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Analysis of the "Best" Positivity and Analyticity Inequalities on a General, Crossing-Symmetric π^0-π^0 Amplitude*

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Inequalities derived previously from rigorous positivity and analyticity for the π^0-π^0 double partial-wave amplitudes (a_{nl}) are reexpressed in terms of Roskies's crossing-symmetric parametrization. Using theorems on moment problems, the "best" inequalities for any value of $\sigma = l + n$ are discussed, and the result is expressed as an allowed region of the otherwise unrestricted parameter space. The $\sigma = 4$ and $\sigma = 5$ "best" inequalities are explicitly calculated and compared with results previously given by Roskies for the case $\sigma = 4$. New inequalities relating the partial-wave amplitudes $f_l(s)$ and their derivatives $f_l'(s)$ are derived, valid for all $l \geq 2$, and all $0 \leq s \leq 1$. They are combined into the general analysis and compared with similar conditions. The moment approach is used to examine the nature of the inequalities resulting from an alternative method due to Roskies and to Piguet and Wanders; we discuss the advantages and disadvantages of this approach. Finally, applications are discussed, to show the constraint that is present in these inequalities. The extension of the method to the case of the π-π amplitudes with isospin is given in an Appendix.

I. INTRODUCTION

In this work we continue the study of the constraint placed on possible π^0-π^0 amplitudes by crossing symmetry, positivity, and analyticity using an approach developed previously.[1] As pointed out by Roskies,[2] these constraints, expressed as inequalities involving only a finite number of the Balachandran and Nuyts[3] double partial waves, a_{nl}, are most economically studied by using a crossing-symmetric parametrization of the amplitude, and expressing all the inequalities in terms of the coefficients of this expansion – this ensures that all results will automatically be consistent with crossing symmetry.

The emphasis in this work is on establishing a method to discuss the "best" inequalities that follow from the positivity of the absorptive part, $A(s, t)$, and of its derivative, $dA(s, t)/ds$, when combined with crossing symmetry. This positivity leads to positive combinations of partial waves and their derivatives [the simplest is the familiar $f_l(s) \geq 0$ for $0 \leq s \leq 4\mu^2$, $l \geq 2$], each of which gives rise to an infinite sequence of inequalities on the a_{nl}'s. Many of these inequalities are redundant, and by using theorems on moment problems, we are able to pick out a "best" set of inequalities, necessary and sufficient for the positivity of each combination. In addition, the method organizes the different inequalities, and clarifies the relationship between them, automatically picking out the "best" set of these. We do not have to examine the many redundant inequalities in detail.[4] The analysis is general and works with equal facility for all $l \geq 2$.

Only constraints on the amplitude within the Mandelstam triangle have been considered; in addition, the consequences of full positivity [$\mathrm{Im} f_l(t) \geq 0$ for $t \geq 4\mu^2$ and all l] have only been partially explored. As a consequence of this, our final results are not the most constraining possible.[5]

In Sec. II we establish our notation, review aspects of our previous work,[1] and collect pertinent results on the Balachandran and Nuyts expansion,[3] on Roskies's parametrization of a crossing-symmetric π^0-π^0 amplitude,[6] and on the relation of our inequalities to moment problems. Section III obtains and examines specific inequalities, utilizing only simple positivity, $A(s, t) \geq 0$, and a comparison with other work is made. The importance of full positivity, and the related derivative condition, $d^nA(s,t)/ds^n \geq 0$, is noted. This leads, in Sec. IV, to a sequence of inequalities relating partial waves and their derivatives, similar to those found by Auberson, Common, Martin, and others.[7-9] These too can be analyzed in the same general fashion to give inequalities on a finite number of a_{nl}'s. Because of certain simplifying approximations made, they are somewhat weak, although of an interesting form. For tighter constraints, depending on full positivity, we turn in Sec. V to an approach developed by Roskies[2] and by Piguet and Wanders.[10] The moment-problem approach developed here and previously[1] is used to extend and refine their method.

Finally, in Sec. VI, the results are discussed and applications indicated. The extension of the method and results to all the isospin amplitudes of the π-π problem is given in the Appendixes.

II. NOTATION AND REVIEW OF PERTINENT DETAILS

Throughout this section and the bulk of the paper, we will discuss only the π^0-π^0 amplitude; the notationally more complicated case with isospin will be considered in Appendix B.

We are studying the π^0-π^0 scattering amplitude

$$F(s,t)=\sum_{l=\text{even}}(2l+1)f_l(s)P_l\left(1+\frac{2t}{s-1}\right), \tag{2.1}$$

with energy units chosen so that $4\mu^2=1$. The partial waves for even $l \geq 2$ are given by the Froissart-Gribov projection

$$f_l(s)=\frac{4}{\pi(1-s)}\int_1^\infty dt A(s,t)Q_l\left(\frac{2t}{1-s}-1\right) \tag{2.2}$$

$$=\frac{2}{\pi}\int_{z_0}^\infty dz A(s,z)Q_l(z), \tag{2.3}$$

with $z_0(s)=(1+s)/(1-s)$. The t-channel absorptive part $A(s,t)$ has the expansion

$$A(s,t)=\sum_{l=\text{even}}(2l+1)\text{Im}f_l(t)P_l\left(1+\frac{2s}{t-1}\right), \tag{2.4}$$

which converges and is positive for $t \geq 1$ and $0 \leq s \leq 1$.

This positivity follows from t-channel unitarity, $\text{Im}f_l^I(t) \geq 0$, $t \geq 1$, and the π^0-π^0 combination $f_l(t) = \frac{1}{3}f_l^0(t)+\frac{2}{3}f_l^2(t)$.

For notational convenience throughout, we use $y=1-s$ and $z=2t/y-1$; we also use the same letter for the function $A(s,t)$, and the corresponding function of z, $A(s,\frac{1}{2}(z+1)y)\equiv A(s,z)$.

In order to examine s-t crossing symmetry $F(t,s)$ in terms of the partial waves, we introduce the expansion over the Mandelstam triangle[3] ($0 \leq s$, $t \leq 1$):

$$f_l(s)=\sum_{n=0}^\infty 2(n+l+1)(1-s)^l P_n^{(2l+1,0)}(2s-1)a_{nl}\,. \tag{2.5}$$

By considering the coefficients a_{nl} as elements of a series of vectors b^σ with $b_l^\sigma=a_{\sigma-l,l'}$, Balachandran *et al.*[3] reduced the problem of crossing symmetry to a series of matrix eigenvector problems, $b^\sigma=X^\sigma b^\sigma$, of dimension $\sigma+1$:

$$b_l^\sigma=\sum_{l'=0}^\sigma X_{ll'}^\sigma b_{l'}^\sigma, \tag{2.6}$$

where the crossing matrix X^σ has $\sigma+1$ orthogonal eigenvectors $E^\sigma(k)$, normalized by

$$E^\sigma(k)\otimes E^\sigma(k')\equiv\sum_{l=0}^\sigma E_l^\sigma(k)(2l+1)E_l^\sigma(k')$$
$$=\delta_{kk'}, \tag{2.7}$$

and with eigenvalues $(-1)^k$. An arbitrary crossing even amplitude will have a_{nl}'s given by

$$b^\sigma=\sum_{k=\text{even}}^\sigma c_k^\sigma E^\sigma(k), \tag{2.8}$$

with the coefficients given simply by

$$c_k^\sigma=E^\sigma(k)\otimes b^\sigma. \tag{2.9}$$

For the π^0-π^0 problem, we have in addition to s-t crossing symmetry, also s-u symmetry, which means in particular that $b_l^\sigma=0$ for odd l; the simultaneous eigenvectors will be some combinations, $\overline{E}^\sigma(p)$, of the $E^\sigma(k)$, with even k.

Roskies,[6] using an alternative approach, established an explicit set of independent π^0-π^0 eigenvectors:

$$(\alpha_p^\sigma)_l=(\sigma-l)!(\sigma+l+1)!$$
$$\times\int_{-1}^1 dz\, P_l(z)(z^2+3)^{3p-\sigma}(1-z^2)^{\sigma-2p}, \tag{2.10}$$

where p runs from $\{\sigma/3\}$ to $[\sigma/2]$, resulting in $n_\sigma = [\sigma/2]-\{\sigma/3\}+1$ independent eigenvectors. (We use $2[\sigma/2]$ to denote the nearest multiple of 2, not greater than σ; and $3\{\sigma/3\}$ to denote the nearest multiple of 3, not less than σ.) If σ has the value $\sigma=(0, 1, 2, 3, 4, 5, 6,$ or $7)$, then n_σ has the value $n_\sigma=(1, 0, 1, 1, 1, 1, 2,$ or $1)$, respectively. Although Roskies did not do this, we may use the set (2.10) to generate the n_σ orthogonal eigenvectors, $\overline{E}^\sigma(k)$, $k=0, \ldots, n_\sigma-1$, normalized according to Eq. (2.7).

An arbitrary crossing-symmetric π^0-π^0 amplitude may then be expanded in the Mandelstam triangle as

$$b^\sigma=\sum_{p=0}^{n_\sigma-1} c_p^\sigma\overline{E}^\sigma(p), \tag{2.11}$$

where the c_p^σ are completely arbitrary real numbers. For $\sigma=0$, 2 to 5, and 7, there is only one eigenvector (and none at all for $\sigma=1$), and the c_p^σ may be obtained from integrals over the S-wave alone, related to

$$a_{\sigma 0}=\int_0^1 ds(1-s)f_0(s)P_\sigma^{(1,0)}(2s-1). \qquad (2.12)$$

For higher σ, higher partial waves are needed to determine all the c_p^σ, which may be simply obtained using the orthonormality of $\bar{E}^\sigma(p)$:

$$c_p^\sigma=\bar{E}^\sigma(p)\otimes b^\sigma. \qquad (2.13)$$

As shown by the author,[1] positivity and analyticity constraints on the partial waves $f_l(s)$, derived by Yndurain,[11] lead to inequalities on the a_{nl} that will now severely restrict the allowed c_p^σ's.

We consider the moments of $f_l(s)$, defined by

$$[k]_l=\int_0^1 ds(1-s)^{k+1}f_l(s). \qquad (2.14)$$

Provided $k\geq l$, the orthogonality of the Jacobi polynomials leads to a finite sum over n:

$$[k]_l=\sum_{n=0}^{k-l}a_{nl}D_{nl}^k$$
$$=\sum_{\sigma=l}^{k}b_l^\sigma D_{\sigma-l,l}^k, \qquad (2.15)$$

with the D_{nl}^k given simply by

$$D_{nl}^k=2(n+l+1)\int_0^1 ds(1-s)^{k+l+1}P_n^{(2l+1,0)}(2s-1). \qquad (2.16)$$

Yndurain,[11] considering a moment problem associated with (2.2), obtained the necessary and "almost sufficient" conditions on the partial waves $f_l(s)$ for $A(s,z)$ to be non-negative:

$$\delta^m f_l(s)\equiv\sum_{j=0}^{m}\binom{m}{j}(-1)^j u_0^{2j}f_{l+2j}(s)\geq 0, \qquad (2.17)$$

for all $m\geq 0$ and even $l\geq 2$, and with

$$u_0(s)=z_0+(z_0^2-1)^{1/2}=(1+s+2\sqrt{s})/y.$$

For $m=0$ this implies $f_l(s)\geq 0$, and thus also $[k]_l\geq 0$. Practically, of course, we cannot use the sufficiency, since we would have to check an infinite number of conditions. For practical work, we must also modify $u_0(s)$ in order to obtain from (2.17) relations involving only a finite number of the a_{nl}'s. The author has shown[1] that (2.17) with $u_0(s)$ replaced by any $u_1(s)\leq u_0(s)$ for $0\leq s\leq 1$ is also a necessary consequence of (2.3). A suitable choice of $u_1(s)$ leads to positive moments

$$[k]\equiv\int_0^1 ds(1-s)^{k+1}\delta^m f_l(s) \qquad (2.18)$$

involving only a finite number of a_{nl}'s. (We will always use $[k]$, without subscripts, to mean some moment in general. The suppressed indexes, such as l, m, and u_1, will always be obvious for a specific case.)

We get in fact a sequence of moments $[k]$, each involving more a_{nl}'s than the previous (higher σ) and using a $u_1(s)$ which is a closer approximation to $u_0(s)$; as we will see, these closer approximations involving "more" of the positivity and analyticity contained in (2.17) give more constraining inequalities on the a_{nl}.

One sequence of approximations is given by $u_1(s)^2=y^n$, where $n\geq -2$. For $n\geq 0$, $u_1\leq 1$; so these do not go to ∞ as $s\to 1$, while of course, u_0 does. However, $u_1^2=1/y$ or $1/y^2$ do go to ∞, and follow the behavior of u_0 much more closely. The corresponding moments (2.23) for $u_1^2=y^n$ are given in terms of $[k]_l$:

$$[k]=\sum_{j=0}^{m}\binom{m}{j}(-1)^j[k+nj]_{l+2j}, \qquad (2.19)$$

where, to ensure that the various sums on n in $[k']_l$ are finite, $k+nj\geq l+2j$ for $j=0, 1,\ldots,m$.

A closer approximation than the above is given by $u_1(s)=z_0(s)=(1+s)/y$, or better still $u_1(s)=(1+3s)/y$; these involve even more values of a_{nl}, and have not been exploited in this paper.

Any choice of m and u_1 leads to a sequence of moments $[k]$ for $k\geq$ some k_0. These moments are all obviously positive, from the positivity of $\delta^m f_l(s)$; in addition, there are various combinations of moments that are positive, too.

Theorems relating to the complete set of inequalities that may be obtained in this way from a moment problem are given by Akhiezer,[12] and have been used by Yndurain,[11] Common,[13] and Griss.[1] We restate two of particular importance here.

Theorem 1. The necessary and sufficient conditions for the "truncated" moment problem,

$$s_k=\int_0^1 du\,\phi(u)u^k,\quad k=0,1,\ldots,n-1$$
$$s_n\geq\int_0^1 du\,\phi(u)u^n, \qquad (2.20)$$

to have a solution with non-negative $\phi(u)$ on $0\leq u\leq 1$ are that for even $n=2m$, the determinants

$$D(0,k)=\begin{vmatrix} s_0 & s_1 & \cdots & s_k\\ s_1 & & & \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ s_k & & & s_{2k}\end{vmatrix},$$

$$\Delta D(1,k)=\begin{vmatrix} (s_1-s_2) & (s_2-s_3) & \cdots & (s_{k-1}-s_k)\\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ (s_{k-1}-s_k) & & \cdots & (s_{2k-1}-s_{2k})\end{vmatrix} \qquad (2.21)$$

be positive, while for odd $n=2m+1$ the determinants

$$D(1,k)=\begin{vmatrix} s_1 & s_2 & \cdots & s_{k+1} \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ s_{k+1} & & \cdots & s_{2k+1} \end{vmatrix},$$

$$\Delta D(0,k)=\begin{vmatrix} (s_0-s_1) & (s_1-s_2) & \cdots & (s_k-s_{k+1}) \\ & & & \\ (s_k-s_{k+1}) & & \cdots & (s_{2k}-s_{2k+1}) \end{vmatrix} \tag{2.22}$$

be positive (for all k such that $m \geq k \geq 0$).

Theorem 2. For a moment problem $s_n=\int_0^a u^n \phi(u)du$, in which $n\to\infty$, alternative (Hausdorff) necessary and sufficient conditions for $\phi(u)$ to be non-negative on $0 \leq u \leq a$ are

$$\delta^m s_n \equiv \sum_{k=0}^{m} \binom{m}{k}(-1)^k s_{n+k} a^{m-k} \geq 0 \tag{2.23}$$

(for all n and $m \geq 0$, and a the upper limit of the moment integral).

The importance of Eqs. (2.21) and (2.22) is that they are sufficient, and are hence the "best" necessary conditions that may be derived. Notice that Eqs. (2.23) are also necessary for the truncated problem, but are not sufficient, because they are based on examining only a certain class, $u^n(a-u)^m$, of all the polynomials which are positive on $[0, a]$. Theorem 1 uses a theorem of Lukacs on the representation of the most general positive polynomial in the interval $0 \leq u \leq 1$. (See Griss[1] and Akhiezer.[12])

The inequalities derived from a given moment problem, with positive weight $\phi(u)$, lead to a characteristic region in the space of the moments $\{s_k\}$. In Sec. III we obtain a number of moment problems, each following from positivity-analyticity, and requiring up to a certain maximum σ; the weight functions, although different, are related by crossing symmetry and lead to relations between the corresponding sets of moments, best displayed by relating all moments, through the a_{nl}'s, to the crossing-symmetric parameters c_p^σ. Our problem is then to superimpose the various regions, distorted and rotated by crossing symmetry, as shapes in the parameter space $\{c_p^\sigma\}$; the resulting allowed region for the parameters c_p^σ is then the intersection of all the superimposed regions, and will be smaller than any separately. It is this aspect of rotation and distortion by crossing symmetry that makes it important to study σ (and hence l) higher than we actually intend to compare with a model; the projection of the resulting "body" onto the space of dimension that we are really interested in will be smaller than if we had considered only this subspace at the start. This is a consequence of the crossing, and not of a single moment problem, as we will now see.

The moment problem and the resulting inequalities are homogeneous in $\phi(u)$; since $s_0 \geq 0$, we study the ratio $r_k=s_k/s_0$ and the characteristic "positivity" region in the space $\{r_k\}$, where of course $r_0=1$.

For $n=1$, Theorem 1 gives the inequalities $r_1 \geq 0$ and $r_0-r_1 \geq 0$, i.e.,

$$1 \geq r_1 \geq 0\,. \tag{2.24}$$

For $n=2$, the inequalities are

$$r_0 \geq 0,\quad r_0 r_2 - r_1^2 \geq 0,\quad \text{and}\quad r_1-r_2 \geq 0,$$

i.e., (2.25)

$$r_1 \geq r_2 \geq r_1^2\,,$$

which is displayed in Fig. 1. Notice that the projection on the r_1 axis is just (2.24).

If we use Theorem 2 with $a=1$, we are led instead to the inequalities for $n+m \leq 2$:

$$\begin{aligned} &r_0 \geq 0,\quad r_1 \geq 0,\quad r_2 \geq 0,\quad r_0-r_1 \geq 0,\\ &r_1-r_2 \geq 0,\quad \text{and}\quad r_0-2r_1+r_2 \geq 0\,. \end{aligned} \tag{2.26}$$

All except the last one are included in (2.24) and (2.25). In fact, as shown by the dotted line in Fig. 1, $1-2r_1+r_2$ is tangent to the parabola of (2.25) and leads to a slightly larger region – it is not as "tight" a constraint. This is true in general; the linear constraints given by Theorem 2 reproduce the content of the nonlinear determinants only as

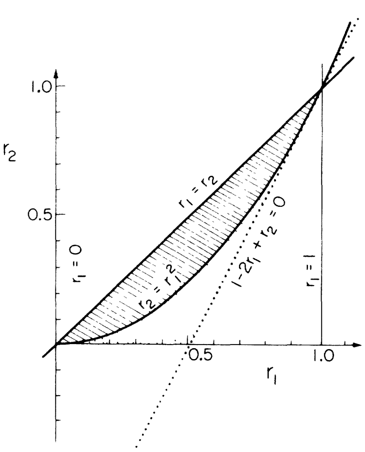

FIG. 1. The basic positivity region in (r_1, r_2) space for the moment problem $r_n=\int_0^1\phi(u)u^n du$, with $r_0=\int_0^1\phi(u)du=1$. The solid lines and hatched region result from the determinants of Theorem 1, while the dotted line and extra stippled region is that due to the weaker Hausdorff conditions of Theorem 2.

the envelope of an infinite sequence of tangent lines – hence the advantage of the determinants.

For $n=3$ the inequalities are

$$\begin{aligned} &r_1 \geq 0, \quad r_1 r_3 - r_2^2 \geq 0, \quad 1 - r_1 \geq 0, \\ &(1-r_1)(r_2 - r_3) - (r_1 - r_2)^2 \geq 0 . \end{aligned} \tag{2.27}$$

Since they imply that r_1 and $1-r_1$ are positive, we obtain

$$r_2 - \frac{(r_1 - r_2)^2}{1 - r_1} \geq r_3 \geq \frac{r_2^2}{r_1} . \tag{2.28}$$

The projection onto (r_1, r_2) space is then obtained by eliminating (r_3), leading to

$$(r_1 - r_2)(r_2 - r_1^2) \geq 0 \tag{2.29}$$

which is again the region specified by (2.25) (since $r_1 - r_2 \geq 0$).

III. EXPLICIT INEQUALITIES FOR $\sigma \leq 5$, USING POSITIVITY AND CROSSING SYMMETRY

In this section we analyze the inequalities that involve a_{nl}'s with $n+l=\sigma \leq 5$, arising from the application of simple positivity, $A(s,t) \geq 0$, to the crossing-symmetric π^0-π^0 amplitude.

For $\sigma \leq 4$, we may use only the double partial waves a_{02}, a_{12}, and a_{22} for the $D(l=2)$ wave, and a_{04} in the $l=4$ wave. The coefficient a_{00} is present only in the S wave, and is not related by crossing symmetry to any wave to which our positivity inequalities apply.

The moments $[k]_l$ [defined in (2.14)] that we may evaluate are then

$$\begin{aligned} &l=2: \ [2]_2, [3]_2, [4]_2, \\ &l=4: \ [4]_4 . \end{aligned} \tag{3.1}$$

Using the explicit values of D_{nl}^k from Table I, we then have

$$\begin{aligned} &[2]_2 = a_{02}, \\ &[3]_2 = \tfrac{1}{7}(6a_{02} - a_{12}), \\ &[4]_2 = \tfrac{1}{36}(27a_{02} - 8a_{12} + a_{22}), \\ &[4]_4 = a_{04} . \end{aligned} \tag{3.2}$$

Our conditions are all homogeneous in $f_l(s)$, and hence in the a_{nl}'s, and because $[2]_2 = a_{02} \geq 0$, we will divide by a_{02} to "normalize" the moments, leaving undetermined the coefficients a_{00} and a_{02}.

In order to compare our results with those of Roskies,[2] we will not use the $\{c_p^\sigma\}$ as basic variables, but an alternative set:

$$x_1 = -a_{12}/a_{02}, \quad x_2 = a_{22}/a_{02} . \tag{3.3}$$

Crossing symmetry, using (2.10), then relates a_{nl}'s with the same values of $\sigma = n+l$, enabling us to relate a_{04} to a_{22}; in addition [see discussion after Eq. (2.11)] we may express all our variables (for $\sigma=4$) in terms of the S-wave components, $a_{\sigma 0}$: $a_{02} = \frac{2}{5}a_{20}$, $a_{12} = -a_{30}$, $a_{22} = a_{04} = \frac{2}{7}a_{40}$, so that

$$\begin{aligned} &x_1 = -a_{12}/a_{02} = \tfrac{5}{2}a_{30}/a_{20}, \\ &x_2 = a_{22}/a_{02} = a_{04}/a_{02} = \tfrac{5}{7}a_{40}/a_{20} . \end{aligned} \tag{3.4}$$

The moments become very simply

$$\begin{aligned} &[2]_2 = 1, \\ &[3]_2 = \tfrac{1}{7}(6 + x_1), \\ &[4]_2 = \tfrac{1}{36}(27 + 8x_1 + x_2), \\ &[4]_4 = x_2 . \end{aligned} \tag{3.5}$$

We now write down the best inequalities for these moments, following from the constraint of positivity on the combinations $[k]$ in (2.19), for a given l, m, and $u_1(s)$.

For $l=2$, $m=0$, $[k]=[k]_2$. There are three moments, and so we apply the determinantal inequalities of (2.21):

$$[2]_2 \geq 0, \tag{3.6a}$$

$$[2]_2[4]_2 - [3]_2[3]_2 \geq 0, \tag{3.6b}$$

$$[3]_2 - [4]_2 \geq 0 . \tag{3.6c}$$

It is interesting to note that these three relations

TABLE I. Values of D_{nl}^k for σ up to 5.

$l=0$

$n \backslash k$	0	1	2	3	4	5
0	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{7}$
1	0	$-\frac{1}{3}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{8}{21}$	$-\frac{27}{70}$
2	0	0	$\frac{1}{10}$	$\frac{6}{35}$	$\frac{3}{19}$	$\frac{5}{21}$
3	0	0	0	$-\frac{1}{35}$	$-\frac{11}{175}$	$-\frac{2}{21}$
4	0	0	0	0	$\frac{11}{1400}$	$\frac{5}{231}$
5	0	0	0	0	0	$-\frac{1}{462}$

$l=2$

$n \backslash k$	2	3	4	5
0	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$
1	0	$-\frac{1}{7}$	$-\frac{2}{9}$	$-\frac{4}{15}$
2	0	0	$\frac{1}{36}$	$\frac{2}{33}$
3	0	0	0	$-\frac{2}{330}$

$l=4$

$n \backslash k$	4	5
0	1	$\frac{10}{11}$
1	0	$-\frac{1}{11}$

already contain all the "chains," $[k] \geq [k+1] \geq [k+2] \geq \cdots \geq 0$, found in a previous paper.[1] In particular, since $[2] \geq 0$, we see that $[4] \geq [3]^2/[2] \geq 0$; hence $[4] \geq 0$, $[2][3] \geq [2][4] \geq [3]^2$, so that $[2] \geq [3]$ giving

$$[2]_2 \geq [3]_2 \geq [4]_2 \geq 0 . \qquad (3.7)$$

For $l=4$ and $m=0$, $[k]=[k]_4$ and all we have is

$$[4]_4 \geq 0 . \qquad (3.8)$$

For $l=2$, $m=1$, and $u_1{}^2 = y^n$, $[k]=[k]_2-[k+n]_4$, leading to a set of relations:

$$\begin{aligned} n=2,&\quad [2]_2-[4]_4 \geq 0 ,\\ n=1,&\quad [3]_2-[4]_4 \geq 0 ,\\ n=0,&\quad [4]_2-[4]_4 \geq 0 . \end{aligned} \qquad (3.9)$$

The chain (3.7) immediately shows that $n=0$ gives the "best" constraint; it corresponds to $u_1(s)=1$, which is the closest of three to $u_0(s)$, confirming our feeling[1] that as u_1 approaches u_0 more closely, more positivity and analyticity are used, giving tighter inequalities.

Expressed in terms of the x_1, the best inequalities are

$$\tfrac{1}{7}(6+x_1) \geq \tfrac{1}{36}(27+8x_1+x_2) , \qquad (3.10a)$$

$$\tfrac{1}{36}(27+8x_1+x_2) \geq [\tfrac{1}{7}(6+x_1)]^2 , \qquad (3.10b)$$

$$x_2 \geq 0 , \qquad (3.10c)$$

$$\tfrac{1}{36}(27+8+x_1+x_2) \geq x_2 . \qquad (3.10d)$$

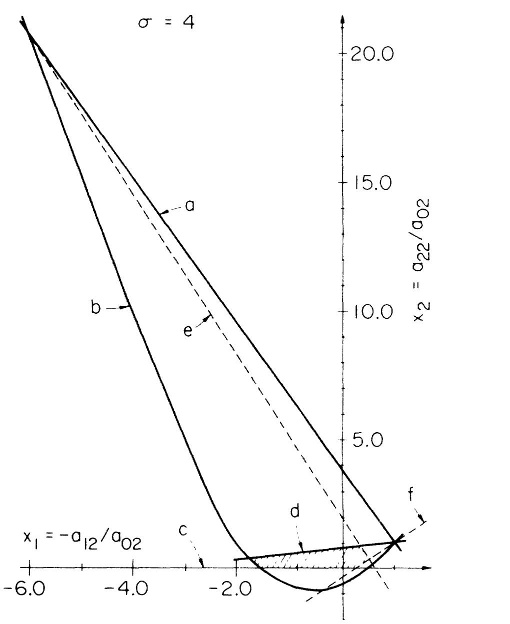

FIG. 2. Allowed region of $x_1 = -a_{12}/a_{02}$ and $x_2 = a_{22}/a_{02}$, determined by positivity and crossing symmetry for $\sigma=4$. The solid lines (a, b, c, and d), corresponding to simple positivity, are given by Eqs. (3.10). In addition, the dashed lines, e and f, derived from Roskies's "one-zero" method in Sec. V, are shown for completeness.

The corresponding curves, labeled a, b, c, and d in Fig. 2, delineate the allowed region. We recognize the characteristic "positivity" shape (lines a and b for $l=2$ positivity); it is distorted and rotated, because we have reexpressed the basic moments $[k]_l$ in terms of the natural crossing-symmetry variables, x_i. Lines c and d are each a single line from separate moment problems, and their characteristic regions only appear in a higher-dimensional space (x_i, $i \geq 3$); superimposed on the above $l=2$ positivity region, they considerably restrict the allowed region – a consequence of combined positivity and crossing symmetry. Figure 3 displays this allowed region more clearly.

The lines b and c were previously found by Roskies[2] using a different, more complicated method based on full positivity [$\mathrm{Im} f_l(t) \geq 0$ for all $l \geq 0$, and $t \geq 1$]. In addition, he obtained lines e, f, and g, shown dotted in Figs. 2 and 3, which restrict the allowed region even more. This improvement uses more than simple positivity, and the method becomes rapidly more complex as higher l and σ are considered. Previous to our study, these lines of Roskies were the best found for $\sigma \leq 4$. Roskies's method will be discussed and extended in Sec. V, showing the connection with our inequalities; phrased in our "moment" language, it becomes simpler to study, and we find a slight improvement to line g, not found by Roskies.

These extra lines, a consequence of more than simple positivity, are shown in order to compare with the improvement that we now find when we consider the extra inequalities that arise when we include σ up to 5. (This is the largest value that still permits expression of the relevant a_{nl}'s in terms of a_{00} of the S wave alone.) We will introduce an extra dimension, x_3, and project the new

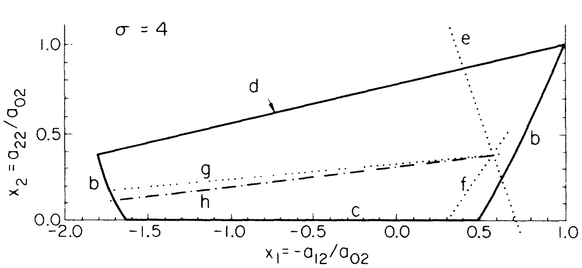

FIG. 3. An enlarged view of the hatched region from Fig. 2. Lines b, c, and d correspond to those of Fig. 3, as do e and f. The extra dotted line g is that due to Roskies, while an improved line h, shown dot-dashed, is also a consequence of the $\sigma=4$, "one-zero" method; it is derived in Sec. V, Eq. (5.38). The "best" region is that contained by lines c, f, h, and b.

region onto the (x_1, x_2) subspace for comparison with previous results.

We define $x_3 = -a_{32}/a_{02}$, and crossing symmetry for $\sigma = 5$ gives $a_{32} = a_{14} = -\frac{1}{2}a_{50}$ so that the new moments $[5]_2$ and $[5]_4$ become

$$[5]_2 = \tfrac{1}{330}(220 + 88x_1 + 20x_2 + 2x_3), \quad [5]_4 = \tfrac{1}{11}(10x_2 + x_3). \tag{3.11}$$

We obtain the best inequalities from the determinants of (2.27). For $l = 2$, $m = 0$:

$$[2]_2 - [3]_2 \geq 0, \tag{3.12a}$$

$$([2]_2 - [3]_2)([4]_2 - [5]_2) - ([3]_2 - [4]_2)^2 \geq 0, \tag{3.12b}$$

$$[3]_2 \geq 0, \tag{3.12c}$$

$$[3]_2[5]_2 - ([4]_2)^2 \geq 0. \tag{3.12d}$$

Again the chain

$$[2]_2 \geq [3]_2 \geq [4]_2 \geq [5]_2 \geq 0 \tag{3.13}$$

is obvious. For $l = 4$, $m = 0$:

$$[4]_4 - [5]_4 \geq 0, \tag{3.14}$$

$$[5]_4 \geq 0. \tag{3.15}$$

For $l = 2$, $m = 1$, and $u_1^2 = y^n$, $[k] = [k]_2 - [k+n]_4$ with $n = 0$, we have

$$[4]_2 - [4]_4 \geq 0, \tag{3.16a}$$

$$[5]_2 - [5]_4 \geq 0, \tag{3.16b}$$

and $n = -1$ gives

$$[5]_2 - [4]_4 \geq 0. \tag{3.17}$$

Again $u_1 = 1/y$ is the best approximation to u_0 permitted here, and in fact, the chains [Eqs. (3.13)-(3.15) and Eq. (3.17)] imply (3.16) as well.

In terms of the x_i, the best inequalities are then

$$\tfrac{1}{7}(1 - x_1) \geq 0, \tag{3.18a}$$

$$\frac{1 - x_1}{7}\left(\frac{165 - 88x_1 - 65x_2 - 12x_3}{1980}\right) - \left(\frac{27 - 20x_1 - 7x_2}{252}\right)^2 \geq 0, \tag{3.18b}$$

$$\tfrac{1}{7}(x_1 + 6) \geq 0, \tag{3.18c}$$

$$\frac{x_1 + 6}{7}\left(\frac{220 + 88x_1 + 20x_2 + 2x_3}{330}\right) - \left(\frac{27 + 8x_1 + x_2}{36}\right)^2 \geq 0, \tag{3.18d}$$

$$\frac{x_2 - x_3}{11} \geq 0, \tag{3.18e}$$

$$\frac{10x_2 + x_3}{11} \geq 0, \tag{3.18f}$$

$$\frac{220 + 88x_1 + 20x_2 + 2x_3}{330} - x_2 \geq 0. \tag{3.18g}$$

These may all be expressed in the form

$$U_i(x_1, x_2) \geq x_3 \geq L_i(x_1, x_2), \quad i = 1, \ldots, 7 \tag{3.19}$$

where we set $U_i(x_1, x_2) = +\infty$ [or $L_i(x_1, x_2) = -\infty$] if the inequality does not give an upper (or lower) bound to x_3.

These inequalities, corresponding to a number of different moment problems are combined through the crossing-symmetric expansion by evaluating the bounds

$$Z_{\rm up}(x_1, x_2) \geq x_3 \geq Z_{\rm dn}(x_1, x_2), \tag{3.20}$$

where

$$Z_{\rm up} = \min_{i=1,\ldots,7} U_i(x_1, x_2), \quad Z_{\rm dn} = \max_{i=1,\ldots,7} L_i(x_1, x_2). \tag{3.21}$$

These upper and lower bounds to x_3, $Z_{\rm up}(x_1, x_2)$, and $Z_{\rm dn}(x_1, x_2)$, are shown in the contour plots of Fig. 4. The boundary corresponds to points (x_1, x_2) for which $Z_{\rm up}(x_1, x_2) = Z_{\rm dn}(x_1, x_2)$; the points outside of the plotted region violate inequality (3.20). One must also realize that although $Z_{\rm up} = Z_{\rm dn}$ on the border of the plots, the "falloff" may be extremely rapid, particularly along the upper edge, and is not visible in the contours. [This falloff is even faster than the rate of change near the edge marked -0.3 in Fig. 4(a).] The resulting curve in (x_1, x_2) space is the "projection" of the three-dimensional body onto the (x_1, x_2) plane, and may be compared with Fig. 3; the major improvement over $\sigma = 4$ oc-

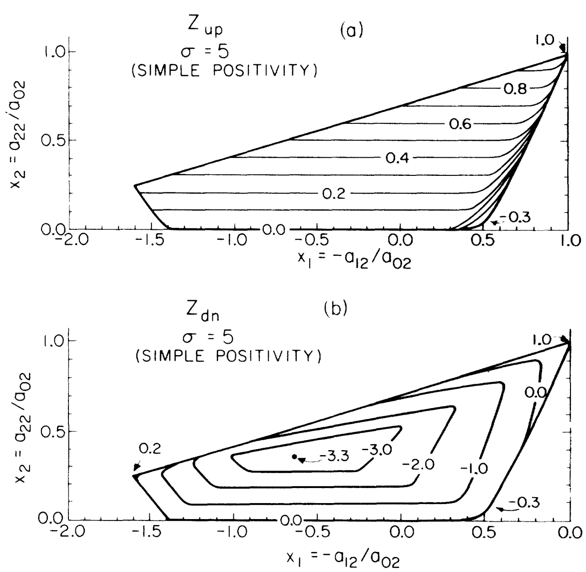

FIG. 4. (a) and (b): Contours of the upper and lower bounds to x_3, $Z_{\rm up}$, and $Z_{\rm dn}$, for the case $\sigma = 5$, following from simple positivity and crossing symmetry. In particular, notice the "projected" improvement near $x_1 = -1.5$, as compared with Fig. 3.

curs near $x_1 = -1.5$, and in a lowering of line d.

It is these "projected" improvements, a consequence of positivity in higher partial waves reflected back to lower waves by crossing symmetry, that makes it important and useful to go to as high an l as possible.

If we were to now examine $\sigma = 6$, which would necessitate an extra two parameters, x_4 and x_5, an extremely large number of extra inequalities would be found. In particular, we could use $u_1^2 = 1/y^2$ which is even closer to u_0, and we anticipate a further improvement to line d in the projection onto (x_1, x_2).[14]

We notice from Figs. 2, 3, and 4 that the point $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$ continues to be an allowed point, and does not seem to be affected by the projection from higher-dimension spaces; this follows from the fact that we are only considering positivity in its simplest form, $A(s, t) \geq 0$, and not full positivity. Examining the inequalities (2.21) and (2.22), we see that a possible solution is $s_k = s_0$ for all k. In terms of the moments $[k]_l$ that we introduced in (2.14) this means that $[k]_l = [l]_l$ for all $k \geq l$. Our choice of x_i then implies that this point corresponds to $x_i = 1$, i.e., double partial-wave amplitudes, a_{nl}, of about the same magnitude, and not dropping off rapidly as σ increases. (This behavior is found in many model calculations.[3,15]) Note, too, that even the condition $\delta^m f_l(s) \geq 0$ does not require that the a_{nl} drop off, since this decrease $f_{l+2}/f_l \to 1/u_0^2(s)$ as $s \to 1$ is simply a consequence of the expected threshold behavior $f_l(s) \sim (1-s)^l$.

The point that is missing is some "smoothness" information on the $f_l(s)$ in the interval $0 \leq s \leq 1$, related to the "smoothness" of $A(s, t)$. Using only simple positivity, the $f_l(s)$ could oscillate wildly, contributing to a_{nl} for very large σ. In fact, $A(s, t)$ is not an arbitrary positive function in the integral (2.2), but has many smoothness properties.[7-9] These can be seen in the expression (2.4), following from the positivity of $\mathrm{Im} f_l(t)$ and the smoothness of $P_l(1 + 2s/(t-1))$.[16]

We are thus led to consider relations involving derivatives of $f_l(s)$; this will be done in Sec. IV, while Roskies's conditions, incorporating this smoothness directly by use of full positivity, will be discussed in Sec. V.

IV. DERIVATIVE CONDITIONS

In this section we will derive and examine a number of inequalities that are a consequence of more than simple positivity, $A(s, t) \geq 0$. It can be shown that not only is $P_l(1+x) \geq 0$ for all l and $x \geq 0$ [this leads to positivity via the expansion (2.4)], but also all the derivatives are positive[7,16]:

$$\frac{d^n P_l(1+x)}{dx^n} \geq 0 \text{ for } x \geq 0 \text{ and } n = 0, 1, \ldots . \qquad (4.1)$$

Using the expansion (2.4), we may then show that

$$\frac{d^n A(s,t)}{ds^n} \geq 0 \text{ for } 0 \leq s \leq 1,\ t \geq 1, \text{ and } n = 0, 1, \ldots . \qquad (4.2)$$

This derivative condition is not sufficient to ensure the full positivity [$\mathrm{Im} f_l(s) \geq 0$ for all l and $t \geq 1$], although "almost" so,[17] but it may be used to derive many valuable inequalities. In particular, we are interested in relations similar to those for the $\pi^0\pi^0$ S wave[7-9]

$$\frac{df_0}{ds} = f_0'(s) \leq 0 \text{ for } 0 \leq s \leq 0.2818, \qquad (4.3)$$

$$\alpha(s) f_0'(s) + f_0(1-s) - f_0(s) \leq 0 \text{ for } 0.65 \leq s \leq 1 \qquad (4.4)$$

[where the prime denotes the derivative with respect to the argument, and $\alpha(s)$ is positive for $0.65 \leq s \leq 1$, but not of simple form], and for the D wave[8]

$$f_2'(s) \leq 0 \text{ for } 0.4 \leq s \leq 1 . \qquad (4.5)$$

The derivations of Eqs. (4.3)–(4.5) are not at all simple, and in addition to (4.2), they utilize the crossing symmetry of the full amplitude, $F(s, t)$.

We will now derive a sequence of similar conditions, using only (4.2). No crossing symmetry is used, and consequently the conditions may be somewhat weaker than the inequalities (4.3)–(4.5). They are valid for all $l \geq 2$, and amenable to the systematic approach developed in Secs. II and III for $\delta^m f_l(s) \geq 0$.

The important point that leads to $\delta^m f_l(s) \geq 0$ is the fact that we can express the $f_l(s)$ as "moments" of a positive function $B(s, w)$,[1,14]

$$f_l(s) = \int_{u_0(s)}^{\infty} dw\, w^{-l-1} B(s, w) \text{ for } 0 \leq s \leq 1 . \qquad (4.6)$$

Application of Theorem 2 (Sec. II) then leads to the complete set of inequalities $\delta^m f_l \geq 0$; any function $m_l(s)$ with this kind of positive-moment representation, may be processed to give inequalities similar to those found previously.

In order to derive (4.6), we make use of the representation for the $Q_l(z)$,[11,13,18]

$$Q_l(z) = \int_1^{\infty} \frac{db}{(b^2-1)^{1/2}} [z + (z^2-1)^{1/2} b]^{-l-1} . \qquad (4.7)$$

Defining $w = z + b(z^2-1)^{1/2}$ and $u = z + (z^2-1)^{1/2}$, this becomes

$$Q_l(z) = \int_u^{\infty} dw \frac{w^{-l-1}}{(w^2 - 2wz + 1)^{1/2}} \qquad (4.8)$$

so that (2.3) becomes

$$f_l(s)=\frac{2}{\pi}\int_{z_0}^{\infty} dz\, A(s,z)\int_u^{\infty} dw\, \frac{w^{-l-1}}{(w^2-2wz+1)^{1/2}}$$

$$=\int_{u_0}^{\infty} dw\, w^{-l-1}B(s,w), \qquad (4.9)$$

$$B(s,w)=\frac{2}{\pi}\int_{z_0}^{(w^2+1)/2w} dz\, A(s,z)/(w^2-2wz+1)^{1/2}, \qquad (4.10)$$

where

$$u_0=z_0+(z_0{}^2-1)^{1/2}=(1+s+2\sqrt{s}\,)/(1-s)\,. \qquad (4.11)$$

Equation (4.9) is the desired moment representation, to which Theorem 2 may be applied after a change of variable $w=1/\sqrt{x}$. The resulting conditions are necessary and sufficient for $B(s,w)$ to be positive, but because of (4.10), only necessary for $A(s,w)$ to be positive. [If $A(s,z)$ does not oscillate violently, the conditions are "almost" sufficient.] Replacing $u_0(s)$ by some $u_1(s)\le u_0(s)$ leads to tractable,[1] necessary conditions for $A(s,z)$ to be nonnegative.

Derivation Conditions for all $l \ge 2$: $A(s,t)$ and $dA(s,t)/ds$ Independent

We now work with a function

$$g_l(s)=\tfrac{1}{4}\pi y f_l(s)=\int_1^{\infty} A(s,t)Q_l(z)dt, \qquad (4.12)$$

and consider (recall $y=1-s$, $z=2t/y-1$)

$$g_l'(s)=\frac{dg_l(s)}{ds}=\tfrac{1}{4}\pi y f_l'(s)-\tfrac{1}{4}\pi f_l(s)$$

$$=\int_1^{\infty} A^{(1)}(s,t)Q_l(z)dt$$

$$+\int_1^{\infty} dt A(s,t)\,\frac{z+1}{y}\,\frac{dQ_l(z)}{dz}, \qquad (4.13)$$

where

$$\frac{d}{ds}=\frac{dz}{ds}\frac{d}{dz}=\frac{z+1}{y}\frac{d}{dz} \quad\text{and}\quad A^{(1)}=\frac{dA}{ds}\ge 0\,. \qquad (4.14)$$

Now

$$\frac{dQ_l(z)}{dz}=-(l+1)\int_1^{\infty}\frac{db}{(b^2-1)^{1/2}}\,w^{-l-2}\frac{dw}{dz} \qquad (4.15)$$

with $dw/dz=1+zb(z^2-1)^{-1/2}$, so that the second integral in (4.14) is clearly negative.

If we assume that $A(s,t)$ and $A^{(1)}(s,t)$ are independent positive functions [in fact they are correlated through (2.4)], we consider

$$k(s)g_l(s)+g_l'(s)=\int_1^{\infty} dt A^{(1)}(s,t)Q_l(z)$$

$$+\int_1^{\infty} dt A(s,t)\left(kQ_l+\frac{z+1}{y}\frac{dQ_l}{dz}\right) \qquad (4.16)$$

as a possible candidate for a positive-moment representation: $k(s)$ should be some function, depending on l and s, simple enough to lead to a tractable set of inequalities. In fact, all we have to show is that

$$k(s)w-(l+1)\frac{z+1}{y}\frac{dw}{dz}\ge 0$$

$$\text{for all } z\ge z_0 \text{ and } b\ge 1\,. \qquad (4.17)$$

Introducing $\bar{k}=yk/(l+1)$ and $r=(z^2-1)^{1/2}$, this is simply

$$\bar{k}(z+rb)-(z+1)(1+zb/r)\ge 0$$

$$\text{for all } b \text{ and any } z\ge z_0\,. \qquad (4.18)$$

As $b\to\infty$, we require

$$\bar{k}\ge\frac{(z+1)z}{r^2}=\frac{z(z+1)}{z^2-1}=\frac{1}{z-1}=1+\frac{1}{z-1}, \qquad (4.19)$$

which has its maximum at $z=z_0(s)$, so that we choose

$$\bar{k}\ge 1+\frac{1}{z_0-1}=\frac{z_0}{z_0-1}=\frac{1+s}{2s}\,. \qquad (4.20)$$

This turns out to be sufficient for the case $b=1$, too, since then we show that

$$\frac{(z+1)(r+z)}{r(r+z)}=\frac{z+1}{(z^2-1)^{1/2}}$$

$$=\left(\frac{z+1}{z-1}\right)^{1/2}\le\left(\frac{z^2}{(z-1)^2}\right)^{1/2}\le\bar{k}\,. \qquad (4.21)$$

So with

$$k(s)=\frac{(l+1)(1+s)}{2s(1-s)}, \qquad (4.22)$$

we obtain as moments of a positive function

$$\frac{(1+s)(l+1)}{2s(1-s)}g_l+g_l'=\frac{\pi}{4}\left(\frac{(l+1)(1+s)-2s}{2s}f_l(s)+(1-s)f_l'(s)\right)$$

$$=\int_1^{\infty} dt\int_1^{\infty}\frac{db}{(b^2-1)^{1/2}}\,w^{-l-2}\left[A^{(1)}w+\frac{(l+1)A}{1-s}\left(\frac{1+s}{2s}w-\frac{zw-1}{z+1}\right)\right]. \qquad (4.23)$$

Multiplying through by some factors, we see that

$$f_{l,1}(s) \equiv [(l+1)(1+s)-2s]f_l(s)+2s(1-s)f_l'(s) \tag{4.24}$$

may be used to generate a sequence of positive functions

$$\delta^m f_{l,1}(s)=\sum_{j=0}^{m}\binom{m}{j}(-1)^j u_0^{2j} f_{l,1}(s) \geq 0\,. \tag{4.25}$$

For example,

$$f_{l,1}(s)\geq 0\,, \tag{4.26}$$

which is very similar to the expressions (4.3)–(4.5).

In fact, for $l=2$, (4.26) becomes

$$(3+s)f_2(s)+2s(1-s)f_2'(s)\geq 0\,, \tag{4.27}$$

which unfortunately does not require $f_2'(s)$ to be negative, although it does give a rather interesting bound to a negative $f_2'(s)$.[19] The reason that this result is so weak is that we did not use any relation between $A(s,t)$ and $A^{(1)}(s,t)$; in fact, we have overestimated the "amount" of positive $g_l(s)$ that must be added to cancel the negative part of dg/ds.

Conditions Using Connection Between $A(s,t)$ and $A'(s,t)$

Since $P_l(1+x)$ may be expanded as a polynomial in x with positive coefficients,[16] it is obvious that

$$\frac{dP_l(1+x)}{dx}\geq \frac{1}{x}[P_l(1+x)-P_l(1)] \quad \text{for } x\geq 0 \text{ and all } l. \tag{4.28}$$

When combined with the expansion (2.4), this leads to the corresponding inequality

$$\frac{dA(s,t)}{ds}\geq \frac{1}{s}[A(s,t)-A(0,t)],\quad 0\leq s\leq 1,\quad t\geq 1\,. \tag{4.29}$$

Using this in the above expression (4.16), and noting that $Q_l(z)$ is a decreasing function of z (i.e., decreasing function of s for fixed t), we finally conclude that

$$\frac{1}{s}g_l(0)+\frac{(l+1)(1+s)-2(1-s)}{2s(1-s)}g_l(s)+g_l'(s) \tag{4.30}$$

has a positive-moment representation. This expression, although "better" than (4.24), involves a constant $g_l(0)=\frac{1}{4}\pi f(0)$. This constant cannot be handled usefully in our approach, because it does not use the explicit, full, partial-wave amplitude.

Analysis of Derivative Conditions in Terms of a_{nl}

We now proceed to analyze the consequences of these new expressions, using the expansion (2.5). With $x=2s-1$, the argument of the Jacobi polynomial, we obtain

$$(1-s)f_l'(s)=-lf_l(s)+h_l(s) \tag{4.31}$$

with the new function

$$\begin{aligned} h_l(s)&=\sum_{n=0}^{\infty}2(n+l+1)a_{nl}(1-s)^{l+1}\frac{d}{ds}P_n^{(\alpha,0)}(2s-1)\\ &=\sum_{n=0}^{\infty}2(n+l+1)a_{nl}(1-s)^{l}\Big[2(1-s)\frac{d}{dx}P_n^{(\alpha,0)}(x)\Big]\,. \end{aligned} \tag{4.32}$$

Using now the relation for the derivative of the Jacobi polynomial[18]

$$\begin{aligned} (2n+\alpha)(1-x^2)\,&\frac{dP_n^{(\alpha,0)}(x)}{dx}\\ &=n[\alpha-(2n+\alpha)x]P_n^{(\alpha,0)}(x)+2n(n+\alpha)P_{n-1}^{(\alpha,0)}(x), \end{aligned} \tag{4.33}$$

and the fact that

$$1-x=2(1-s),\quad 1+x=2s,\quad x=(1-2y), \tag{4.34}$$

we obtain

$$2sh_l(s)=\sum_{n=0}2(n+l+1)(1-s)^l a_{nl}\left(\frac{n}{2n+\alpha}[2y(2n+\alpha)-2n]P_n^{(\alpha,0)}(x)+\frac{2n(n+\alpha)}{2n+\alpha}P_{n-1}^{(\alpha,0)}(x)\right). \tag{4.35}$$

We then define moments of $2sh_l(s)$, analogously to (2.14):

$$\begin{aligned} [k]_{l,1}&\equiv\int_0^1 ds(1-s)^{k+1}(2sh_l)\,,\quad k\geq l\\ &=\sum_{n=0}^{k+1-l}2na_{nl}\left(D_{nl}^{k+1}+\frac{n+2l+1}{2n+nl+1}\,\frac{n+l+1}{n+l}D_{n-1,l}^{k}\right)-\sum_{n=0}^{k-l}\frac{2n^2D_{nl}^{k}a_{nl}}{2n+2l+1} \end{aligned} \tag{4.36}$$

$$\equiv\sum_{n=0}^{k+1-l}{}_1D_{nl}^{k}a_{nl}\,. \tag{4.37}$$

Values of ${}_1D^k_{nl}$ for σ up to 5 are given in Table II. From (4.31), we see then that the moments $[k]_l{}'$ of

$$2s(1-s)f_l{}'(s) = 2(1-y)yf_l{}'(s)$$
$$= -2l(1-y)f_l(s) + [2sh_l(s)] \qquad (4.38)$$

are

$$[k]_l{}' \equiv -2l([k]_l - [k+1]_l) + [k]_{l,1} \quad \text{for } k \geq l. \qquad (4.39)$$

Consequences of Our Derivative Conditions

We now examine the content of (4.25), which we have shown to be valid for all s in(0, 1). Using (4.31), $f_{l,1}(s)$ becomes

$$f_{l,1}(s) = (l+1)(1-s)f_l(s) + 2sh_l(s), \qquad (4.40)$$

leading to moments

$${}_1[k]_l = (l+1)[k+1]_l + [k]_{l,1} \text{ for } k \geq l \qquad (4.41)$$

which obey the usual determinantal inequalities: (for $l=2$, $\sigma=4$)

$${}_1[3]_2 = \tfrac{1}{36}(81 - 32x_1 - 13x_2) \geq 0, \qquad (4.42a)$$

$${}_1[2]_2 - {}_1[3]_2 = \tfrac{1}{252}(81 - 172x_1 + 91x_2) \geq 0. \qquad (4.42b)$$

These lines are not shown, as the improvement is imperceptible; (4.42b) does produce a slight improvement in the region $x_1 \approx 1$.

For $\sigma=5$, we have the additional moments (for $l=2$ and $l=4$)

$${}_1[4]_2 = 3[5]_2 + [4]_{2,1}$$
$$= 2 - \tfrac{4}{9}x_1 - \tfrac{46}{99}x_2 - \tfrac{1}{11}x_3, \qquad (4.43a)$$

$${}_1[4]_4 = 5[5]_4 + [4]_{4,1}$$
$$= \tfrac{1}{11}(50x_2 - 17x_3), \qquad (4.43b)$$

which give rise to the set of inequalities for $l=2$ and $m=0$:

$${}_1[2]_2 = \tfrac{1}{7}(18 - 11x_1) \geq 0, \qquad (4.44a)$$

$${}_1[2]_2\,{}_1[4]_2 - ({}_1[3]_2)^2 = \tfrac{2}{7}(18 - 11x_1) - \tfrac{4}{9}x_1 - \tfrac{46}{99}x_2 - \tfrac{1}{1296}(81 - 32x_1 - 13x_2)^2 \geq 0, \qquad (4.44b)$$

$${}_1[3]_2 - {}_1[4]_2 = \tfrac{9}{36} - \tfrac{16}{36}x_1 + \tfrac{41}{396}x_2 + \tfrac{1}{11}x_3 \geq 0. \qquad (4.44c)$$

For $l=4$ and $m=0$,

$${}_1[4]_4 = \tfrac{1}{11}(50x_2 - 17x_3) \geq 0, \qquad (4.45)$$

while $l=2$, $m=1$, and $u_1=1$ give

$${}_1[4]_2 - {}_1[4]_4 = 2 - \tfrac{4}{9}x_1 - \tfrac{49}{99}x_2 + \tfrac{16}{11}x_3 \geq 0. \qquad (4.46)$$

If we now combine these derivative inequalities, with those found previously for $\sigma=5$ [Eq. (3.18)], and reevaluate the bounds on x_3, Z_{up}, and Z_{dn}, we discover a marked improvement shown in the con-

TABLE II. Values of ${}_1D^k_{nl}$ for σ up to 5.

$l=0$

$n \backslash k$	0	1	2	3	4
0	0	0	0	0	0
1	2	$\frac{6}{5}$	$\frac{4}{5}$	$\frac{4}{7}$	$\frac{3}{7}$
2	0	$-\frac{4}{5}$	$-\frac{64}{70}$	$-\frac{6}{7}$	$-\frac{16}{21}$
3	0	0	$\frac{2}{7}$	$\frac{10}{21}$	$\frac{4}{7}$
4	0	0	0	$-\frac{2}{21}$	$-\frac{16}{77}$
5	0	0	0	0	$\frac{1}{33}$

$l=2$

$n \backslash k$	2	3	4
0	0	0	0
1	2	$\frac{14}{9}$	$\frac{112}{90}$
2	0	$-\frac{4}{9}$	$-\frac{64}{99}$
3	0	0	$\frac{12}{110}$

$l=4$

$n \backslash k$	4
0	0
1	2

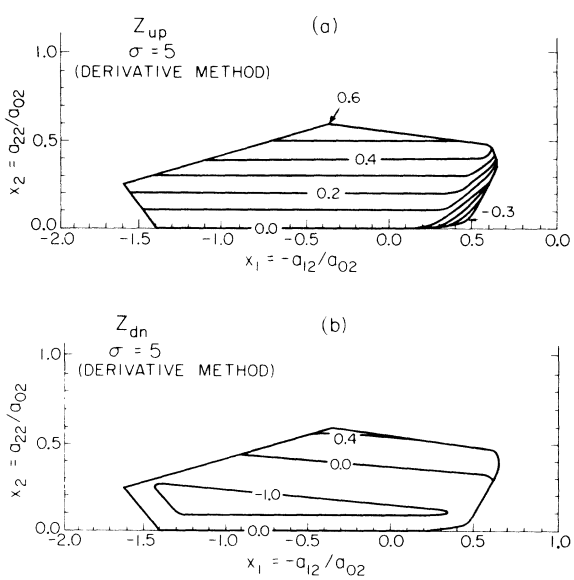

FIG. 5. (a) and (b): Contours of the upper and lower bounds to x_3, Z_{up}, and Z_{dn}, for $\sigma=5$, using simple positivity augmented with the derivative conditions of Eq. (3.18). Compare the improvement in Z_{dn} and the "projected" improvement with Fig. 4 and Fig. 3, respectively.

tours of Fig. 5. In particular, the (x_1, x_2) projection is improved near the vertex at $x_1 = 1$, $x_2 = 1$, although not as much as by Roskies's lines, and the lower bound Z_{dn} is considerably improved.

V. ROSKIES'S ONE-ZERO METHOD AND EXTENSIONS

We are able to derive some additional constraints, following from more than just simple positivity $A(s, t) \geq 0$, by using an alternative approach, formulated by Piguet and Wanders[10] and by Roskies.[2] In order to obtain stronger inequalites using this method, we will have to satisfy certain subsidiary conditions related to the positivity of certain "test" functions. This will prove to be a disadvantage, hindering a simple application of the method to higher values of σ. Nevertheless, use of the moment approach developed in the previous sections will enable an organized, systematic analysis of these new conditions, and allow us to combine them simply with our previous results.

With (2.4) reexpressed in the form

$$A(s, t) = \tfrac{1}{4}\pi \sum_l \beta_l(t) P_l(x), \qquad x = x(s, t) = \frac{2s}{t-1} + 1, \tag{5.1}$$

where $\beta_l(t) \geq 0$ for $t \geq 1$ from t-channel unitarity (this is "full" positivity), and the definitions of $f_l(s)$ and a_{nl} given by (2.2) and (2.5), we express

$$a_{nl} = \int_1^\infty dt \sum_{l'} \beta_{l'}(t) \int_0^1 ds (1-s)^l Q_l\left(\frac{2t}{1-s} - 1\right) \times P_n^{(\alpha,0)}(2s-1) P_{l'}(x) \tag{5.2}$$

(where $\alpha = 2l+1$).

We define $[\eta]$ for an arbitrary set of numbers $\{\eta_{nl}\}$ by

$$[\eta] = \sum_{n,l} \eta_{nl} a_{nl} \,. \tag{5.3}$$

In particular, our moments $[k]_l$ are given with $\eta_{nl} = D_{nl}^k$ for fixed l and k by

$$[k]_l = \int_1^\infty dt \sum_{l'} \beta_{l'}(t) \int_0^1 ds\, y^k Q_l(z) P_{l'}(x) \,. \tag{5.4}$$

We now define the functions "generated" by the set η_{nl},

$$B_\eta^{l'}(t) = \sum_{nl} \eta_{nl} \int_0^1 ds (1-s)^l Q_l(z)\, P_n^{(\alpha,0)}(2s-1) P_{l'}(x) \,. \tag{5.5}$$

The method then proceeds by observing that (5.2) and (5.3) may be considered as a generalized-moment problem, where the function that is to be positive is $\beta_l(t)$ and the moments are a_{nl}. A necessary, and sufficient condition for $\beta_l(t)$ to be positive for all $t \geq 1$ and all even l,[10,12] is that $[\eta] \geq 0$ for all $B_\eta^{l'}(t) \geq 0$ – i.e., all possible sets of $\{\eta_{nl}\}$ are to be found that give positive $B_\eta^{l'}(t)$. The resulting inequalities are $[\eta] \geq 0$.

If $\eta_{nl} = D_{nl}^k$ which generates $[k]_l$, then the corresponding function

$$B_{kl}^{l'}(t) = \int_0^1 ds (1-s)^k Q_l(z) P_{l'}(x) \tag{5.6}$$

is obviously positive, since $z \geq z_0 \geq 1$, and so $Q_l(z) \geq 0$; likewise for the $P_l(x)$. This then implies that $[k]_l \geq 0$; certain combinations of the $Q_l(z)$, given by [20]

$$\delta^m Q_l(z) = \sum_{j=0}^m \binom{m}{j} (-1)^j u_1^{2j} Q_{l+2j}(z) \tag{5.7}$$

for any $u_1 \leq u_0(s) = z_0 + (z_0^2 - 1)^{1/2}$, are positive, too. In fact, for any polynomial, $P(s)$, positive in (0,1), the corresponding $B_{Pl}^{l'}(t)$ is positive;

$$B_{Pl}^{l'}(t) = \int_0^1 P(s)(1-s)^l Q_l(z) P_{l'}(x) ds \geq 0 \,. \tag{5.8}$$

Expanding $P(s)$ in terms of $y = 1-s$, $P(s) = \sum_k P_k y^k$, leads immediately to $\sum_k P_k [k]_l \geq 0$. The best inequalities are then given by Theorems 1 or 2, exactly as in Secs. II, III, and IV.

Roskies[2] observed that it is not necessary for the argument of the integral in (5.8) to be positive, to ensure that $B_{Pl}^{l'}(t)$ be positive for all l and $t \geq 1$. The product $Q_l(z) P_{l'}(x)$ tends to emphasize values of s near 1, and so the polynomial $P(s)$ may be negative in part of the interval, provided it is positive near $s = 1$. The exact statement of these properties is given in the following theorems, somewhat modified from those given by Roskies, and the proofs follow.

Lemma. If $g(s)$ is an arbitrary function, and $I(s)$ an increasing function, with $I(s) = \int_0^s i(s')ds' + I(0)$, where $i(s)$ and $I(0)$ are non-negative, define the function $G(s) = \int_0^s g(s')ds'$, $G(0) = 0$. Then the integral

$$\begin{aligned} J &= \int_0^1 g(s) I(s) ds \\ &= [G(s)I(s)]_0^1 - \int_0^1 G(s) i(s)\, ds \\ &= G(1)I(1) - \int_0^1 G(s) i(s)\, ds \\ &\geq [G(1) - \max G] I(1) + \max G I(0) \\ &\geq 0 \text{ if the maximum of } G \text{ on } (0,1) \text{ occurs at } s = 1 \,. \end{aligned}$$

We then notice that if $g(s)$ has but one zero in (0, 1) and $g(1) \geq 0$, then the maximum of $G(s)$ occurs either at 0 or at 1; simply test $G(1) \geq G(0) = 0$. This leads immediately to Roskies's theorems.[2]

Theorem 3. If $h(s, t)$ has but one zero in $0 \leq s \leq 1$, and $h(1^-, t) \geq 0$ for all $t \geq 1$ and if $\int_0^1 h(s,t)ds \geq 0$ for all $t \geq 1$, then $\int_0^1 h(s,t)P_l(1+2s/(t-1))\,ds \geq 0$ for all l and $t \geq 1$.

Proof. $P_l(1) \geq 0$ and $P_l{}'(1+x) \geq 0$; identify with $I(s)$. Similarly identify $h(s,t)$ as $g(s)$. Then $G(1) = \int_0^1 h(s,t)ds \geq 0$ is the condition of the above lemma.

Theorem 4. If $k(s)$ has but one zero in $0 \leq s \leq 1$, and $k(1^-) \geq 0$, and if $\int_0^1 k(s)Q_l(z_0)ds \geq 0$, then $\int_0^1 k(s)Q_l(z)dz \geq 0$ for all $t \geq 1$ ($z = 2t/y - 1$ and $z_0 = 2/y - 1$).

Proof. Roskies has shown that $Q_l(z)/Q_l(z_0)$ is an increasing function of s for $t \geq 1$;[2] identify this with $I(s)$, while $k(s)Q_l(z_0)$ is then $g(s)$. The condition is that of the above lemma.

The lemma and above proofs indicate that we may, in fact, consider functions with more than one zero. Possible extrema of $G(s)$ are at the zeros of $g(s)$, and at $s=0$, $s=1$; some are minima, but the others are to be checked; the condition is then $\int_{s_i}^1 g(s)ds \geq 0$ for all s_i that are maxima of G. [Some resulting "two-zero" conditions have been investigated by the author, but the resulting inequalities, similar to (4.42), are too slack to justify the extra work required.]

The problem of choosing a set of η_{nl} that will give a polynomial with a zero in it, yet still ensure that the corresponding $B^{l'}_{Pl}(t)$ is positive, is now most easily expressed in terms of our moments. The content of the condition, as well as of the resulting inequalities, becomes extremely transparent.

We choose a polynomial $R(s)$ that automatically satisfies the "one-zero, positive near $s=1$" condition of Theorems 3 and 4 as

$$R(s) = (a-y)P(s), \tag{5.9}$$

where $P(s)$ is any positive polynomial, and $0 \leq a \leq 1$ to ensure that there is a zero between $0 \leq s \leq 1$. The function that must be positive is then $B^{l'}_{Pl}(t)$; by setting

$$h(s,t) = R(s)y^l Q_l(z) \tag{5.10}$$

(the extra factor of y^l ensures that only a finite number of a_{nl} enter), Theorems 3 and 4 assure us that the inequality

$$[R(s)]_l \equiv \int_1^\infty dt \sum_{l'} \beta_{l'}(t) \int_0^1 ds\, y^l R(s) Q_l(z) \times P_{l'}(1+2s/t-1) \geq 0 \tag{5.11}$$

is true provided

$$\langle R(s)\rangle_l \equiv \int_0^1 ds\, y^l R(s) Q_l(z_0) \geq 0\,. \tag{5.12}$$

(Note that $\langle\ \rangle_l$ is an average over a known function, while $[\ ]_l$ is essentially a functional with unknown weight function $\beta_{l'}(t)$.) The linear character of both operations $\langle\ \rangle$ and $[\ ]$ then shows that

$$a[P(s)]_l \geq [yP(s)]_l \tag{5.11$'$}$$

provided

$$a\langle P(s)\rangle_l \geq \langle yP(s)\rangle_l\,. \tag{5.12$'$}$$

The expansion of $P(s)$ as $\sum y^i P_i$ shows the connection with our moments $[k]_l \equiv [y^{k-l}]_l$,

$$[P(s)]_l = \sum_i P_i[i+l]_l\,. \tag{5.11$''$}$$

The average $\langle\ \rangle_l$ is easily calculated in terms of the numbers

$$A_l^p \equiv \int_0^1 ds\, s^p y^l Q_l(z_0), \tag{5.13}$$

where for $l=2$, the expression is extremely simple:

$$A_2^p = 2/[(p+1)(p+2)(p+3)]^2\,. \tag{5.14}$$

The resulting averages $\langle y^n\rangle_l$ for $l=2$ and 4 are given in Table III.

The simplest polynomials lead to lines found by Roskies[2] previously. If $P(s)=1$, $R(s)=(a-y)$ which leads to the inequality

$$a[y^0]_l - [y^1]_l = a[l]_l - [l+1]_l \geq 0 \tag{5.15}$$

and the condition

$$a\langle y^0\rangle_l \geq \langle y^1\rangle_l\,. \tag{5.16}$$

Inequality (5.15) is clearly an improvement over the "chain" that we had previously, and we clearly want a as close to 0 as possible; this is given by

$$a = \langle y^1\rangle_2/\langle y^0\rangle_2 = \tfrac{15}{16} \tag{5.17}$$

using Table III. The inequality $a[2]_2 \geq [3]_2$ is then

$$\tfrac{15}{16} \geq \tfrac{1}{7}(6+x_1), \quad \text{i.e.,} \quad x_1 \leq \tfrac{9}{16}\,. \tag{5.18}$$

This is a considerable improvement over $x_1 \leq 1$, found from $[2]_2 \geq [3]_2$.

If $P(s)=y$, and we determine a by

$$a = \langle y^2\rangle_2/\langle y^1\rangle_2 = \tfrac{118}{125} = 0.944\,, \tag{5.19}$$

then the inequality

$$a[3]_2 - [4]_2 = 0.944\tfrac{1}{7}(6+x_1) - \tfrac{1}{36}(27+8x_1+x_2) \geq 0 \tag{5.20}$$

leads to line e in Figs. 2 and 3, a clear improvement over line a, which corresponds to $[3]_2 \geq [4]_2$.

If $P(s)=s=1-y$, and we determine a by

$$a = \frac{\langle y\rangle_2 - \langle y^2\rangle_2}{\langle y^0\rangle_2 - \langle y\rangle_2} = 0.840\,, \tag{5.21}$$

then the inequality

$$a([2]_2 - [3]_2) - ([3]_2 - [4]_2) = 0.120(1-x_1) - \tfrac{1}{252}(27-20x_1-7x_2) \geq 0\,, \tag{5.22}$$

TABLE III. The averages $\langle y^n \rangle_l$ for $\sigma = 5$.

$l=2$	
$\langle y^0 \rangle_2 = \frac{1}{18}$	$\langle y^0 \rangle_2 - \langle y^1 \rangle_2 = \frac{1}{288}$
$\langle y^1 \rangle_2 = \frac{5}{96}$	$\langle y^1 \rangle_2 - \langle y^2 \rangle_2 = \frac{7}{2400}$
$\langle y^2 \rangle_2 = \frac{59}{1200}$	
$\langle y^3 \rangle_2 = \frac{14}{300}$	
$l=4$	
$\langle y^0 \rangle_4 = \frac{1}{50}$	$\langle y^0 \rangle_4 - \langle y^1 \rangle_4 = \frac{1}{1800}$
$\langle y^1 \rangle_4 = \frac{35}{1800}$	

shown as line f in Figs. 2 and 3, improves on line b, which arose from the determinant (3.10b).

Now the most general positive polynomial, of order one, is a combination of the previous two

$$P_\lambda(s) = (1-\lambda)y + \lambda(1-y) = \lambda + (1-2\lambda)y, \tag{5.23}$$

where $0 \leqslant \lambda \leqslant 1$ and an over-all constant has been divided out. Using this expression leads to an $a = a(\lambda)$; the inequalities depend on λ as well, and the problem of deciding which value of λ gives the "best" inequality is the major disadvantage of this method. As polynomials of higher order are considered, there are more parameters to vary, and the condition (5.12) depends rather critically on them. Nevertheless, we can go ahead and pick some polynomial; satisfying the subsidiary condition is easy, and we then have a necessary inequality, possibly redundant. It is the question of redundancy and "best" inequalities that complicate this method. It appears, in fact, that the best inequalities occur for $\lambda = 0$ and $\lambda = 1$, but for higher orders there still remain parameters that have to be varied. (See the discussion in Appendix A.) For the case $\sigma = 5$, not considered by Roskies, the most general positive polynomial that involves y^2 [so that $yP(s) \to x_3$] is

$$P(s) = (1-\lambda)y(1-y) + \lambda(y+\alpha)^2, \tag{5.24}$$

where α is an arbitrary real parameter and $0 \leqslant \lambda \leqslant 1$ as before. It appears again that $\lambda = 0$ and $\lambda = 1$ give the best inequalities, but the optimum value of α, around -0.5, will in fact depend on the values of x_1 and x_2. The resulting inequalities are combined with others in the calculation of Z_{up} and Z_{dn}, displayed in Fig. 6, and no simple form may be given to the optimum inequalities; for different values of x_1 and x_2 over the plot, different inequalities become the limiting ones [different values of α in (5.24)].

For $l = 4$, the first inequality corresponds to $P(s) = 1$, contributing to $\sigma = 5$. We fix

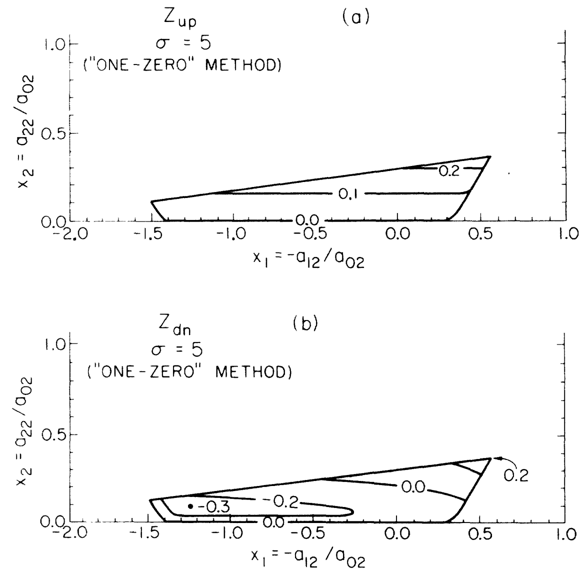

FIG. 6. (a) and (b): Contours of the upper and lower bounds to x_3, Z_{up}, and Z_{dn}, for $\sigma = 5$, using the "one-zero" method of Roskies (Sec. V) in addition to the simple positivity of Sec. III.

$$a = \langle y^1 \rangle_4 / \langle y^0 \rangle_4 = \tfrac{35}{36}, \tag{5.25}$$

and the inequality becomes

$$a[4]_4 - [5]_4 = \tfrac{35}{36} x_2 - \tfrac{1}{11}(10x_2 + x_3) = \tfrac{1}{396}(25x_2 - 36x_3) \geqslant 0, \tag{5.26}$$

which improves on $[4]_4 - [5]_4 \geqslant 0$; this is the single major improvement in the size of Z_{up} and Z_{dn}, shown in Fig. 6, as compared with the "no-zero" case in Fig. 4.

We will now derive line g, which is Roskies's improvement to our inequality $[4]_2 - [4]_4 \geqslant 0$, line d in Fig. 3. Our derivation will be simpler than that given by Roskies, and in fact we discover that g is not the optimum that may be obtained, even for the case $\sigma = 4$, considered by Roskies.

We consider the function

$$R(s,t) = P(s) y^l [Q_l(z) - (1/a) u_1^2 Q_{l+2}(z)], \tag{5.27}$$

where $P(s)$ is positive on $0 \leqslant s \leqslant 1$. If $u_1(s) \leqslant u_0(s) = z_0 + (z_0^2 - 1)^{1/2}$, and $a = 1$, then $R(s,t) \geqslant 0$, so that

$$[R] = \int_0^1 dt \sum_{l'} \beta_{l'}(t) \int_0^1 ds\, R(s,t) P_{l'}\left(1 + \frac{2s}{t-1}\right) \geqslant 0, \tag{5.28}$$

which leads to our inequalities using $\delta^m f_l(s) \geqslant 0$ with $m = 1$. If $a \leqslant 1$, $R(s,t)$ will have a zero in $(0 \leqslant s \leqslant 1)$, and to ensure $[R] \geqslant 0$, we must ensure that

$$B_R^{l'}(t) = \int_0^1 ds\, R(s,t) P_{l'}\left(1 + \frac{2s}{t-1}\right) \geqslant 0 \tag{5.5'}$$

for all $t \geq 1$ and all l'. [See discussion after Eq. (5.5).]

Now

$$R(s, t) = P(s) y^l Q_l(z)\left(1 - \frac{u_1^2 Q_{l+2}(z)}{a Q_l(z)}\right) \tag{5.29a}$$

$$\geq P(s) y^l Q_l(z)\left(1 - \frac{u_1^2 Q_{l+2}(z_0)}{a Q_l(z_0)}\right) \tag{5.29b}$$

because[7]

$$\frac{Q_{l+n}(z)}{Q_l(z)} \leq \frac{Q_{l+n}(z_0)}{Q_l(z_0)} \tag{5.30}$$

for $z \geq z_0$ and $n \geq 0$.

So now

$$B_R^{l'}(t) \geq \int_0^1 ds\, r(s, t) P_{l'}\left(1 + \frac{2s}{t-1}\right), \tag{5.31}$$

where

$$r(s, t) = P(s) y^l \frac{Q_l(z)}{Q_l(z_0)}\left(Q_l(z_0) - \frac{u_1^2}{a} Q_{l+2}(z_0)\right). \tag{5.29$'$}$$

It is clear that $r(s, t)$ has no more than one zero in $0 \leq s \leq 1$, for any t, and because[18]

$$u_1^2 \frac{Q_{l+2}(z)}{Q_l(z)} \to (u_1 y)^2 \left(\frac{(l+2)!}{l!}\right)^2 \frac{(2l+1)!}{(2l+5)!}$$
$$= y^2 G_l \to 0 \text{ as } s \to 1,$$

we see that $r(s, t)$ is non-negative near $s = 1$ (provided that $u_1^2 y^2 \to 0$). We may then apply Theorems 3 and 4 to see that the integral in (5.31) is non-negative, provided that the condition

$$\langle r \rangle = \int_0^1 ds\, P(s) y^l [Q_l(z_0) - (u_1^2/a) Q_{l+2}(z_0)] \geq 0 \tag{5.32}$$

is ensured. This implies that $B_R^{l'}(t) \geq 0$, as required, and we obtain the inequality

$$a[P(s)]_l - [(u_1/y)^2 P(s)]_{l+2} \geq 0, \tag{5.33}$$

with the subsidiary condition

$$a\langle P(s)\rangle_l - \langle (u_1/y)^2 P(s)\rangle_{l+2} \geq 0. \tag{5.34}$$

Of course, $(u_1/y)^2 P(s)$ must be a polynomial if only a finite number of a_{nl} are to enter.

If we put $P(s) = y$ and $u_1^2 = y$, we obtain for $l = 2$,

$$a[3]_2 - [4]_4 \geq 0 \tag{5.35}$$

and

$$a \geq \langle y^0\rangle_4 / \langle y^1\rangle_2 = \tfrac{48}{125}. \tag{5.36}$$

This is a clear improvement on $[3]_2 - [4]_4 \geq 0$, and is in fact line g, found by Roskies.[2] Our method now makes it obvious that we should also examine $P(s) = y^2$ and $u_1^2 = 1$, since $[3]_2 \geq [4]_2$:

$$a[4]_2 - [4]_4 \geq 0,$$
$$a \geq \langle y^0\rangle_4 / \langle y^0\rangle_2 = \tfrac{24}{59} = \tfrac{48}{118}, \tag{5.37}$$

which is an improvement on our line d, in Fig. 3, corresponding to $[4]_2 - [4]_4 \geq 0$. This line

$$\tfrac{24}{59}\, \tfrac{1}{36}(27 + 8x_1 + x_2) - x_2 = \tfrac{1}{177}(54 + 16x_1 - 175x_2)$$
$$= \tfrac{1}{885}(270 + 80x_1 - 875x_2) \geq 0, \tag{5.38}$$

shown as h in Fig. 3, is also a clear improvement over line g due to Roskies, and also uses only $\sigma = 4$.

For $\sigma = 5$, we have in addition to (5.26) the inequality with $P(s) = y^3$, $u_1^2 = 1/y$ in (5.33):

$$a[5]_2 - [4]_4 \geq 0 \tag{5.39}$$

and

$$a \geq \langle y^0\rangle_4 / \langle y^3\rangle_2 = \tfrac{48}{109}. \tag{5.40}$$

Finally, $P(s) = y^2(1-y)$ and $u_1^2 = 1$ give

$$a([4]_2 - [5]_2) \geq ([4]_4 - [5]_4), \tag{5.41}$$

with

$$a \geq \frac{\langle y^0\rangle_4 - \langle y^1\rangle_4}{\langle y^2\rangle_2 - \langle y^3\rangle_2} = \tfrac{2}{9}. \tag{5.42}$$

With these extra "one-zero" inequalities, the improvement in $Z_{\rm up}$ and $Z_{\rm dn}$ is dramatic: Compare Figs. 4 and 6. Not only is the projection onto (x_1, x_2) improved (in fact, it is now superior to the original region found by Roskies,[2] shown in Fig. 3), but also the bounds on x_3, $Z_{\rm up}$, and $Z_{\rm dn}$ are tighter. (We may slightly improve the "one-zero" bounds of Fig. 6, near $x_1 = 0.3$, $x_2 = 0$, by including the derivative constraints of Fig. 5.)

VI. CONCLUSIONS AND APPLICATIONS

We have seen that the moment approach provides a very simple and practical method of combining the positivity and analyticity properties of the different partial waves with the constraint of crossing symmetry. It permits the evaluation, in a graphic way, of the content of various proposed relations between partial waves and their derivatives, and of the effect of crossing symmetry on relations that at first sight appear rather weak [for example, Eq. (4.25)]. Also we can see how important it is to consider the effect of the positivity of higher partial waves; this will be reflected, through crossing symmetry, as restrictions on the lower waves which are accessible in a particular investigation.

It is important to notice the clear separation in our method between the inequalities derived from positivity and analyticity, and the connections established by use of crossing symmetry, that "tighten" the constraints. The defect, of course, is that we are restricted to studying only a finite number of a_{nl}'s ($n+l$ up to σ) and cannot derive relations that directly incorporate positivity and crossing symmetry on the partial waves (and their derivatives) themselves, as has been done for some of the lower waves.[7-9] However, in deriving these S-, P-, and D-wave inequalities,[7-9] certain compromises have been made in order to get reasonable conditions, and so these conditions may not lead to very tight constraints when analyzed in terms of our moments. By combining the resulting moment inequalities, with our inequalities found above, we may be able to obtain even better bounds than those of Fig. 6.

For any application that goes only to some maximum order in s and t, we can in fact use our graphic approach to see where in the parameter space our model "lies." By analyzing all the available inequalities in terms of our moments, we can then decide which are likely to be the limiting ones.

As we go to higher values of l and consider the positivity of these higher partial waves, many more parameters x_i (or c_p^σ) will enter, and it will become prohibitive to evaluate "contours" in these higher-dimensional spaces; rather we must use the inequalities with a particular application in mind. The numbers D_{nl}^k and the eigenvectors $\overline{E}^\sigma(k)$ (Sec. II) are very simply generated numerically by recursion relations, while the a_{nl} may be found for a given amplitude with reasonable accuracy by Gauss-point integration. We project out the appropriate c_p^σ using Eq. (2.13), and simply test whether the appropriate set of inequalities is satisfied and in what range the "last" variable is permitted to lie.

There are essentially two classes of application where we feel that these inequalities may be of interest or of use in determining parameters, and we will briefly comment on them, although full results will be described elsewhere.

In the first class, we are given an explicitly crossing-symmetric amplitude, and some form of "approximate unitarity" is enforced on the resulting partial waves; in the second, explicitly unitary parametrizations are used for certain partial waves (the lowest, usually), and an approximate form of crossing symmetry enforced.

Explicitly Crossing-Symmetric Amplitudes

We are given an amplitude, $F(s,t)$, that is crossing-symmetric, $F(s,t)=F(t,s)$. It will enable us to check the accuracy of the numerical method, since the a_{nl} must satisfy the crossing relations, Eq. (2.6). We then compare various methods of determining the free parameters; in particular, to see how restrictive our inequalities are. We have in mind a "unitarization" of a current-algebra-type amplitude, given by Iliopoulos,[21] and similar studies by Dilley,[22] and by Arbab and Donahue.[23]

In these models, the π-π amplitude (a brief discussion of the effects of isospin is given in Appendix B) is expanded in a crossing-symmetric way in terms of the variables $k_s=0.5(4\mu^2-s)^{1/2}$, $k_t=0.5(4\mu^2-t)^{1/2}$, and $k_u=0.5(4\mu^2-u)^{1/2}$; this ensures the existence of cuts compatible with unitarity, and enables us to examine the partial waves a short distance out of the triangle and compare with the physical phase shifts. Approximate unitarity is enforced by equating coefficients of $(k_s)^n$ to some order, in the relation $\text{Im} f_0(s)=\rho(s)|f_0(s)|^2$,[21] or by satisfying unitarity on a set of energies just above threshold.[22,23] Some of the remaining parameters are determined by physical input, such as current algebra,[21] and the use of the S- and P-wave inequalities.[7-9] There nevertheless remains some freedom, and a number of "solutions" appear. By applying our inequalities, we are able to reduce some of the ambiguity.

The amplitude $A(s,t,u)$ of the π-π problem with isospin [Eq. (B1)] is parametrized as

$$A(s,t,u)=C_0+C_1\frac{k_s}{2\mu}+C_1'\frac{k_u+k_t}{2\mu}+C_2\frac{k_s^2}{(2\mu)^2}+C_3\frac{k_s^3}{(2\mu)^2}+C_3'\frac{k_u^3+k_t^3}{(2\mu)^3}+\cdots. \tag{6.1}$$

The π^0-π^0 amplitude is then

$$F(s,t,u)=\overline{C}_0+\overline{C}_1\frac{k_s+k_t+k_u}{2\mu}+\overline{C}_3\frac{k_s^3+k_t^3+k_u^3}{(2\mu)^3}+\cdots, \tag{6.2}$$

where

$$\overline{C}_0=3C_0+\tfrac{1}{2}C_2;\quad \overline{C}_1=C_1+2C_1',\quad\text{and}\quad \overline{C}_3=C_3+2C_3'. \tag{6.3}$$

We numerically obtain the a_{nl} as

$$\begin{aligned}
a_{02}&=10^{-5}(-211.6\overline{C}_1+72.2\overline{C}_3),\\
a_{12}&=10^{-5}(43.3\overline{C}_1-4.16\overline{C}_3),\\
a_{22}&=10^{-5}(-13.3\overline{C}_1+0.666\overline{C}_3),\\
a_{32}&=10^{-5}(5.15\overline{C}_1-0.163\overline{C}_3).
\end{aligned} \tag{6.4}$$

In terms of the solutions (a), (b), (c), and (d) given by Iliopoulos,[21] we evaluate C_i, C_i', $\overline{C}_1$, and $\overline{C}_3$ in Table IV, along with the corresponding x_i, and the bounds Z_{up} and Z_{dn} that are given for these x_1 and x_2 in Fig. 6. We see that solution (b) is in fact eliminated by our inequalities, while the others are satisfactory, but quite close to a boundary.

TABLE IV. Parameters for Iliopoulos model, Eqs. (6.1) and (6.2).

Solution[a]	C_0	C_1	C_1'	C_2	C_3	C_3'	$\overline{C}_1$	$\overline{C}_3$	x_1	x_2	x_3	$Z_{\rm up}$	$Z_{\rm dn}$
a	8.4	−0.28	−0.28	−56.0	32.0	2.88	−0.84	37.7	0.066	0.013	0.003	0.008	−0.125
b	68.5	−284	−0.064	554.0	−512	5.9	−284	−500	0.425	0.143	0.057	0.039	−0.036[b]
c	842.0	312	−526	−2868	2248	−1484	−740	−720	0.277	0.089	0.035	0.062	−0.079
d	806.0	194	−520	−2160	1344	−1240	−846	−1136	0.33	0.108	0.043	0.071	−0.053

[a]See Ref. 21.

[b]This violates $Z_{\rm up} \geq x_3 \geq Z_{\rm dn}$.

In fact, the limiting inequality that removes case (b) is that given by our derivative condition Eq. (4.45). Iliopoulos[21] was also able to eliminate this solution with the help of one of the S-wave derivative conditions. Possible improvements in the inequalities, obtained by going to higher σ or tighter derivative constraints, might eliminate the other unwanted solutions (c) and (d). Solution (a), corresponding to Weinberg's solution,[24] lies closer to the point $x_i = (0, 0, 0)$ and is less likely to be eliminated.

So far we have ignored isospin, and our $\pi^0\pi^0$ inequalities restrict only the $\overline{C}_i$ combination of the original C_i and C_i' parameters. Roskies has extended his parametrization of crossing-symmetric amplitudes to include the case with isospin as well[25]; in Appendix B we show that a further $\{\sigma/3\}$ parameters, e_p^σ, are required to fully describe the amplitude $A(s, t, u)$ [Eq. (6.1)]. By considering the positivity properties of the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude, we derive a further set of inequalities. As shown in Appendix B, these new inequalities are very restrictive for the Iliopoulos model,[21] and we are able to eliminate the solutions (b), (c), and (d) at the level of $\sigma = 4$, without having to resort[10] to the derivative or "one-zero" inequalities of Secs. IV and V.

One of the calculations performed by Dilley[22] considers (6.2) as the amplitude for scalar, neutral pions; following Iliopoulos,[21] he enforces S-wave unitarity at threshold, obtaining two relations between the $\overline{C}_i$, which enable us to express the $\overline{C}_i$ in terms of one variable, α:

$$\begin{aligned} \overline{C}_1 &= -2\alpha^2, \\ \overline{C}_3 &= 8\alpha^2(\alpha^2 - \alpha - 0.5)/(1 - 3\alpha), \qquad (6.5) \\ \overline{C}_0 &= \alpha - \overline{C}_1 - \overline{C}_3/4. \end{aligned}$$

When one evaluates the deviation from S-wave unitarity on a set of points just above threshold (up to $s = 5$), one discovers that unitarity is satisfied "best" for two regions[22]: α about −0.5, and a somewhat less favorable solution at α about 1.0. This shows that S-wave unitarity at threshold is not enough to determine the amplitude entirely. The question then arises whether our inequalities, valid below threshold, are sufficiently sensitive to "detect" gross violations of unitarity above threshold. [The sensitivity of course must be related to the analytic structure of the amplitude, and perhaps we should not use (6.2) too far above threshold.]

We now examine the effect of our inequalities, which are simply obtained from (6.5) and the a_{nl} given in Eq. (6.4). We discover that the range of α from about 0.1 to about 0.4, as well as the neighborhood of $\alpha = 3.0$, are forbidden by our inequalities. This restriction is not as strong as that due to unitarity above threshold, but it does indicate that there is some useful constraint in these inequalities, even for so low a value of σ.

Explicitly Unitary Partial Waves

An alternative type of application that has received much attention recently[26] is the direct parametrization of some of the partial waves for l up to $l_{\rm max}$ (usually only the S, P, and possible D waves). Exact unitarity is imposed and by using a flexible form with correct left- and right-hand cut structure,[27] one can hope to extend the results further into the physical region than with the full-amplitude case. The parameters to be determined either enter into a K matrix,[27] or in a pole approximation to the left-hand cut.[28]

Now, of course, crossing symmetry has to be ensured for the waves that we have; this aspect has been discussed by Roskies,[25] and a somewhat different approach, more suited to numerical work, is given in Appendix B. (The "size" of the parameters c_p^σ, allowed by positivity, may aid in determining how significant a certain deviation from crossing symmetry is.)

We then assume these partial waves "belong" to a fully crossing-symmetric amplitude, whose partial waves for l greater than $l_{\rm max}$ "exist," but are not directly accessible; we may nevertheless consider the effect of the various positivity inequalities that may be related to the a_{nl} for $l \leq l_{\rm max}$. This is in addition to the various inequalities on these accessible partial waves that we may test without having to introduce the a_{nl}; in particular, we may use the S-wave inequalities[7-9] as well as the inequality of Sec. IV that required $f_l(0)$ [Eq. (4.30)], which is believed to be better than (4.25); again our moment technique will help us to "pinpoint" the significant inequalities.

Finally, another aspect that may be investigated is the shape of the S wave itself. Results derived by more involved methods[7-9] indicate that the S wave has a unique minimum at $s \simeq 0.4$. If it is positive, we get a whole sequence of moments $[k]_0$ for $k \geq 0$ that are positive, and apart from the one coefficient a_{00}, all variables are related to our x_i. (The appropriate D^k_{n0}, and ${}_1D^k_{n0}$ are given in Tables I and II for these investigations.) If it is not positive, then it may have one or two zeros, s_1 and s_2, and $(s-s_1)(s-s_2)f_0(s)$ is positive, again leading to inequalities that may be examined by our method, giving relations between a_{00}, s_1, and s_2, and the x_i.

Particularly interesting will be relations involving the derivative of the π^0-π^0 S wave: The coefficient a_{00} will be removed by the derivative, and a_{10} is not present as a consequence of crossing symmetry [$n_1 = 0$ in Roskies's parametrization, Eq. (2.10)], so that the first coefficient will be a_{20}, connected to a_{02} by crossing symmetry. This means that these relations can be directly represented on our contour plots, provided that the expression is given for all s in (0, 1).

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APPENDIX A: NATURE OF THE "ONE-ZERO" INEQUALITIES

For a given positive polynomial of order n, $P_n(y)$, the development in Sec. V leads to the inequality

$$a[P_n(y)]_l \geq [yP_n(y)]_l , \tag{A1}$$

subject to the condition

$$a = \frac{\langle yP_n(y)\rangle_l}{\langle P_n(y)\rangle_l} \tag{A2}$$

[Eqs. (5.11) and (5.12) in the most constraining form].

The most general polynomial which is positive on $0 \leq y \leq 1$ is given by a theorem of Lukacs[1,12] as

$$P_n(y) = [A(y)]^2 + y(1-y)[B(y)]^2 \text{ if } n \text{ is even}$$

or as

$$P_n(y) = y[A(y)]^2 + (1-y)[B(y)]^2 \text{ if } n \text{ is odd}, \tag{A3}$$

where $A(y)$ and $B(y)$ are arbitrary polynomials of the appropriate order.

Since (A1) and (A2) are homogeneous in the polynomial $P_n(y)$, we may divide out an over-all constant in $P_n(y)$, and set the leading coefficients of $A(y)$ and $B(y)$ to 1.

We may then describe our (normalized) positive polynomial as

$$P_n(y) = (1-\lambda)(y^m + \alpha_1 y^{m-1} + \cdots + \alpha_m)^2 + \lambda(y)(1-y)(y^{m-1} + \beta_1 y^{m-2} + \cdots + \beta_{m-1})^2 \quad \text{for } n \text{ even}, \ m = [n/2], \tag{A4}$$

and similarly for n odd, $m = [(n-1)/2]$,

$$P_n(y) = (1-\lambda)y(y^m + \alpha_1 y^{m-1} + \cdots + \alpha_m)^2 + \lambda(1-y)(y^m + \beta_1 y^{m-1} + \cdots + \beta_m)^2, \tag{A5}$$

where clearly $0 \leq \lambda \leq 1$, and the parameters α_i, β_i are arbitrary real numbers. Collecting powers of y we get finally

$$P_n(y) = (1-2\lambda)y^n + \lambda Q_1(\alpha, \beta, y) + Q_0(\alpha, \beta, y), \tag{A6}$$

where Q_1 and Q_0 are polynomials of order $n-1$ in y and depend on $\{\alpha_i\}$ and $\{\beta_i\}$. For example, the simplest polynomials are

$$\begin{aligned} P_0(y) &= 1, \\ P_1(y) &= \lambda + (1-2\lambda)y, \\ P_2(y) &= \lambda y(1-y) + (1-\lambda)(y+\alpha)^2 \\ &= (1-2\lambda)y^2 + \lambda(y - 2\alpha y - \alpha^2) + (2\alpha y + \alpha^2). \end{aligned} \tag{A7}$$

The inequality (A1) becomes

$$a[P_n(y)]_l - \lambda[yQ_1(y)]_l - [yQ_0(y)]_l \geq (1-2\lambda)[y^{n+1}]_l, \tag{A8}$$

which leads to both upper and lower bounds on the moment $[y^{n+1}]$, in terms of λ, α_i, β_i, and the lower moments $[y^k]$, $k = 0, \ldots, n$.

Defining

$$B(\lambda) = \frac{a[P_n] - \lambda[yQ_1] - [yQ_0]}{1-2\lambda} = \frac{aN(\lambda) + M(\lambda)}{1-2\lambda} \tag{A9}$$

(where N and M are linear in λ and depend on all the other variables $[y^k]$, α_i, β_i), we see that

$$[y^{n+1}] \leq B(\lambda) \quad \text{for} \quad 0 \leq \lambda < \tfrac{1}{2}, \tag{A10}$$

while the inequality reverses for $\lambda > \frac{1}{2}$,

$$B(\lambda) \leq [y^{n+1}] \quad \text{for} \quad \tfrac{1}{2} < \lambda \leq 1. \tag{A11}$$

Setting $\lambda = \frac{1}{2}$ eliminates $[y^{n+1}]$, and we have the inequality that arises from a (non-normalized) polynomial $P_{n-1}(y)$.

It is important to notice that a depends on λ in the following way:

$$a = a(\lambda) = \frac{a_0 + a_1\lambda}{a_2 + a_3\lambda}, \tag{A12}$$

where a_i depend on the parameters α_i and β_i.

The best inequalities on the moment $[y^{n+1}]_l$ are

$$L(\alpha, \beta, [y^k]) \leq [y^{n+1}] \leq U(\alpha, \beta, [y^k]), \tag{A13}$$

with

$$U = \min_{0 \leq \lambda < \frac{1}{2}} B(\lambda), \quad L = \max_{\frac{1}{2} < \lambda \leq 1} B(\lambda), \tag{A14}$$

for given fixed α_i, β_i, and $[y^k]$.

For a fixed point in the moment space $[y^k]$, we vary α_i, β_i to give the optimum bounds; the optimum of course depends on the values $[y^k]$ and so the boundary will be "curved," corresponding to our determinants, for the simple positivity covered by Theorem 1. To find L and U, we evaluate $B(\lambda)$ at the end points and at possible extrema at the zeros of the derivative

$$\frac{\partial B}{\partial \lambda}(\lambda) = 0 \text{ at } \lambda_i, \text{ provided } 0 \leq \lambda_i \leq 1. \tag{A15}$$

With

$$N(\lambda) = n_0 + n_1\lambda, \quad M(\lambda) = m_0 + m_1\lambda,$$

we see that

$$(1-2\lambda)^2 \frac{\partial B}{\partial \lambda} = (1-2\lambda)\frac{da}{d\lambda}(n_0 + n_1\lambda) + an_1 + m_1 + 2an_0 + 2m_0. \tag{A16}$$

Now if a did not depend on λ (as is the case for simple positivity, equivalent to $a = 1$) we see that $(1-2\lambda)^2 \partial B/\partial\lambda$ would also be independent of λ. This means that for best bounds we could use $B(0)$ and $B(1)$; this is in fact the separation into two sets of determinants in Theorem 1, Sec. II. The lower bounds are given by determinants with s_k as elements, and upper bounds by determinants with $s_k - s_{k+1}$. The determinant itself corresponds to varying α_i, β_i over their whole allowed range (see Ref. 1).

However, a does depend on λ (A12) and

$$\frac{da}{d\lambda} = \frac{a_2 a_1 - a_0 a_3}{(a_2 + a_3\lambda)^2}, \tag{A17}$$

where $(a_2 + a_3\lambda) \geq 0$.

Finally, we obtain

$$(a_2 + a_3\lambda)^2 (1-2\lambda)^2 \frac{\partial B}{\partial \lambda} = C_0 + C_1\lambda + C_2\lambda^2, \tag{A18}$$

a quadratic in λ, where the C_i depend on α_i, β_i, and the moments $[y^k]$, $k = 0, \ldots, n_0$. There are at most two zeros that may lead to better bounds on $[y^{k+1}]$ than $B(0)$, $B(1)$, but an actual numerical investigation of (A18) for the simple cases given in (A7) indicates that these zeros λ_i rarely give a significant contribution; either there are no zeros, or they do not occur in $(0 \leq \lambda \leq 1)$, or, when they do, the resulting improvement to the set of inequalities is negligible.

APPENDIX B: THE π-π AMPLITUDE WITH ISOSPIN, CROSSING SYMMETRY, AND POSITIVITY

The π-π amplitude is described by the usual invariant functions $A(s, t, u)$, $B(s, t, u)$, and $C(s, t, u)$[29] in terms of which the s-channel isospin amplitudes are

$$\begin{aligned} F^0(s,t,u) &= 3A(s,t,u) + B(s,t,u) + C(s,t,u), \\ F^1(s,t,u) &= B(s,t,u) - C(s,t,u), \\ F^2(s,t,u) &= B(s,t,u) + C(s,t,u). \end{aligned} \tag{B1}$$

The π^0-π^0 amplitude examined in the above work is given by $F(s,t,u) = \frac{1}{3}(F^0 + 2F^1) = A + B + C$. Crossing symmetry relates A, B, and C:

$$A(s,t,u) = A(s,u,t) = B(t,s,u) = C(u,t,s). \tag{B2}$$

The s-channel partial waves of $F^I(s,t,u)$, $f_l^{I_s}$, are given by the Froissart-Gribov projection for $l \geq 2$:

$$f_l^{I_s}(s) = \frac{(-1)^l}{1-s}\frac{4}{\pi}\int_1^\infty dt\, Q_l(z) \sum_{I_t=0} \beta^{I_s I_t} A^{I_t}(t,s), \tag{B3}$$

where $A^{I_t}(t,s)$ is the absorptive part of the t-channel amplitude and is positive for $t \geq 1$ and $0 \leq s \leq 1$; $\beta^{I_s I_t}$ is the usual isospin crossing matrix.[29] Because of Bose symmetry ($l + I_s$ even), we have only two independent combinations of the $f_l^{I_s}(s)$, corresponding to positive $A(t,s)$:

$$f_l^{00}(s) = \tfrac{1}{3}(f_l^{(0)} + 2f_l^{(2)}) = \frac{4}{(1-s)\pi}\int_1^\infty dt\, Q_l(z)\tfrac{1}{3}[A^{I_t=0}(t,s) + 2A^{I_t=2}(t,s)] \tag{B4}$$

and

$$f_l^{+-}(s) = \tfrac{1}{3}(f_l^{(0)} - f_l^{(2)}) = \frac{4}{(1-s)\pi}\int_1^\infty dt\, Q_l(z)\tfrac{1}{2}[A^{I_t=1}(t,s) + A^{I_t=2}(t,s)] \tag{B5}$$

for even $l \geq 2$ and $0 \leq s \leq 1$.

The $\pi^0\pi^0 \to \pi^0\pi^0$ partial waves $f_l^{00}(s)$ are actually the $f_l(s)$ used in the above work; we will now get a further set of inequalities by similarly using the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude, $f_l^{+-}(s)$.

Crossing-Symmetric Parametrization

Roskies[25] has extended his parametrization to include all the isospin amplitudes; we will follow a different approach to clarify some properties of the resulting parametrization.

We project out the double partial waves, A_{nl}, of $A(s, t, u)$; these make up vectors $(A^\sigma)_l = A_{\sigma-l,l}$ of dimension $\sigma+1$. With X^σ the s-t crossing matrix and U^σ the u-t crossing matrix, with elements $U^\sigma_{ll'} = (-1)^{l+l'}$, we observe that the s-u crossing matrix is given by $\tilde{X}^\sigma = U^\sigma X^\sigma U^\sigma$. The crossing symmetry of A, B, and C [Eq. (B2)] then gives the double partial waves of B and C (B_{nl}, C_{nl}) from those of $A(s, t, u)$:

$$B^\sigma = X^\sigma A^\sigma , \quad C^\sigma = \tilde{X}^\sigma A^\sigma . \tag{B6}$$

Since $A(s, t, u)$ is even under t-u exchange, A^σ has only even l:

$$U^\sigma A^\sigma = A^\sigma . \tag{B7}$$

Of the $\sigma+1$ eigenvectors of U^σ, $n_e = [\sigma/2]+1$ "even," with eigenvalue +1 corresponding to (B7). We will divide these eigenvectors into two groups, α_e^σ and β_e^σ, according to whether they are also "even" eigenvectors of X^σ or not. The set $\{\alpha_e^\sigma\}$, obeying $X^\sigma \alpha_e^\sigma = \alpha_e^\sigma$, is simply the set $\{\alpha_p^\sigma\}$ of Eq. (2.10), $n_\sigma = [\sigma/2]+1-\{\sigma/3\}$ in number.[6] The remaining $n_\beta = n_e - n_\sigma = \{\sigma/3\}$ vectors $\{\beta_e^\sigma\}$ are not eigenvectors of X^σ.

We now expand A^σ in terms of these n_e vectors $\{\alpha_e^\sigma, \beta_e^\sigma\}$:

$$A^\sigma = \sum_p c_p^\sigma (\alpha_p^\sigma)_e + \sum_{p'} e_{p'}^\sigma (\beta_{p'}^\sigma)_e , \tag{B8}$$

where c_p^σ and $e_{p'}^\sigma$ are $n_e = n_\sigma + n_\beta$ arbitrary real coefficients. We abbreviate (B8) as

$$A = \alpha_e + \beta_e , \tag{B9}$$

where $\{\alpha_e, \beta_e\}$ obey

$$U\alpha_e = \alpha_e , \quad U\beta_e = \beta_e , \quad \text{and} \quad X\alpha_e = \tilde{X}\alpha_e = \alpha_e . \tag{B10}$$

The double partial waves of $f_l^I(s)$ are then $(a_{nl}^{I\sigma}) \equiv a^I$:

$$\begin{aligned} a^0 &= 3(\alpha_e+\beta_e) + X(\alpha_e+\beta_e) + \tilde{X}(\alpha_e+\beta_e) \\ &= 5\alpha_e + [3\beta_e + (1+U)]X\beta_e , \\ a^1 &= X(\alpha_e+\beta_e) - \tilde{X}(\alpha_e+\beta_e) \\ &= (1-U)X\beta_e , \\ \text{and} \\ a^2 &= X(\alpha_e+\beta_e) + \tilde{X}(\alpha_e+\beta_e) \\ &= 2\alpha_e + (1+U)X\beta_e . \end{aligned} \tag{B11}$$

Now the totally symmetric π^0-π^0 amplitude corresponds to

$$\tfrac{1}{3}(a^0 + 2a^2) = 3\alpha_e + [\beta_e + (1+U)X\beta_e] \tag{B12}$$

and since $\{\alpha_e\}$ includes all the totally symmetric eigenfunctions, β_e must obey

$$\beta_e = -(1+U)X\beta_e . \tag{B13}$$

We define a set of vectors β_o, with only odd l,

$$\beta_o = -(1-U)X\beta_e . \tag{B14}$$

These may be considered as the even and odd parts of a vector β, which obeys

$$\beta = -2X\beta_e = -X(1+U)\beta . \tag{B15}$$

An important consequence of this definition of β and of the fact[3] that $X = FL$, where $F = F^T$ (a symmetric matrix), and that L and U are diagonal matrices [with $L_{ll'} = (2l+1)\delta_{ll'}$], is the orthogonality relation

$$\beta \otimes \beta' = \beta^T L \beta' = 4\beta_e \otimes \beta'_e = \tfrac{4}{3}\beta_o \otimes \beta'_o . \tag{B16}$$

If β_e and β'_e are orthogonal, then so are β and β', and also β_o and β'_o.

Equations (B11) then become

$$\begin{aligned} a^0 &= 5(\alpha)_e + 2(\beta)_e , \\ a^1 &= -(\beta)_o , \\ a^2 &= 2(\alpha)_e - (\beta)_e , \end{aligned} \tag{B17}$$

and all the relations among the a_{nl}^I due to crossing symmetry are contained in these equations.

Roskies,[25] using his alternative approach, has given explicit expressions for the independent set of vectors, $\{\beta_p^\sigma\}$ [$\{\alpha_p^\sigma\}$ is given by Eq. (2.10)]:

$$\{\beta_p^\sigma\}_l = (\sigma-l)!(\sigma+l+1)! \times \left\{ \begin{array}{l} \displaystyle\int_{-1}^{1} dz (3+z^2)^{3p'-\sigma'}(1-z^2)^{\sigma'-2p'}(1+z) \\ \displaystyle\frac{1}{2}\int_{-1}^{1} dz (3+z^2)^{3p''-\sigma''}(1-z^2)^{\sigma''-2p''}(3-z^2-6z) , \end{array} \right. \tag{B18}$$

where $\sigma' = \sigma-1$ and $\sigma'' = \sigma-2$; p' and p'' are restricted in the same way as p in Eq. (2.10), giving $n_{\sigma'} + n_{\sigma''}$ vectors, respectively. (We may verify that $n_\beta = n_{\sigma'} + n_{\sigma''}$.)

We now orthonormalize this set of vectors, $\{\alpha\}$ and $\{\beta\}$, giving $n_e = [\sigma/2]+1$ orthogonal vectors:

$$\overline{E}^{\sigma}(p),\ p=0,\ldots,n_{\sigma}-1$$

and

$$\overline{H}^{\sigma}(p),\ p=0,\ldots,n_{\beta}-1,$$

so that the isospin amplitudes are

$$a^0=5\sum c_p^{\sigma}\overline{E}^{\sigma}(p)+2\sum e_p^{\sigma}[\overline{H}(p)]_e,$$

$$a^1=-\sum e_p^{\sigma}[\overline{H}(p)]_0, \tag{B19}$$

$$a^2=2\sum_p c_p^{\sigma}\overline{E}^{\sigma}(p)-\sum_p e_p^{\sigma}[\overline{H}(p)]_e.$$

Because of the orthonormality of $\overline{E}$, $\overline{H}$, and Eq. (B16), we have

$$c_p^{\sigma}=\tfrac{1}{5}\overline{E}^{\sigma}(p)\otimes a^0 \tag{B20a}$$

$$=\tfrac{1}{2}\overline{E}^{\sigma}(p)\otimes a^2,\quad p=0,\ldots,n_{\sigma}-1 \tag{B20b}$$

$$e_p^{\sigma}=\tfrac{1}{2}4\overline{H}_e(p)\otimes a^0 \tag{B21a}$$

$$=-\tfrac{4}{3}\overline{H}_0(p)\otimes a^1 \tag{B21b}$$

$$=-4\overline{H}_e(p)\otimes a^2,\quad p=0,\ldots,n_{\beta}-1. \tag{B21c}$$

It is of course important that (B20a) and (B20b) should give the same values of c_p^{σ}, and (B21a), (B21b), and (B21c) give the same e_p^{σ}'s. This is one test of whether the a_{nl}^I are crossing-symmetric; in addition, there should be no further components in (B19) corresponding to vectors not in the set $\overline{E}^{\sigma}(p)$, $\overline{H}^{\sigma}(p)$. In particular, of the $n_0=\sigma+1-n_e$ vectors with only odd l, only the $\{\sigma/3\}$ vectors, $\overline{H}_0(p)$, appear in the expansion of a^1.

A convenient measure of "deviation from crossing symmetry," which will also indicate the accuracy of the numerical method will be given by determining c_p^{σ} and e_p^{σ} from (B20) and (B21), and evaluating the deviations from (B19).

$$\delta^{\sigma}=\sum_l\{[a_l^0-5c_p^{\sigma}\overline{E}_e^{\sigma}(p)_l-2e_p^{\sigma}\overline{H}_e^{\sigma}(p)_l]^2+[a_l^1+e_p^{\sigma}\overline{H}_0^{\sigma}(p)_l]^2+[a_l^2-2c_p^{\sigma}\overline{E}^{\sigma}(p)_l+e_p^{\sigma}\overline{H}_e^{\sigma}(p)_l]^2\}(\sigma+1)^{-1/2}. \tag{B22}$$

Inequalities from Positivity

The two amplitudes with positivity properties are now (B4) and (B5).

$\pi^0\pi^0\to\pi^0\pi^0$:

$$F^{00}(s,t,u)=\tfrac{1}{3}(F^0+2F^2)=A+B+C,$$

with double partial-wave amplitudes

$$(a_{nl})=3c_p^{\sigma}\overline{E}^{\sigma}(p). \tag{B23}$$

$\pi^+\pi^-\to\pi^0\pi^0$:

$$F^{+-}(s,t,u)=\tfrac{1}{3}(F^0-F^2)=A(s,t,u),$$

with double partial waves which we call b_{nl},

$$(b_{nl})=c_p^{\sigma}\overline{E}^{\sigma}(p)+e_p^{\sigma}\overline{H}^{\sigma}(p)_e. \tag{B24}$$

The various moments $[k]_l$ and $[k]$ derived from (B23) depend on c_p^{σ} only, while the corresponding moments with a_{nl} replaced by b_{nl} from (B24) depend on the $\{c_p^{\sigma}\}$ and the $\{e_p^{\sigma}\}$. The parameters which remain unrestricted by the positivity inequalities for $l\geq 2$ are a_{00}, a_{02}, and b_{10}, while b_{02} is partially restricted.

It is particularly important to notice that the subsidiary conditions of the "one-zero" method (Sec. V), do not depend on the variables used in the moments, so that the numerical values of the a's are unchanged; e.g., Eq. (5.18) gives for $\sigma=3$ the sets $[3]_2\geq 0$ and $\frac{15}{16}[2]_2\geq[3]_2$, which become

$$\tfrac{15}{16}b_{02}\geq\tfrac{1}{7}(6b_{02}-b_{12})\geq 0. \tag{B25}$$

As an example of the application of these new inequalities, we consider the Iliopoulos model with isospin[21] [$A(s,t,u)$ given by Eq. (6.1)]. Because the amplitude is explicitly crossing-symmetric, we do not have to introduce the (α_e) and (β_e), but can directly evaluate the b_{nl}'s:

$$\begin{aligned}
b_{02}&=10^{-5}(-211.6C_1'+72.2C_3'),\\
b_{12}&=10^{-5}(43.3C_1'-4.16C_3'),\\
b_{22}&=10^{-5}(-13.3C_1'+0.066C_3'),\\
b_{32}&=10^{-5}(5.15C_1'-0.163C_3'),\\
b_{04}&=10^{-5}(-13.3C_1'+0.666C_3'+1.67C_3).
\end{aligned} \tag{B26}$$

[Although the coefficients are the same as for the a_{nl}, Eq. (6.4), the C_i and C_i' take on values different from the $\overline{C}_i$, so that of course $b_{nl}\neq a_{nl}$. Notice also that $b_{04}\neq b_{22}$.]

Using the values of C_i and C_i' given in Table IV for the Iliopoulos solutions (a), (b), (c), and (d), we discover that all but solution (a) violate the simple positivity inequalities on the

$$[k]_l=\sum D_{nl}^k b_{nl}/a_{02}\quad \text{for } \sigma\leq 4.$$

For example, solution (b) gives $[4]_4=b_{02}/a_{02}=-0.1058$, which is significantly negative. Solution (c) is the worst of all, and fails even at $\sigma=3$: $[2]_2-[3]_2=-0.0505$ and $[3]_2-[4]_2=-0.03$. Solution (d) violates the $\sigma=4$ determinant, $[2]_2[4]_2-[3]_2^2=-0.0041$, as well as the "one-zero," $\sigma=3$ result $\frac{15}{16}[2]_2-[3]_2=-0.0143$.

Solution (a) satisfies all the inequalities, but is quite close to the limit: $[4]_2=2.06\times10^{-3}$ is bounded by the inequalities $0.944[3]_2-[4]_2\geq 0$ and $[2]_2[4]_2-[3]^2\geq 0$, giving the fairly tight constraint

$$2.36\times10^{-3}\geq[4]_2\geq 2.02\times10^{-3}. \tag{B27}$$

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[20]This follows immediately from the integral representation (4.9), and the observation that $(w^{-2})^{l+1}(u_1^{-2} - w^{-2})^m \geq 0$ for $u_1 \leq u_0 \leq w \leq \infty$.

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