[4]J. Weber, *General Relativity and Gravitational Waves* (Interscience, New York, 1962), Chap. 8.

[5]R. Ruffini and J. A. Wheeler, *The Significance of Space Research for Fundamental Physics* (European Space Research Organization, Paris, France, 1970, to be published).

[6]By proper choice of orientations and locations on earth, three mutually orthogonal detectors can be obtained in practice. However, only two detectors are really needed if use is made of the rotation of detectors with that of the earth and if the source of the signals is believed to be the same at different times.

[7]The three 4-h blocks chosen by Weber (Ref. 2) in our coordinates are (−2)−(+2) h, 2−6 h, and 6−10 h. The latter two blocks of low intensity are now identical. The anisotropy, however, is not as pronounced as with our choice.

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Amplitudes with Schwarz-Type Regge Trajectories*

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Amplitudes containing Schwarz-type Regge trajectories must satisfy certain constraints. Some recent examples in the literature do not do so.

Recently several authors have suggested that Regge trajectories of the type discussed by Schwarz[1] may be present in nature. These are

$$\alpha_\pm(t) = 1 \pm i\gamma(-t)^{1/2} + O(t),$$
$$t(j) = \left(\frac{j-1}{\gamma}\right)^2 + O((j-1)^3), \tag{1}$$

where $t(j)$ is the moving singularity in the energy plane. Such trajectories have occurred in bootstrap models of the Pomeranchukon,[1,2] in the Regge-eikonal model,[3] in models of the violation of the Pomeranchuk theorem,[4-6] and in models of diffraction scattering in which the diffraction pattern shrinks faster than $(\ln s)^{-1}$.[6] In the papers dealing with violation of the Pomeranchuk theorem, there are Schwarz trajectories of negative as well as positive signature. In the present paper we point out that general considerations rule out amplitudes containing *only* Schwarz trajectories, unless the latter are poles. In particular, model amplitudes presented in Refs. 2, 3, and 4 are unacceptable as they stand.

For simplicity we consider the scattering of identical spinless particles in the t channel. Then one of the following alternatives hold:

(1) The partial-wave amplitude in the t channel is infinite at $j=\alpha_\pm(t)$ for an interval of t or j. In this case there must be additional singularities that cross $\alpha_\pm(t)$ at $t=4\mu^2$. This is required by t-channel elastic unitarity, and the conclusion holds except when the Schwarz singularities are poles. Since $4\mu^2$ can be very small compared to the energy over which Regge singularities move, these additional singularities are potentially as important for diffraction scattering as the original Schwarz cuts. All the amplitudes presented in Refs. 2, 3, and 4 have only Schwarz cuts and are infinite at $j=\alpha_\pm(t)$, so they are in violation of t-channel unitarity. Oehme recognizes the need for additional singularities under the circumstances discussed here.[6,6a]

(2) The partial-wave amplitude in the t channel is finite at $j=\alpha_\pm(t)$. In this case the secondary, tertiary, and subsequent diffraction maxima become successively higher. This possibility arises because $\mathrm{Re}\,\alpha_\pm(t)\cong 1$ for $t\leq 0$, and the trend of the differential cross section with t is controlled by interference between the contributions of the two Schwarz singularities to the Sommerfeld-Watson integral. Here again, only the presence of additional singularities can modify our conclusion, and Oehme provides an example.[6] A diffraction pattern with increasing subsidiary maxima is incompatible with experimental data presently available.[7]

To study these questions we examine the family of partial-wave amplitudes

$$f(t,j) = a[(j-1)^2 - \gamma^2 t]^p + R, \tag{2}$$

where R is smaller near $t=t(j)$ than the first term, and p is an arbitrary real number. Here we have suppressed possible t or j dependence of a, p, and γ^2, and we shall mention how such dependence affects our conclusions as we proceed. Our family is obviously not exhaustive,[7] but it contains enough possibilities to expose the difficulty in constructing amplitudes with Schwarz singularities only.

The Froissart bound requires $p \geq -\frac{3}{2}$.[8] The Regge-eikonal model gives $p=-\frac{3}{2}$, as do some of the bootstrap models.[2,3]

The consequences of elastic t-channel unitarity for the threshold behavior of Regge-cut discontinuities have been discussed,[6,9] with the general conclusion that the discontinuity must vanish except at isolated points and under special circumstances. Here we shall pursue the investigations further to see what happens if we insist on the behavior indicated in Eq. (2), with $p<0$, $p\neq -1$. The partial-wave amplitude has the representation

$$f(t,j)=[W(t,j)+Y(t,j)]^{-1}, \tag{3}$$

where $W(t,j)$ has the elastic unitarity cut and $Y(t,j)$ has all the other cuts, including the cut $t(j)$ in the energy plane.

$$\mathrm{Disc}\,W(t,j)=-2t^{-1/2}(\tfrac{1}{4}t-\mu^2)^{j+1/2} \quad (t>4\mu^2). \tag{4}$$

Comparing Eqs. (2) and (3), we see that for $p<0$, $p\neq -1$, Y has a leading singularity at $t(j)$ of the form $[(j-1)^2-\gamma^2 t]^{-p}$, which permits a subtracted dispersion representation

$$Y(t,j)=\frac{t-t(j)}{\pi}\int_{t(j)}^{t_1}\frac{dt'\,\mathrm{Disc}\,Y(t',j)}{[t'-t(j)](t'-t)}+Y_R(t,j). \tag{5}$$

t_1 is chosen large enough so that Y_R is a real meromorphic function of t (for real j) in a region that includes $t(j)$ and $4\mu^2$. For $-1<p<0$, the only way to obtain the behavior of Eq. (2) is to demand that

$$\mathrm{Disc}\,Y(t',j)=\frac{1}{\sin\pi p}[\gamma^2 t'-(j-1)^2]^{-p}, \tag{6a}$$

$$Y_R(t(j),j)+W(t(j),j)=0. \tag{6b}$$

For $-\frac{3}{2}\leq p<-1$, there is an additional condition that only strengthens our conclusions. Equation (6b) is the crucial equation that makes Eq. (2) inconsistent with elastic unitarity for $p<0$, $p\neq 1$, because W has explicit singularities at $j_\pm=\alpha_\pm(4\mu^2)$.

There are three possibilities entailed by these explicit singularities:

(1) $t(j)$ is analytic at $j_\pm$, as in Eq. (2), and there are no additional moving cuts in the energy plane. The explicit singularities of $W(t(j),j)$ are canceled by fixed j cuts in $Y_R(t,j)$ at $j_\pm$, with the fixed cuts having just the properties required to keep $t(j)$ analytic at $j_\pm$.

(2) $Y_R(t,j)$ has no fixed branch point in j at $j_\pm$, and $t(j)$ is the only moving cut in the energy plane, but $t(j)$ is modified so that it has branch points at $j_\pm$.

(3) $t(j)$ and $Y_R(t,j)$ have no fixed branch points in j at $j_\pm$, but there is an additional moving cut in the energy plane, $\bar{t}(j)$, with $\bar{t}(j_\pm)=4\mu^2$. This additional moving cut in Y_R serves the same purpose as the fixed j cuts in possibility (1).

The presence of CDD (Castillejo-Dalitz-Dyson) poles in Y_R at or near $t=4\mu^2$ and $j=j_\pm$ does not affect our argument since $W(t(j),j)$ has branch points at $j=j_\pm$. We do not consider an infinite accumulation of poles.

Possibilities (1) and (2) put fixed cuts into $f(t,j)$ at $j_\pm$. These are absent in Eq. (2), and probably violate the Froissart bound[8] because $j_+>1$ if the $O(t)$ term in Eq. (1) can be neglected over the range $0<t<4\mu^2$. Possibility (3) does not share the problem of the Froissart bound, but it also introduces singularities that are both absent in Eq. (2) and potentially important for diffraction scattering. Note that t or j dependence of a, p, and γ^2 in Eq. (2) does not affect these conclusions as long as $p<0$ for some range of j, for then Eq. (6b) must be an identity in j.

Let us now return to the case $p=-1$. In this instance Y has no cut term in Eq. (5), which means there is no explicit dependence on $t(j)$. Accordingly, possibility (2) no longer leads to fixed j cuts in $f(t,j)$, and we see that Schwarz trajectories can occur in isolation if they are poles, and if the trajectories $\alpha_\pm(t)$ have cuts starting at the elastic threshold $4\mu^2$. The presence of such $O(t)$ terms in Eq. (1) rules out an exactly self-reproducing set of Mandelstam cuts, however.[1,2] This concludes our analysis of the consistency of the case $p<0$ with t-channel unitarity.

For $p>0$ the scattering amplitude in the crossed channel at high energy is

$$F(s,t)\underset{s\to\infty}{\sim}\frac{isb}{2\pi i}\int dj\,e^{-i\pi(j-1)/2}s^{j-1}[(j-1)^2-\gamma^2 t]^{p}$$

$$\underset{s\to\infty}{\sim}\frac{isb\,\pi^{1/2}}{\Gamma(-p)}(\ln s-\tfrac{1}{2}i\pi)^{-2p-1}$$

$$\times\frac{J_{-p-1/2}(\gamma(-t)^{1/2}(\ln s-\frac{1}{2}i\pi))}{[2\gamma(-t)^{1/2}(\ln s-\frac{1}{2}i\pi)]^{-p-1/2}}. \tag{7}$$

Here various kinematic factors nonsingular at $j=1$ have been incorporated into b. For positive-signature trajectories, Eq. (7) is straightforward to obtain. For negative-signature trajectories there is a pole at $j=1$ in the Sommerfeld-Watson integral that comes from the signature factor. This pole is canceled by a linear factor $(j-1)$ that must be present in Eq. (2) so that the physical p-wave amplitude $f_-(t,1)$ does not have a branch point at $t=0$.[10] Altogether, Eq. (7) also gives the asymptotic contribution of Schwarz cuts in the negative-signature partial-wave amplitude provided b is taken to be pure imaginary. The subsidiary diffractive maxima in Eq. (7) grow in amplitude for $p>0$. This result holds if a, p, and γ^2 are t- or j-dependent because the diffraction pattern shrinks like $(\ln s)^{-2}$, and the maxima we are discussing are crowded into an arbitrarily small range of t at high energy. This concludes our comments on the case $p>0$.

Looking back over our argument, we see that too strong a branch point is inconsistent with t-channel elastic unitarity unless high trajectories other than those of Eq. (1) are introduced. Too weak a branch point produces an anomalous diffraction pattern. We cannot rule out exceptions, but we expect these qualitative conclusions to be valid for quite general partial-wave amplitudes. The constraints we have discussed should be kept in mind when constructing model amplitudes.

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[7]There is a third possibility, which is that the amplitude approaches no limit at $j=\alpha_\pm(t)$, but oscillates for example. Such behavior might be compatible with the presence of Schwarz cuts only, but this only confirms our principal contention, which is that "simple" model amplitudes are unsatisfactory.

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Integral Representation for the Forward Scattering Amplitude*

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The two-variable perturbation-theoretical integral representation (the Deser-Gilbert-Sudarshan representation) for the off-the-mass-shell forward scattering amplitude is explicitly proved to all orders in perturbation theory under certain natural spectral conditions.

Recently, much attention has been paid to inelastic electron-proton scattering. Its structure functions can be expressed in terms of the forward virtual-photon-proton scattering amplitude. A number of authors[1] have discussed this amplitude by using the so-called DGS (Deser-Gilbert-Sudarshan) representation or the two-variable PTIR, where PTIR is an abbreviation of perturbation-theoretical integral representation.

The two-variable PTIR for the *vertex function* was used first in the Bethe-Salpeter equation by Wick,[2] Cutkosky,[3] and Wanders.[4] Its derivation based on the axiomatic field theory was made independently by Faĭnberg,[5] Deser, Gilbert, and Sudarshan,[6] and Ida,[7] but the incorrectness of their reasoning was pointed out by Minguzzi and Streater.[8] The present author,[9] however, proved that the two-variable PTIR for the vertex function holds to all orders in perturbation theory.

The two-variable PTIR for the *forward scattering amplitude* was heuristically derived also by Deser, Gilbert, and Sudarshan.[10] On the other hand, the present author[11] proposed PTIR's for the general off-the-mass-shell scattering amplitude and for the on-the-mass-shell one, and investigated their support properties on the basis of Feynman integrals. A perturbation-theoretical proof of the two-variable PTIR of Ref. 10, however, has never been given *explicitly,* though it is implicitly contained in the present author's previous