(1-3\epsilon) = 0. \tag{16}$$

The solutions to the quadratic equation (16) are easily found, and upon substitution of the two values of l into Eqs. (14) and (15) we obtain two sets of solutions for r, l, and n. However, Eq. (12), which relates the arbitrary positive constant $\varphi_0^{-1}\rho_0 t_0^2$ to ϵ, ω, and α, may be used to eliminate one set of solutions since it implies $\varphi_0^{-1}\rho_0 t_0^2 < 0$ for this set. Thus we are left with the following solution:

$$\begin{aligned} r &= [2/3(1+\epsilon)]d^{-1}[(6\omega+9)(1-\epsilon^2) - 3\alpha(1+\epsilon)(1-3\epsilon)], \\ l &= 4(1-3\epsilon)d^{-1}, \\ n &= -2d^{-1}[(6\omega+9)(1-\epsilon^2) + (1-3\epsilon)(1-3\epsilon-4\alpha)], \\ \varphi_0^{-1}\rho_0 t_0^2 &= 4(2\omega+3)d^{-2}[(6\omega+9)(1-\epsilon)^2 - (1-3\epsilon)^2], \end{aligned} \tag{17}$$

where

$$d \equiv [(6\omega+9)(1-\epsilon^2) + (1-3\epsilon)(1-3\epsilon-2\alpha)].$$

Since the last of Eqs. (17) must be positive, we see that the solution (17) fails whenever this expression vanishes. This occurs for values of ϵ given by

$$\epsilon = 1 - \omega^{-1}[(1+\tfrac{2}{3}\omega)^{1/2} - 1]. \tag{18}$$

From Eq. (18) we see that as we vary ω over the range $(1,\infty)$ ϵ takes on the values 0.7 ($\omega \sim 1$), 0.8 ($\omega \sim 5$), and 1.0 ($\omega \to \infty$). Thus for any value of ω we have a valid solution for reasonable equations of state ($0 \leq \epsilon \leq \frac{1}{3}$). The solution also holds for harder equations of state as the parameter ω becomes larger, but holds for $\epsilon = 1$ only as $\omega \to \infty$, which is the general-relativity (GR) limit of the BD theory.

In Table I we have displayed the specific exponents [see Eq. (17)] for the three unit formalisms of interest, relative to the corresponding exponents in the Einstein theory. We first note that in the limit as $\omega \to \infty$ we recover the Einstein exponents. ($l \to 0$ in this limit so the scalar field is constant.) For the pressure-free case ($\epsilon = 0$) the results agree with the solutions of Brans and Dicke[3] in the original units ($\alpha = 1$) and in the transformed units ($\alpha = 0, -1$).

For $0 \leq \epsilon < \frac{1}{3}$ we see from Table I that λ increases for all α, and that compared to GR a increases less rapidly for $\alpha = 1, 0$ but increases more rapidly for $\alpha = -1$, and that ρ decreases less rapidly for $\alpha = 1$, is the same for $\alpha = 0$, and decreases more rapidly for $\alpha = -1$.

For $\epsilon = \frac{1}{3}$, λ becomes constant and all the exponents attain their GR values. Finally, if $\epsilon > \frac{1}{3}$ we find that λ decreases for all α, and that compared to GR a increases more rapidly for $\alpha = 1$ and less rapidly for $\alpha = 0, -1$ and that ρ decreases more rapidly for $\alpha = 1$, is the same for $\alpha = 0$, and decreases less rapidly for $\alpha = -1$.

These results, which are summarized in Table II, may be qualitatively interpreted quite simply in terms of the first integral given by Eq. (9). The cosmological expansion is shared by the three quantities ρ, a, and λ in such a manner that Eq. (9) is satisfied. The effect of the different unit formalisms (i.e., the parameter α) is to distribute the expansion of ρ, a, and λ in different ways according to the value of α.

This effect is evident in the last two columns of Table I where we see that the functions $f_\alpha(\epsilon)$ tend to increase or decrease the time dependence of ρ and a relative to the GR case depending upon the particular formalism (and the value of ϵ).

IV. CONCLUSIONS

When expressed in terms of a common universal time, say t, the cosmologies obtained above are

TABLE I. The exponents as a function of ϵ.

α [a]	f_α	l_α	(r_α/r_E) [b]	(n_α/n_E) [b]
1	$\dfrac{1-3\epsilon}{3\omega(1-\epsilon^2)+4}$	$2f_1(\epsilon)$	$1-f_1(\epsilon)$	$1-f_1(\epsilon)$
0	$\dfrac{\frac{1}{2}(1-3\epsilon)^2}{3\omega(1-\epsilon^2)+5-3\epsilon}$	$\dfrac{4f_0(\epsilon)}{1-3\epsilon}$	$1-f_0(\epsilon)$	1
−1	$\dfrac{-3\epsilon(1-3\epsilon)}{3\omega(1-\epsilon^2)+6(1-\epsilon)}$	$\dfrac{-2f_{-1}(\epsilon)}{3\epsilon}$	$1-f_{-1}(\epsilon)$	$1-\dfrac{f_{-1}(\epsilon)}{3\epsilon}$

[a] α denotes the particular units. $\alpha = 1$ corresponds to atomic units, $\alpha = 0$ corresponds to gravitational units (G = const), and $\alpha = -1$ corresponds to Schwarzschild units ($2Gm/c^2$ = const).

[b] $r_E \equiv \frac{2}{3}(1+\epsilon)^{-1}$ and $n_E \equiv -2$ are the exponents in the Einstein theory.

TABLE II. Nature of solutions in different units as a function of ϵ.

Equation of state $p=\epsilon\rho$	Atomic units $\alpha=1$	Gravitational units $\alpha=0$	Schwarzschild units $\alpha=-1$
$0\le\epsilon<\frac{1}{3}$	λ incr.	λ incr.	λ incr.
	$r/r_E<1$	$r/r_E<1$	$r/r_E>1$
	$n/n_E<1$	$n/n_E=1$	$n/n_E>1$
$\epsilon=\frac{1}{3}$	λ const	λ const	λ const
	$r/r_E=1$	$r/r_E=1$	$r/r_E=1$
	$n/n_E=1$	$n/n_E=1$	$n/n_E=1$
$\frac{1}{3}<\epsilon<\epsilon_{max}<1$[a]	λ decr.	λ decr.	λ decr.
	$r/r_E>1$	$r/r_E<1$	$r/r_E<1$
	$n/n_E>1$	$n/n_E=1$	$n/n_E<1$

[a] ϵ_{max} is determined from Eq. (18) of the text.

identical[5]; that is, λ is the same function of the common time t and the numerical quantities are related by the UT given in Eqs. (4). However, the cosmologies are distinct in the sense that each observer views the universe as isotropic and homogeneous and then chooses whatever units he pleases to be constant by taking Eq. (2) as his line element in comoving coordinates. In so doing, he chooses a particular universal time $\bar{t}$ which depends upon his choice of units (α). When expressed in terms of a given universal time $\bar{t}$, the BD cosmologies yield $\bar{\rho}(\bar{t})$, $\bar{a}(\bar{t})$, and $\lambda(\bar{t})$ according to Eqs. (17) (restoring barred notation), which are different functions of $\bar{t}$ for each set of units.[6] This will of course have an effect on astrophysical observations of cosmological origin.

For example, by employing Eq. (17) and restoring barred notation, the Hubble parameter $\bar{H}(\bar{t})$ is given by the following expression:

$$\bar{H}(\bar{t})\equiv\frac{\dot{\bar{a}}}{\bar{a}}=\frac{2}{3(1+\epsilon)\bar{t}}\,\frac{(6\omega+9)(1-\epsilon^2)-3\alpha(1+\epsilon)(1-3\epsilon)}{(6\omega+9)(1-\epsilon^2)+(1-3\epsilon)(1-3\epsilon-2\alpha)}. \tag{19}$$

The first factor is just the GR result and the remaining factors bring about a further dependence upon ϵ as well as a dependence upon α (units). We note that this latter dependence disappears for $\epsilon=\frac{1}{3}$ as might be expected since then the coupling of the scalar field to matter becomes zero (except perhaps for initial conditions).

Other effects of such a conscious choice of units of observation, say on the temperature history and particularly its relation to the universal 3°K background radiation and the hydrogen-helium ratio, would seem to be worth investigation. In a future paper we hope to look into these problems in some detail.

APPENDIX

We shall give two derivations of Eq. (8) of the text. If we multiply Eq. (6b) by three and add it to Eq. (6a), we obtain

$$\begin{aligned}6[(\ddot{a}/a)+(\dot{a}/a)^2+ka^{-2}]=&\tfrac{1}{2}[2\omega+3(1-\alpha)][6(\dot{a}/a)\dot{\Lambda}]\\&+\tfrac{1}{2}[2\omega+3(1-\alpha)](2\alpha\dot{\Lambda}^2+2\ddot{\Lambda})\\&-\tfrac{1}{2}[2\omega+3(1-\alpha^2)]\dot{\Lambda}^2.\end{aligned} \tag{20}$$

Expanding the conservation law (7), we find

$$\begin{aligned}0=&\varphi_0{}^{-1}\lambda^{-\alpha}[(d/dt)(\rho a^3)+p(d/dt)a^3]\\&-\varphi_0{}^{-1}\lambda^{-\alpha}\rho a^3\alpha\dot{\Lambda}+\tfrac{1}{2}[2\omega+3(1-\alpha^2)]a^3\ddot{\Lambda}\dot{\Lambda}\\&-3a^3\{\alpha\dot{\Lambda}[(\ddot{a}/a)-(\dot{a}/a)^2]+\alpha\ddot{\Lambda}(\dot{a}/a)\}\\&+3a^3(\dot{a}/a)\{\tfrac{1}{2}[2\omega+3(1-\alpha^2)]\dot{\Lambda}^2-\alpha(\dot{a}/a)\dot{\Lambda}\\&+\alpha^2\dot{\Lambda}^2+\alpha\ddot{\Lambda}\}.\end{aligned} \tag{21}$$

Also, by solving Eq. (20) for $\ddot{a}/a$ and Eq. (6a) of the text for $\varphi_0{}^{-1}\lambda^{-\alpha}\rho$ and substituting the resulting expressions into Eq. (20) above, we are left with

$$\begin{aligned}0=&\varphi_0{}^{-1}\lambda^{-\alpha}[(d/dt)(\rho a^3)+p(d/dt)a^3]\\&+\tfrac{1}{2}(2\omega+3)(1-\alpha)a^3\dot{\Lambda}[3(\dot{a}/a)\dot{\Lambda}+\ddot{\Lambda}+\alpha\dot{\Lambda}^2].\end{aligned} \tag{22}$$

Using Eq. (6c) for the last bracketed expression above, Eq. (21) yields Eq. (8) of the text.

An alternative derivation involves simply applying the units transformation to the known first integral in the original BD formalism where the matter tensor is conserved independently of the scalar field. Thus, noting that the density ρ scales as $\lambda^{-2(1-\alpha)}$ and a scales as $\lambda^{1-\alpha}$, we easily find that $a^{3(1+\epsilon)}=\text{const}$ becomes precisely Eq. (9). Although this last derivation of the result is certainly valid, it would perhaps not be so convincing had we not first derived the results directly from the field equations.

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[1]R. E. Morganstern, Phys. Rev. D 3, 2946 (1971).

[2]R. E. Morganstern, Phys. Rev. D 3, 616 (1971).

[3]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[4]The rather odd-looking transformation on the differential $d\bar{t}$ in Eq. (4) results from the fact that we insist on a "universal time" $\bar{t}$ as our fourth coordinate so that it can no longer be dimensionless, but must carry the units of time. This may be remedied by introducing the new dimensionless parameter η according to $d\bar{t} = \bar{a}_l(\eta)d\eta$ and $dt = a d\eta$ so that the line elements are then related by the second of Eqs. (4). However, we still obtain the same relation between the two universal times t and $\bar{t}$ if we wish to reintroduce them.

[5]This statement is obviously true because it was just the UT of Eq. (4) which generated the other formalisms and their new cosmic times.

[6]In fact, the gravitational constant may be considered as either increasing or decreasing, depending upon the units of measure. Since $\varphi_{(\alpha)} = \varphi_{(0)}\lambda^{\alpha}$, and $G \sim \varphi_{(\alpha)}^{-1}$, it is evident that for $\alpha = 1$, the gravitational constant is decreasing, while for $\alpha = -1$, it is increasing.

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Exact Solutions to Radiation-Filled Brans-Dicke Cosmologies*

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Using the Robertson-Walker metric, exact general solutions to the Brans-Dicke cosmologies for $p = \frac{1}{3}\rho$ are found. The two first integrals which exist for this case reduce the problem to quadratures. In the case of flat space the quadratures may be integrated explicitly in terms of elementary functions, while in the curved spaces ($k = \pm 1$), the solutions may be expressed in terms of elliptic integrals. The limit as a approaches ∞ (or $a_{\max}$ for a closed universe) may be evaluated for positive, negative, and zero curvature, and leads to the rather surprising result that the scalar field λ approaches a constant.

I. INTRODUCTION

In a previous paper[1] we have written down the cosmological equations for the Brans-Dicke (BD) theory in arbitrary units under the usual assumptions of a Robertson-Walker metric and an isotropic matter tensor in comoving coordinates. Here we find exact solutions for flat ($k=0$) and curved spaces ($k = \pm 1$) in the case of a radiation-filled universe. We initially restrict ourselves to the gravitational unit formalism[2] ($\alpha = 0$ of Ref. 1), and find the limiting behavior of the solutions for $t \to 0$ and $t \to \infty$. The rather unexpected result appears that the scalar field λ asymptotically approaches a constant in a radiation-filled universe. Finally, we outline a general procedure for obtaining the solutions in other units, i.e., in terms of their appropriate cosmological time.[3] In general, this procedure cannot be carried out analytically except in the limiting behavior as $t \to 0, \infty$. In the limit as $t \to 0$ the scalar field $\lambda[t_{(\alpha)}]$, the expansion parameter $a_{(\alpha)}[t_{(\alpha)}]$, and the density $\rho_{(\alpha)}[t_{(\alpha)}]$ display somewhat different time dependences for the three natural unit systems,[4] while in the limit as $t \to \infty$ there is no difference, since $\lambda \to$ constant.

II. FIELD EQUATIONS AND SOLUTIONS

For $\alpha = 0$ and $\epsilon = \frac{1}{3}$ the field equations of Ref. 1 reduce to

$$(\dot{a}/a)^2 = \tfrac{1}{3}\varphi_0^{-1}\rho + \tfrac{1}{12}(2\omega+3)\dot{\lambda}^2 - ka^{-2}, \tag{1a}$$

$$a^3\dot{\lambda} = B \equiv a_0^3\dot{\lambda}_0, \tag{1b}$$

$$\rho a^4 = q \equiv \rho_0 a_0^4, \tag{1c}$$

where we have suppressed bracketed subscripts (0) referring to the units, and where a dot denotes differentiation with respect to the cosmic time $t_{(0)}$. We have also absorbed a factor of 8π into the definition of ρ and have set $c^2 = 1$. Substituting the two first integrals (1b) and (1c) into (1a), we obtain

$$\frac{dt}{da} = (\tfrac{1}{3}\beta B^2 + \tfrac{1}{3}\varphi_0^{-1}qa^2 - ka^4)^{-1/2}a^2, \tag{2}$$

where $\beta = \frac{1}{4}(2\omega+3)$. Equation (3) immediately yields the quadrature

$$t(a) = \int_{u=0}^{u=a} (\tfrac{1}{3}\beta B^2 + \tfrac{1}{3}\varphi_0^{-1}qu^2 - ku^4)^{-1/2}u^2 du, \tag{3}$$

where we have set the integration constant equal to zero so that $a=0$ when $t=0$. Equation (3) gives a implicitly as a function of t and, therefore, Eq. (1c) immediately gives ρ implicitly as a function of t. Equation (1b) may now be solved for $\Lambda(a)$ by using the expression for dt/da in Eq. (2). We obtain[5]