obtained from the following approximations:

$$v \approx \tfrac{1}{4}e^2 z + \tfrac{1}{2}, \qquad z \to \infty$$

$$v \approx 4(\ln z)^{-2}, \qquad z \to 0 \qquad \text{(A8)}$$

$$v \approx -1 - [\tfrac{3}{2}(-1-z)]^{2/3}, \qquad z \to -1.$$

The final mapping onto the interior of a unit circle in the y plane is given by the function

$$y = (v^{1/2} - V^{1/2})/(v^{1/2} + V^{1/2}), \qquad \text{(A9)}$$

where $V = v(w=0)$.

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Cosmic-Ray Experiments and the Scaling Hypothesis*

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Angular distributions obtained in cosmic-ray experiments are examined with a view to test the scaling hypothesis. Using appropriately scaled parametrizations of accelerator production data, we critically analyze the experimental results obtained at the Echo Lake facility in the range of 150–300 GeV. We find that the scaling hypothesis accounts for the main features of the data. However, both uncertainties in the scaled parametrizations used and the many experimental biases that need to be corrected preclude a critical test. We conclude that direct measurements at the new accelerators are necessary for a definitive test of scaling.

I. INTRODUCTION

Recently there has been much interest in reactions of an inclusive nature, that is, reactions in which only one of the final particles is observed, regardless of whatever else may happen. It has been conjectured by Feynman,[1] and by Benecke, Chou, Yang, and Yen[2] that such reactions should have many simple features at sufficiently high energies. The main aspect is the expectation that inclusive cross sections should scale at high energies. By this we mean that inclusive cross sections, when studied as functions of appropriate variables, should become energy-independent as the energy increases. A number of theoretical models for inclusive reactions have been constructed,[3] and many of them exhibit this scaling feature. However, no model gives a value of the energy beyond which scaling can be expected to hold with good accuracy. It was with the purpose of determining this energy that we have analyzed 12-, 19-, and 30-GeV pion-production data in proton-proton collisions.[4] The tentative conclusion, reached after studying the behavior of integrated quantities such as the multiplicity and the inelasticity, was that scaling seemed to be approximately valid at laboratory energies of 30 GeV for this particular reaction. Once this is established, the scaling hypothesis then allows detailed predictions of π production in pp collisions at higher energies. At this time, the only existing data at very high energy come from cosmic-ray studies. Of these, the cleanest are, without doubt, those obtained by Jones *et al.*[5] at the Echo Lake facility. The chief advantage of this experiment is that it has a hydrogen target rather than the more usual emulsion stack. In an attempt to further explore the scaling hypothesis and its implications, we have analyzed[6] the angular distributions of charged particles produced at Echo Lake. The comparison between theory and experiment obtained in Ref. 6 was satisfactory, but not spectacular. We would like here to critically reexamine both our work and the information provided by the experiment, in order to probe in more detail into the validity of scaling.

II. GENERAL CONSIDERATIONS

In the reaction $a+b\to c+\text{anything}$, we can write the differential cross section as

$$d\sigma_{ab}^{c}=\frac{d^{3}k}{k^{0}}\,\sigma_{ab}(s)N_{ab}^{c}(\vec{k},s)\,, \tag{1}$$

where $\vec{k}$ is the momentum of the detected particle c, k^0 is its energy, s is the square of the c.m. energy, and σ_{ab} is the total cross section. With this normalization, $(d^3k/k^0)N$ gives the average number of particles in a momentum interval d^3k in a single collision. We use invariant phase space d^3k/k^0 so that the number density N is a relativistic scalar. Scaling implies that at high energy, $s\to\infty$,

$$N_{ab}^{c}(\vec{k},s)\to N_{ab}^{c}(k_\perp,x)\,, \tag{2}$$

where $x=2k_\parallel/\sqrt{s}$ and $k_\perp$ and $k_\parallel$ are the perpendicular and parallel components of $\vec{k}$ in the center-of-mass system. We assume here that no multiplicative $\ln s$ terms are present.

The average multiplicity of the detected particle c is generally

$$\langle n_{ab}^{c}(s)\rangle=\int\frac{d^{3}k}{k^{0}}\,N_{ab}^{c}(\vec{k},s)\,. \tag{3}$$

If N_{ab}^c scales, with $N_{ab}^c(k_\perp,x)$ regular at $x=0$, we have

$$\begin{aligned}\langle n_{ab}^{c}(s)\rangle&=\int d^{2}k_\perp N_{ab}^{c}(k_\perp,0)\ln s+\text{constant}\\&=C_{ab}^{c}\ln s+D_{ab}^{c}\,.\end{aligned} \tag{4}$$

In the work of Ref. 4, one important piece of evidence for the scaling hypothesis was the near agreement of the coefficient C_{ab}^c obtained from the integral of N_{ab}^c at accelerator energies with the direct measurement of the growth of the multiplicity at the Echo Lake experiment.

The particular parametrization of $N_{ab}^c(k_\perp,x)$ used in Refs. 4 and 6 was

$$N_{pp}^{\pi^\pm}(k_\perp,x)=N_0^\pm\exp[-b(k_\perp+k_\perp{}^2/M)]e^{-a^\pm x^2}\,. \tag{5}$$

It provided a good fit to the data with $a^-=12.1$, $a^+=7.6$, and $b^+=b^-=b=2.44$ (GeV/c)$^{-1}$. The normalization factor $N_0^\pm$ still exhibited some energy dependence at accelerator energies. If we assume that the transverse momentum distribution is essentially energy-independent, as appears to be the case, then any energy dependence in $N_0^\pm$ will reflect itself in an energy dependence of the coefficient of the logarithm

$$C_{pp}^{\pm}=\int d^{2}k_\perp N_{pp}^{\pm}(k_\perp,0)\,. \tag{6}$$

A Regge analysis along the lines of the work of Mueller[3] indicates that one should expect a residual energy dependence in C which is controlled by some fractional power of s. Abarbanel[3] argues, more specifically, that the approach to the scaling limit should be like $s^{-1/4}$. However, more recent considerations based on duality[7] tend to favor a perhaps more rapid approach to the scaling limit for proton-proton reactions than the one indicated by Abarbanel. To get a feeling of how fast the scaling limit is attained, we have made a least-squares fit of the energy dependence of $C_{pp}^\pm$ obtained in Ref. 4,[8] assuming that the approach to the scaling limit goes like $s^{-1/4}$, $s^{-1/2}$, and s^{-1}. Our uncertainty in the value of C is such that we cannot pretend to extract the "correct" energy dependence. Figure 1 displays the result of such a fit. We constrained out fit so that $C_{pp}^+(\infty)=C_{pp}^-(\infty)$, since we must conserve charge. As can be seen, none of the plots in Fig. 1 are particularly good, although the faster approach to scaling is somewhat "more likely." We should remark that if the approach to the asymptotic region is indeed given by $s^{-1/4}$, contrary to what was said in Ref. 4, we are still far away from the scaling limit at 30 GeV/c. Furthermore, the region of 150–700 GeV/c, characteristic of the Echo Lake experiment, is not yet itself asymptotic. However, the value of $C_{pp}^\pm$ obtained with this energy behavior is far too high to be in agreement with the multiplicity obtained by Jones *et al.*[5] This value also implies that the pions carry away a much greater fraction of the total energy ~60–70% than is observed in cosmic-ray experiments. We shall use in this work a value of $N_0^\pm(\infty)$ that corresponds to $C_{pp}^\pm(\infty)=0.42$, i.e., $N_0^\pm(\infty)=1.1$ (GeV/c)$^{-2}$. We feel that this is a reasonable compromise between the values obtained in Fig. 1, for the $s^{-1/2}$ and s^{-1} dependence. We should point out that this value of C is somewhat in disagreement with the multiplicity

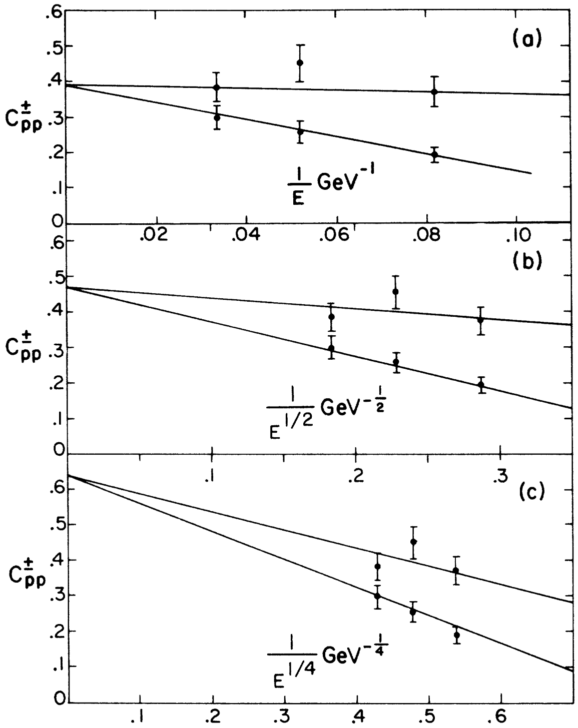

FIG. 1. Variation of $C^{\pm}_{pp}$ with energy assuming an approach to the scaling limit like (a) E^{-1}, (b) $E^{-1/2}$, (c) $E^{-1/4}$. We have indicated our uncertainty in the values of $C^{\pm}_{pp}$ by assuming, somewhat arbitrarily, a uniform error of 10%. The actual error may in fact be more, especially for the 19.2-GeV data where a large extrapolation was necessary to obtain the value at $x=0$ in Ref. 4. Here E is the lab energy.

determination of the Echo Lake experiment, but we feel, and shall comment on it later, that the authors of Ref. 5 underestimate their multiplicity. We should emphasize that the value of the normalization parameter is the one which we know with the least accuracy. Thus our error in N_0 could well be as much as 15%.

The parametrization of Eq. (5) involves an assumption of separation of the dependence of N on x and $k_\perp$ into two separate factors, an assumption that has nothing to do with scaling. Although it does not follow from any theoretical model,[3] it seems to be remarkably well satisfied by pp data at various accelerator energies.[9] We shall call this feature "longitudinal decoupling." The number function $N^c_{ab}(k_\perp, x)$ can, however, be expected to have "factorization" properties characteristic of Regge exchange. In an appropriate range of x, the produced particle does not "remember" the initial projectile and/or target.[10] Thus, if various pairs of initial particles are selected from the set a, b, d, e, and $x>0$ indicates that the detected particle c moves in the direction of the incident particle (first subscript below),

$$\begin{aligned} N^c_{ab}(k_\perp, x) &= N^c_{ad}(k_\perp, x)\,, \quad x>0 \\ N^c_{ab}(k_\perp, x) &= N^c_{db}(k_\perp, x)\,, \quad x<0 \\ N^c_{ab}(k_\perp, x) &= N^c_{de}(k_\perp, x)\,, \quad x=0\,. \end{aligned} \tag{7}$$

These results follow from the multiperipheral model or from Mueller's analysis[3] with a factorizable Pomeranchuk singularity.

In order to gain some insight into the form of the angular distribution, it is convenient to study the dependence of N on the rapidity rather than on x. We define the rapidity of a paticle of mass μ and four-momentum k as

$$\begin{aligned} R &= \tfrac{1}{2}\ln\left(\frac{k^0+k^3}{k^0-k^3}\right) = \tfrac{1}{2}\ln\left(\frac{k_+}{k_-}\right) \\ &= \ln\left[\frac{k_+}{(k_\perp{}^2+\mu^2)^{1/2}}\right] = \ln\left[\frac{(k_\perp{}^2+\mu^2)^{1/2}}{k_-}\right], \end{aligned} \tag{8}$$

where we choose the third axis as the incident direction. Under Lorentz boosts along the incident direction, R simply translates by the boost parameter. The phase-space factor now becomes

$$\frac{d^3k}{k^0} = dR\, d^2k_\perp \tag{9}$$

so that $d\sigma^c_{ab}/dR$ is invariant under longitudinal boosts (e.g., boosts going from the lab to c.m. frame or to the target rest frame). If r is the rapidity in the c.m. frame, then

$$x = 2\,\frac{(k_\perp{}^2+\mu^2)^{1/2}}{\sqrt{s}}\sinh r = \frac{2m_\perp}{\sqrt{s}}\sinh r\,, \tag{10}$$

with $m_\perp$ a "transverse mass." The scaling hypothesis now reads

$$\frac{1}{\sigma_{ab}(s)}\frac{d\sigma^c_{ab}}{dr} = \frac{dn^c_{ab}}{dr} = \int d^2k_\perp N^c_{ab}\left(k_\perp, \frac{2m_\perp}{\sqrt{s}}\sinh r\right). \tag{11}$$

In particular, we note that the density at $r=0$ is the coefficient of the logarithm in the multiplicity

$$\left.\frac{dn^c_{ab}}{dr}\right|_{r=0} = C^c_{ab}\,. \tag{12}$$

If the factorization described in Eq. (7) holds, C^c_{ab} is independent of both the type of projectile and target, and we can write simply C^c. The well-established experimental fact that the produced particles are limited in their transverse momentum $k_\perp$ allows us to obtain a qualitative picture of dn^c/dr as a function of r. For large enough s, $(2m_\perp/\sqrt{s})\sinh r$ is practically zero until $r\approx \ln s$. Thus we expect that in this region, $dn^c/dr \approx C^c$. As r reaches $\sim\ln s$, we expect that dn^c/dr should rapidly go to zero, since the integral constraint

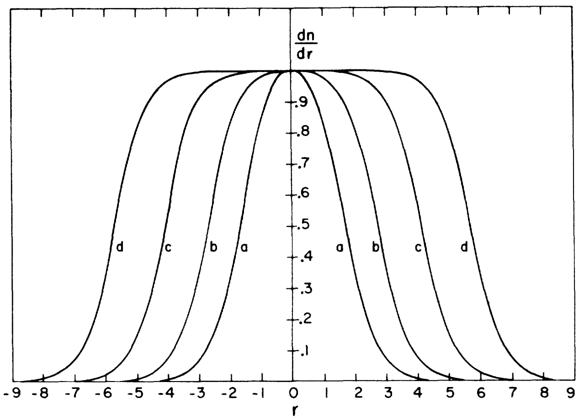

FIG. 2. Rapidity plots at various energies (unnormalized). The incident energies are (a) $E=30$ GeV, (b) $E=250$ GeV, (c) $E=4000$ GeV, (d) $E=100\,000$ GeV.

$$\int dr \frac{dn^c}{dr} \approx C^c \ln s \tag{13}$$

must be satisfied. The picture that emerges is a boxlike shape of the number distribution dn^c/dr with a height C^c and width $\sim\ln s$. As the energy s increases, the width of the box increases like $\ln s$ but its height remains constant. In Fig. 2 we exhibit rapidity plots obtained using the parametrization of Eq. (5) for π^- production.

III. ANGULAR DISTRIBUTIONS IN COSMIC-RAY EXPERIMENTS

The qualitative features of the high-energy inclusive cross section discussed in Sec. II have direct implications for cosmic-ray experiments. In this discussion we shall limit ourselves to the cosmic-ray data of the Echo Lake experiment,[5] because they are the only high-energy data currently available which have a proton target. Although there is a wealth of information on the interaction of high-energy protons with emulsions, it is very hard to extract from this data the behavior of the nucleon-nucleon inelastic cross sections. The chief reason for this is, of course, the complications due to rescattering in large nuclei. This complexity can be readily understood once one realizes that the secondaries of a primary collision have a mean free path in nuclear matter of only ~1 F. Thus the incoming energy is not pumped directly into a set of secondaries and a fast and a slow proton, but rather it is further degraded by multiple collisions inside the nucleus. The net result is a marked increase of the multiplicity (at 1000 GeV, where $\langle n\rangle$ in protons is 7–8, the measured multiplicity in nuclei can be 13 or more) and a distortion of the angular distribution, with the slower particles multiple scattering into relatively large angles. We have made Monte Carlo estimates of these effects, and have achieved a qualitative understanding of emulsion data at about 1000 GeV, but considerable work remains to be done in order to fully understand emulsion data.

Because cosmic-ray measurements cannot, in general, determine the energy of a produced particle, but only the scattering angle, the data are usually presented in terms of the Castagnoli variables[11] $u=\ln\tan\frac{1}{2}\theta_{\text{c.m.}}$ and $v=\ln\tan\theta_{\text{lab}}$. These variables, for most of the range of x, are closely related to the rapidity,

$$r=\ln\left[\frac{k_+}{(k_\perp{}^2+\mu^2)^{1/2}}\right]=\ln\left[\frac{k^0/k+\cos\theta_{\text{c.m.}}}{(\sin^2\theta_{\text{c.m.}}+\mu^2/k^2)^{1/2}}\right]. \tag{14}$$

Clearly, for $\mu\to 0$, we have

$$r\to -\ln\tan\tfrac{1}{2}\theta_{\text{c.m.}}=-u \tag{15}$$

so that, if the mass of the produced particle may be neglected, these variables are equivalent. For forward or backward angles in the c.m. system, where $k^2\gg\mu^2$, the mass is clearly negligible, and the plots vs r or $-u$ are identical. For $\theta_{\text{c.m.}}$ in the neighborhood of 90° the produced particle may be slow, its mass cannot be neglected, and there will be a difference between the two plots, as was pointed out by Lyon, Risk, and Tow[12] (see Fig. 3). Neglecting for the moment this effect, we can expect that the angular distribution in u should be quite similar to the distribution in r discussed in Sec. II. We thus infer that the angular distributions will also exhibit a boxlike shape, with a width growing like $\ln s$. This last feature, although very important, also indicates that cosmic-ray angular

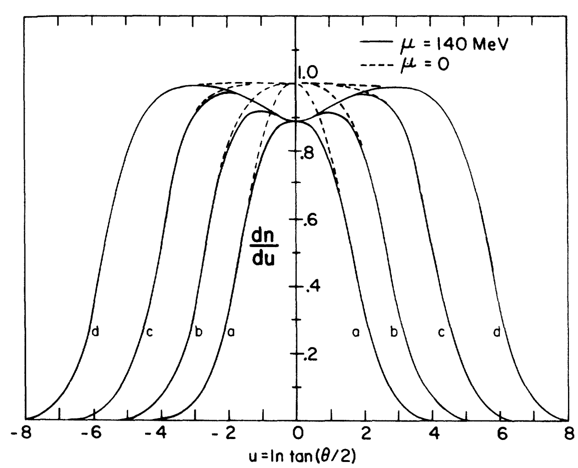

FIG. 3. The effect of the pion mass in plots of dn/du versus u is illustrated at the same energies as in Fig. 2. The dashed lines are the corresponding zero-mass plots.

distributions should vary very slowly with the total energy, and in general exhibit little structure. Thus we do not expect these distributions to provide very delicate tests of the function N^c as a function of x, $k_\perp$, and s. It is only with exceedingly large changes of the energy, such as going from $s = 60$ to 6000 GeV^2, that marked changes in these angular distributions should be manifest.

In order to compare the parametrization of N_{ab}^c given in Eq. (5) with the Echo Lake data, a number of rather crucial experimental biases have to be taken into account. The principal ones are the following:

(i) The experiment cannot determine the type of the produced particle, and only distributions for *all* charged particles are obtained. Thus produced π's, K's, and recoiling nucleons are all lumped together. Their angular distributions can be expected to differ from one another.

(ii) The data contains a sizable contamination of e^+e^- pairs arising from γ-ray conversion in the walls of the chamber. Moreover, the spectra is contaminated by hadron rescattering.

(iii) The experiment cannot determine real space angular distributions but only distributions on a "projected" plane.

(iv) There is a strong variation of the detection efficiency with production angle due to limitations of the spark chambers at large angles.

(v) Small angles (less than 2 mrad) cannot be resolved.

(vi) There is contamination due to the fact that the incident "beam" contains 30% pions.

We shall now briefly discuss each of these effects.

The inability to determine the type of produced particle is a serious difficulty, as it entails the knowledge of N_{pp}^c for all c. At the energies of the Echo Lake experiment (150–700 GeV), the majority of the produced particles are pions; nevertheless there is 20–25% proton and about 10% K production. It is well known from accelerator energies that the K distributions are substantially the same as the pion ones, except possibly at very large and very small angles in the c.m. frame. For the protons, however, the picture is quite different. On the one hand, we do not expect substantial $N\bar{N}$ production and thus N_{pp}^p should nearly vanish at $x = 0$. On the other hand, the two nucleons produced in the collision carry away, on the average, about 60% of the energy of the incident particle. These two reasons indicate that the p distribution is markedly different from the π and K distributions, and should not look at all like Eq. (5). In order to estimate the proton contribution to the spectrum, we have assumed that the proton distribution, like the π distribution, nearly scales at accelerator energies. We can then take the Anderson *et al.*[13] parametrization at 30 GeV and "scale" it to the desired energy. We have from Ref. 12 at $s = 30$ GeV:

$$\frac{d^3\sigma_{pp}^p}{dk^3} = 610 k_\perp e^{-6.03 k_\perp} \text{ mb}\,(\text{GeV}/c)^{-3} \tag{16}$$

or

$$dn_{pp}^p = \frac{d^3k}{k^0}\left[\frac{1}{\sigma_{pp}}\ 610 k_\perp k^0 e^{-6.03k_\perp}\right]. \tag{17}$$

Scaling implies that the bracketed quantity is s-independent, or

$$dn_{pp}^p = \frac{d^3k}{k^0}\,\frac{1}{\sigma_{pp}}\,4.67\times 10^3 k_\perp\,\frac{k^0}{\sqrt{s}}\,e^{-6.03k_\perp}. \tag{18}$$

The above formula scales everywhere except for very small x, where an s dependence is introduced through the $k^0/\sqrt{s}$ factor. The above parametrization does a reasonable job in fitting the data at 18 and 30 GeV and is consistent with preliminary data at much higher energies.[14] For produced kaons we adopt, for lack of anything better, a parametrization like Eq. (5) with $N_0 = 0.1$ (GeV/c)$^{-2}$, or about 10% of the charged-pion normalization.

The pair contamination comes chiefly from γ rays arising from the decay of the π^0. As the produced pairs have a very small opening angle, they are not resolved. Thus the final pair tracks will correspond to the folding of the π^0 distribution with its decay into 2γ. The result of these effects is a pair distribution with very nearly the same form as the charged-π distribution.[6] Since 8% of the π^0's give rise to pairs in the apparatus, this effect can be accounted for roughly by an appropriate change in the pion normalization. The hadron regeneration in the chamber is of the order of 3% per track, and thus much larger per event. It can be taken into consideration in similar fashion as the π^0 effects.

We turn now to point (iii) above. The Echo Lake experiment is not able to measure the real space angle between the incident particle and a secondary particle, but rather it measures $dn/d(\ln\cos\theta_p)$, where θ_p is the projected angle between the incident particle and the secondary particle in a plane containing the incident track.[5] The real angle between the incident and secondary tracks cannot be reconstructed. Thus in order to match the experimental distributions, it is necessary to subject the theoretical distributions to an analogous projection. There is no unique relation between the real scattering angle and the projected angle, unless the incident proton is coming vertically, which is not the case in the experiment. Fortunately, cosmic-ray protons are strongly peaked about the zenith,[15] so that one can assume that approximately the projection procedure is unique, and hope that the errors introduced will be small.

The last major correction to consider in the data is due to the presence of a strong bias against large laboratory angles produced by the peculiarities of spark-chamber triggering. The efficiency of the detection chambers as a function of the angle is not completely known to us at this time, except that it is essentially 100% for zenith angles less than 40° in the lab frame and zero for angles greater than 55°. In particular, this means that most of the slow-proton tracks are not detected, as well as the very large angle π's and K's. In correcting for this bias, it should be noted that the varying cosmic-ray proton flux as a function of the incident angle gets folded into it, as the cutoff is effective in real lab angle, and not in the projected scattering angle. We have corrected for this effect by putting a sharp cutoff at 45° (real angle). Varying the cutoff to 50° does not affect our distributions markedly.

The corrections due to lack of resolution at very small angles are negligible for all practical purposes.

Finally, at the altitude of Echo Lake, the cosmic-ray flux can be expected to contain a contamination of 30% high-energy pions whose products cannot be resolved from the ones produced by protons. As discussed in Sec. II, the pion-originated events can be expected to differ from the proton ones for $x>0$, but should be essentially the same for $x\leq 0$, if the factorization hypothesis holds. We have not corrected for this effect.

IV. COMPARISON WITH ECHO LAKE DATA

When all the corrections described in Sec. III are taken into account, it becomes possible to calculate the angular distributions of cosmic-ray secondaries *with no adjustable parameters*. The procedure is, of course, to take the parameters describing N_{pp}^{π} and N_{pp}^{p} and N_{pp}^{K}, which were fixed by our analysis of accelerator data and the scaling hypothesis, and use them to calculate the distributions at the much greater energies of the Echo Lake experiment, taking care to include all the biases and corrections discussed in Sec. III.[16] A very primitive attempt in this direction was described in Ref. 6. However, in that work not all the corrections and biases were included, and, moreover, the mass of the secondary particle was neglected. This last point is well worth emphasizing. In Fig. 3 we exhibit the pion distributions that follow from the parametrization of Eq. (5) for several energies. The dashed curves are the corresponding zero-mass distributions. The effect of the "pion-mass hole" is significant. Of course, we do not expect this "hole" to be present in the laboratory distributions, since the approximation of neglecting the mass is valid everywhere except for extremely large angles. We show laboratory pion distributions in Fig. 4. Figure 5 exhibits the proton distributions in the laboratory, obtained by using Eq. (18). The two-"horned" distribution is typical of the strong forward and backward peaking of the proton distribution.

In Figs. 6 and 7 we present our final projected charged secondary distributions at the two energies given by the Echo Lake experiment. We stress once more that essentially *no* free parameters go into the construction of these curves. The agreement of the distributions with experiment is reasonably good. The calculated distributions are somewhat larger than the experimental ones at very small angles. There may be an experimental bias involving the resolution of final trajectories by the apparatus when extremely

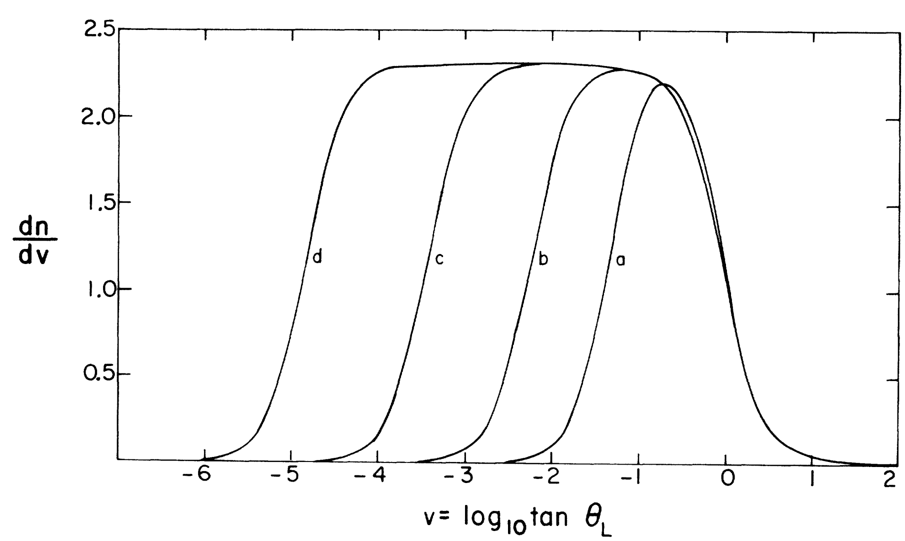

FIG. 4. Distributions of pions in the laboratory frame at the same energies as in Fig. 2.

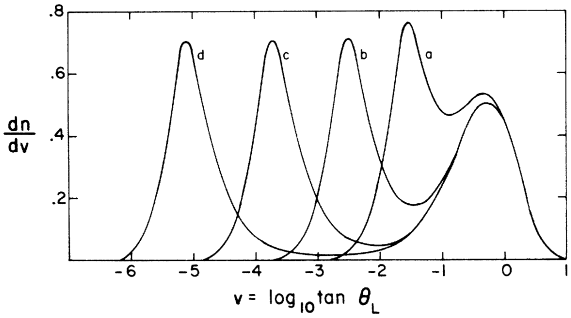

FIG. 5. Proton distributions in the laboratory frame. Again the energies are as in Fig. 2. The two-horned structures and the limiting behavior of the large-angle part of the spectrum are evident.

small angles are involved, but this does not seem to be a large effect. The theoretical peak slightly undershoots the experimental one. This discrepancy is not understood. The largest discrepancy between theory and experiment occurs at large angles, where the theoretical curve lies substantially above the experimental points. It is in this region, however, that the rapid drop in detection efficiency can be expected to dominate the distribution. In fact his effect practically controls the shape of the distribution for $\theta_p > 45°$.

The detailed nature of this cutoff is not completely known to us at this time, but it would seem to be more severe than presently estimated by the experimentalists. It is very unlikely that the large-angle overshoot can be accounted for by nonscaling effects or changes in the parameters of our distributions, because these would also imply discrepancies in the small-angle region, where the fit is reasonable. Calculations by Caneschi, Lyon, and Risk,[17] of these same distributions, using a different form for N_{pp}, but taking

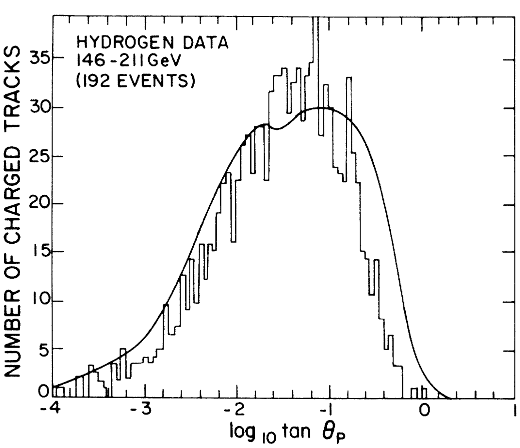

FIG. 6. Comparison of the theoretical distributions with the experimental data of Ref. 5 (E = 146–211 GeV).

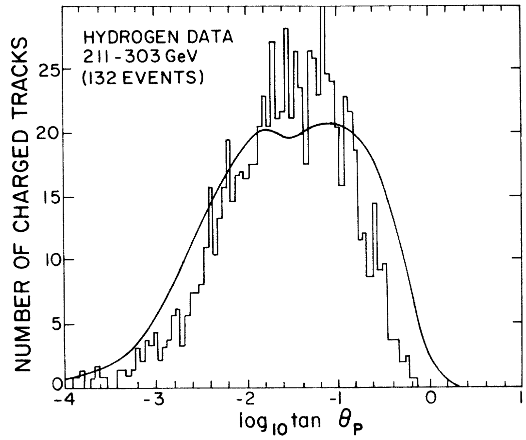

FIG. 7. Comparison of the theoretical distributions with the experimental data of Ref. 5 (E = 211–303 GeV).

into account most of the experimental biases seems to bear this out.[17] It should be pointed out that with our value of $N_{pp}^{\pi} \approx 1.1/(\mathrm{GeV}/c)^2$ the multiplicity we obtain is about 15% higher than that measured by the Echo Lake experiment. However, we believe that the missing experimental multiplicity is in the large-angle region, where, it would seem, the details of the cutoff are not completely understood.

In spite of these troubles, the scaled N_{pp} functions seem to reasonably account for the angular distributions at cosmic-ray energies. This we take as a qualified support for the scaling hypothesis.[18] The gross change in the angular distribution as the energy increases from 30 to 250 GeV are well accounted for by our calculations, and this change is much greater than the discrepancies between theory and experiment. The size of this change is indicated by the curves at 30 and 250 GeV that are displayed in Fig. 4.

On the other hand, it should be pointed out that the angular distributions, once the scaling hypothesis is accepted, depend little on the detailed parametrizations of the N_{pp} functions. Rather large alterations in the parameters produce minimal changes in the angular distributions. Thus only very high-precision experiments on angular distributions can be expected to yield detailed information on the scaling functions N_{pp}. For this purpose, cosmic-ray experiments are probably not as well suited as are the very high-energy laboratory experiments which will be soon carried out with accelerators.

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$$\frac{\sigma_{\text{tot inelastic}}}{\sigma_{\text{tot}}}=\frac{30.5}{38.0}.$$

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[15]The incoming proton flux variation with zenith angle is further reduced by the geometrical acceptance of the apparatus. We took the zenith dependence of the proton flux to be

$$\begin{aligned} I(\theta) &= \exp\left(\frac{-x}{\lambda\cos\theta}\right)\left(1-\frac{\theta}{0.4}\right), \quad \theta\le 0.4 \\ &= 0, \qquad\qquad \theta>0.4\,, \end{aligned}$$

where $x=710$ g/cm^2 and $\lambda=140$ g/cm^2. The last bracket accounts approximately for the geometrical acceptance of the apparatus.

[16]We should mention that since elastic events are not included in the data of Echo Lake, we have adjusted our normalization by the factor

$$\frac{\sigma_{\text{tot}}}{\sigma_{\text{tot inelastic}}}=\frac{38.0}{30.5}.$$

(This was the original motivation for normalizing to $\sigma_{\text{tot inelastic}}$ in both Refs. 4 and 6. However, we feel that the convention adopted in the present paper is more meaningful.)

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