range of t at high energy. This concludes our comments on the case $p>0$.

Looking back over our argument, we see that too strong a branch point is inconsistent with t-channel elastic unitarity unless high trajectories other than those of Eq. (1) are introduced. Too weak a branch point produces an anomalous diffraction pattern. We cannot rule out exceptions, but we expect these qualitative conclusions to be valid for quite general partial-wave amplitudes. The constraints we have discussed should be kept in mind when constructing model amplitudes.

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Integral Representation for the Forward Scattering Amplitude*

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The two-variable perturbation-theoretical integral representation (the Deser-Gilbert-Sudarshan representation) for the off-the-mass-shell forward scattering amplitude is explicitly proved to all orders in perturbation theory under certain natural spectral conditions.

Recently, much attention has been paid to inelastic electron-proton scattering. Its structure functions can be expressed in terms of the forward virtual-photon-proton scattering amplitude. A number of authors[1] have discussed this amplitude by using the so-called DGS (Deser-Gilbert-Sudarshan) representation or the two-variable PTIR, where PTIR is an abbreviation of perturbation-theoretical integral representation.

The two-variable PTIR for the *vertex function* was used first in the Bethe-Salpeter equation by Wick,[2] Cutkosky,[3] and Wanders.[4] Its derivation based on the axiomatic field theory was made independently by Faĭnberg,[5] Deser, Gilbert, and Sudarshan,[6] and Ida,[7] but the incorrectness of their reasoning was pointed out by Minguzzi and Streater.[8] The present author,[9] however, proved that the two-variable PTIR for the vertex function holds to all orders in perturbation theory.

The two-variable PTIR for the *forward scattering amplitude* was heuristically derived also by Deser, Gilbert, and Sudarshan.[10] On the other hand, the present author[11] proposed PTIR's for the general off-the-mass-shell scattering amplitude and for the on-the-mass-shell one, and investigated their support properties on the basis of Feynman integrals. A perturbation-theoretical proof of the two-variable PTIR of Ref. 10, however, has never been given *explicitly*, though it is implicitly contained in the present author's previous work.[9,11,12] Therefore, it is probably not futile to present an explicit proof of the two-variable PTIR for the forward scattering amplitude.

Let $g=\{a, b, c, d\}$ be the set of four external lines. Any nontrivial division $(h|g-h)$ of g defines a channel. There are seven channels corresponding to $h=\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, and $\{a,d\}$. (For simplicity, we shall hereafter omit parentheses and commas in writing the elements of h.) Let M_h be the smallest possible total mass of intermediate states belonging to the channel $(h|g-h)$. Furthermore, let p_h be the outgoing external momentum of the channel $(h|g-h)$. Since we are considering the forward scattering, we can set $p_a=-p_b=p$ and $p_c=-p_d=q$. The momentum p is put on the nucleon mass shell, that is, $p^2=M^2$.

With the above notation, the integral representation of Ref. 10 reads

$$\int_{-1}^{+1} dz \int_{-\infty}^{+\infty} d\gamma \, \frac{\varphi(z,\gamma)}{\gamma - q^2 - 2zpq - i\epsilon}, \tag{1}$$

where the weight distribution $\varphi(z,\gamma)$ vanishes unless

$$\gamma \geq \max[(M_{ac}-M)^2+2M(M_{ac}-M)z,\ (M_{ad}-M)^2-2M(M_{ad}-M)z] \tag{2}$$

(the maximum should be taken for each z fixed) if

$$2M \leq M_{ac}+M_{ad}, \tag{3}$$

and

$$\gamma \geq \max[\tfrac{1}{2}(M_{ac}^2+M_{ad}^2)-M^2+\tfrac{1}{2}(M_{ac}^2-M_{ad}^2)z,\ -M^2z^2] \tag{4}$$

otherwise. In what follows, we prove (2) under the assumption (3) to all orders in perturbation theory, apart from the subtraction problem.

As is well known, the analyticity of a Feynman integral can be analyzed by means of the denominator function, called V, in the Feynman-parametric formula. The general form of V is[13]

$$V=\sum_l \alpha_l m_l^2 - \sum_h \zeta_h p_h^2, \tag{5}$$

where α_l and m_l (≥ 0) denote a Feynman parameter and an internal mass of a line l, respectively, ζ_h is a function of Feynman parameters such that $\zeta_h \geq 0$ for $\alpha_l \geq 0$ (all l), and the summation $\sum_h$ runs over all possible channels $(h|g-h)$. The Feynman-parametric integral can be transformed into the form of (1) by introducing the following integration variables:

$$z=\frac{\zeta_{ac}-\zeta_{ad}}{\zeta_c+\zeta_d+\zeta_{ac}+\zeta_{ad}}, \tag{6}$$

$$\gamma=\frac{\sum\alpha_l m_l^2-(\zeta_a+\zeta_b+\zeta_{ac}+\zeta_{ad})M^2}{\zeta_c+\zeta_d+\zeta_{ac}+\zeta_{ad}}. \tag{7}$$

Now, the basic tool of our proof is the following theorem[9,14]:

When $\alpha_l \geq 0$, V is positive semidefinite for p_h satisfying

$$p_h=\lambda_h p \quad \text{with} \quad p^2=1, \tag{8}$$

$$\lambda_a+\lambda_b+\lambda_c+\lambda_d=0, \tag{9}$$

$$|\lambda_h| \leq M_h. \tag{10}$$

We also need the following auxiliary theorem[15]: If h and h' are two *disjoint* subsets of g, then M_h, $M_{h'}$, and $M_{h\cup h'}$ satisfy the triangular inequalities (including the equality cases) in *any Lagrangian field theory* (involving real masses only).

From the stability condition for the nucleon, we can set $M_a=M_b=M$. Then the auxiliary theorem implies

$$\max(|M_{ac}-M|,|M_{ad}-M|) \leq \min(M_c, M_d), \tag{11}$$

because $M_{ac}=M_{bd}$ and $M_{ad}=M_{bc}$. Under the assumption (3), we can also show that

$$|2M-M_{ac}| \leq M_{ad}, \tag{12}$$

$$|2M-M_{ad}| \leq M_{ac}. \tag{13}$$

In (5), we insert (8) with

$$\lambda_a=-\lambda_b=M, \quad \lambda_c=-\lambda_d=M_{ac}-M; \tag{14}$$

then (9) is obvious, and (11) and (12) guarantee (10). Therefore, the main theorem implies

$$\sum_l \alpha_l m_l^2-(\zeta_a+\zeta_b)M^2-(\zeta_c+\zeta_d)(M_{ac}-M)^2 - \zeta_{ac}M_{ac}^2-\zeta_{ad}(2M-M_{ac})^2 \geq 0. \tag{15}$$

By using (6) and (7), the inequality (15) is rewritten as

$$\gamma \geq (M_{ac}-M)^2+2M(M_{ac}-M)z. \tag{16}$$

Likewise, by setting

$$\lambda_a=-\lambda_b=M, \quad \lambda_c=-\lambda_d=-M_{ad}+M, \tag{17}$$

we find

$$\gamma \geq (M_{ad}-M)^2-2M(M_{ad}-M)z. \tag{18}$$

Thus (2) has been established.

In the case $2M>M_{ac}+M_{ad}$, which is not needed for the practical application, we are unable to prove (4). We can only prove that

$$\gamma \geq -M^2 \tag{19}$$

from the support property of the off-the-mass-shell PTIR[11,15] and the stability condition.

Finally, we remark that the asymptotic behavior and the subtraction problem of the two-variable PTIR were already investigated in detail.[16] Two remarkable results are as follows:

(a) The unsubtracted PTIR can describe the Regge-type asymptotic behavior.

(b) The subtraction term of PTIR is a polynomial. No single spectral terms are necessary to be added.

Of course, some modifications are necessary for (1), because it is not of the standard form of PTIR.

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Validity of Eikonal-Type Approximations for Potential Scattering*

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It is shown that for a wide class of potentials the slightly modified form of the Glauber eikonal approximation, recently proposed by Abarbanel and Itzykson, is essentially equivalent to the simplified form of the Saxon-Schiff formula obtained by Sugar and Blankenbecler. Insight into the relationship among several high-energy approximations is thereby gained. It is then suggested that the above modified Glauber approximation may possess a rather large angular range of validity and may be as reliable as several of the recently proposed "improved" high-energy approximations. This is also numerically illustrated.

The eikonal approximation of Glauber[1] provides an attractive method for studying high-energy scattering. The elastic scattering amplitude for a particle of mass m and momentum $\vec{k}_0$ incident on a potential V is given in this approximation by

$$\langle \vec{k}_f | T_G | \vec{k}_0 \rangle = \frac{ik_0}{m} \int d^2 b\, e^{-i\vec{\Delta}\cdot\vec{b}} \{\exp[(im/k_0)\chi(\vec{b})] - 1\}, \tag{1a}$$

where

$$\chi(\vec{b}) = -\int_{-\infty}^{\infty} V(\vec{b}, z)\, dz, \tag{1b}$$

$$\vec{\Delta} = \vec{k}_f - \vec{k}_0, \quad \vec{r} = \vec{b} + \vec{z}. \tag{1c}$$

Since the above expression can easily be obtained by assuming high-energy and small-angle scattering, it is believed to have a rather limited range