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Bootstrap Models and a Special Property of the Unitary Lie Algebras. II. The Minimal Bootstrap

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We prove the following statement: In any of the standard bootstrap frameworks, if one begins with a meson world chosen so that (a) it is symmetric with respect to some simple, compact Lie algebra, $\mathfrak{L}$, (b) all particles belong to N-fold degenerate families, each classifiable under the identical representation $\underline{R}$, and (c) the discrete quantum numbers are such that both symmetric and antisymmetric trilinear couplings exist, then for a given choice of l (l = the number of mutually commuting conserved charges) the minimal degeneracy turns out to be $N=(l+1)^2$ and the corresponding minimal representation $\underline{R}$ = $\underline{\text{adjoint}}\oplus\underline{\text{singlet}}$ of SU($l+1$).

I. INTRODUCTION

In this paper we extend a result we have derived previously[1] dealing with self-consistent worlds composed of mesons only.

Our assumptions are as follows:

(a) The meson world in question is symmetric with respect to some simple, compact Lie algebra, $\mathfrak{L}$.

(b) All particles belong to N-fold degenerate families. Each such family is classifiable under the same (in general reducible) representation $\underline{R}$ of $\mathfrak{L}$.

(c) The dynamical framework is that of one of the known bootstrap schemes: N/D, finite-energy sum rules (FESR), $Z=0$, narrow-resonance models, etc.

(d) The discrete quantum numbers are such that trilinear couplings of both antisymmetric and symmetric types exist.

(e) The number of particles, N, in each family is minimal.

In I we considered assumptions (a)–(d), and, in addition, assumed that $\underline{R}$ = $\underline{\text{adjoint}}\oplus\underline{\text{singlet}}$ of some arbitrary $\mathfrak{L}$. We then showed that $\mathfrak{L}$ necessarily was SU($l+1$), where l is the number of mutually commuting conserved charges, (i.e., the rank) of the algebra $\mathfrak{L}$.[1]

Here we drop our previous assumption about the nature of $\underline{R}$ and replace it by the minimality assumption (e). Our new result is that, under assumptions (a)–(e), $\mathfrak{L}$ is SU($l+1$), $\underline{R}$ = $\underline{\text{adjoint}}\oplus\underline{\text{singlet}}$, and $N=(l+1)^2$. In the present framework we see absolutely no way to determine l and/or N.

In the following section we will briefly review the rules of the game and indicate how the result arises. Mathematical details can be found in the Appendix below, and a more detailed discussion of the general framework, with illustrative examples, can be found in I.

II. EXPLANATION OF RESULTS

A. Dynamical Framework

Let ξ be the amplitude for the elastic process

$$\underline{R}^{(i)}\oplus\underline{R}^{(i)}\to\underline{R}^{(i)}\oplus\underline{R}^{(i)}, \tag{1}$$

where all particles belong to a representation $\underline{R}$ of some Lie algebra $\mathfrak{L}$, as in (a) and (b) above, and the superscript (i) represents all labels other than those associated with $\mathfrak{L}$. ($\underline{R}=\underline{A}\oplus\underline{B}\oplus\underline{C}\oplus\cdots$, where the terms on the right-hand side are irreducible representations of $\mathfrak{L}$.)

By assumption (c) we are in the dynamical framework of one of the known bootstrap schemes. In these schemes an amplitude such as ξ must satisfy a self-consistency equation of the schematic form

$$\xi=CM\xi, \tag{2}$$

where C is a crossing operator associated with the internal-symmetry algebra $\mathfrak{L}$, and M is a crossing matrix in ordinary helicity space.

All simple bootstrap schemes have a crucial property in common which amounts to a generalized pole approximation. In each case, amplitudes such as ξ can be expressed as a sum over simple "tree diagrams" in all channels, with the intermediate states having definite internal quantum numbers and external labels (i). Of course, self-consistency then requires that the intermediate states have internal-symmetry properties identical to the external scattering states, so that any one generalized pole diagram must be proportional to a bilinear form in the trilinear coupling of $\underline{R}$ to itself.

Consider any one particular reaction (1) and its associated amplitude ξ. In general, in the bootstrap framework it is always possible to write ξ as a product

$$\xi = \xi_I K \qquad (3)$$

of internal-symmetry and kinematic factors, so that (2) takes the schematic form

$$\xi_I = C' \xi_I \qquad (4)$$

for the internal-symmetry piece, where C' is related to C by certain phase factors, and channel labels have been suppressed throughout. If it is desired one can extend these considerations to a set of particles of various masses, parities, spins, etc., for example, as in the narrow-resonance-model bootstrap.[2] The price that one is forced to pay for factorizing out the internal-symmetry dependence as in Eq. (3) is that certain intermediate states in general acquire negative norms. We will ignore that problem here, treating all reactions as though all spins were zero and dealing with the pure algebraic constraints (4).

B. Algebraic Results

Let the trilinear couplings of $\underline{R}$ to itself be g^P_{ijk}, where (i, j, k) are labels for particles in $\underline{R}$, each index running from 1 to N, and where $P = +$ or $-$, depending on whether $g_{jik} = +g_{ijk}$ or $-g_{ijk}$. (The first two subscripts in g will always refer to external scattering particles.) By assumption (d), some g^+ and some g^- are not zero. Let the s-channel process (1) be $a + b \to c + d$, and for illustrative purposes let $g^+_{ijk} = 0$ and $g^-_{ijk} = F_{ijk}$, the structure constants of any $\mathfrak{L}$. In this simple case, Eq.(4) can be equivalently written as the Jacobi identity[3]

$$F_{abs} F_{cds} + F_{adu} F_{bcu} = F_{act} F_{bdt}. \qquad (5)$$

To implement assumptions (d) and (e) and prove the main assertion above, it is sufficient to find the set of $\underline{R}$'s such that

(I) $\dim(\underline{R}) \equiv N \leq (l+1)^2$,

(II) $\underline{R}$ is self-conjugate,

(III) $\underline{R} \otimes \underline{R}$ contains at least a part of $\underline{R}$ in both the symmetric and antisymmetric Kronecker products,

(IV) $\underline{R}$ is a solution of the crossing constraint (4).

Besides these algebraic conditions, we also will require that the amplitudes involved are globally nonpathological in that essential singularities and/or spurious poles and cuts are forbidden.[4]

In I, we considered the special case $\underline{R} = \underline{R}_0$ = adjoint$\oplus$singlet of SU$(l+1)$, and this choice was shown to be consistent with assumptions (a)–(d) above. Here we replace our assumption $\underline{R} = \underline{R}_0$ by the minimality condition (I).

Condition (III) arises from assumption (d), $g^{\pm}_{ijk}$ just being the numbers which appear in the antisymmetrized and symmetrized Clebsch-Gordan series for $\underline{R} \otimes \underline{R}$.

Condition (IV) has two crucial features. First, it requires that if $\underline{R} = \underline{A} \oplus \underline{B} \oplus \underline{C} \oplus \cdots$ is a sum of irreducible representations, the subreactions contained in $\underline{R} + \underline{R} \to \underline{R} + \underline{R} \cdots$ must each be self-consistent (e.g., $\underline{A} + \underline{A} \to \underline{B} + \underline{B}$, $\underline{C} + \underline{C} \to \underline{C} + \underline{C}$, etc.). Secondly, it requires that the "force" between s-channel scattering particles generated by a t-channel exchange be attractive rather than repulsive. This last requirement will be crucial in the elimination of $\{7\} \oplus \{1\}$ of G_2. In fact, it turns out that conditions (I)–(III) are so tight that we will need to invoke condition (IV) only to deal with G_2.

To prove our assertion we proceed by exhaustion, first finding all self-adjoint representations of all $\mathfrak{L}$'s satisfying condition (I), and then computing the Kronecker square of each and checking condition (III).

Cartan's list of all $\mathfrak{L}$'s is A_l, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, G_2, F_4, E_6, E_7, E_8. In the Appendix we give a detailed verification of our assertion for $l > 2$. For clarity and to illustrate what is involved we will consider A_2[SU(3)], B_2, and G_2 here. Condition (I) states that $\dim(\underline{R}) \leq 9$ in this case.

A_2

The lowest dimensional possibilities are $\{1\}$, $\{3\}$, $\{3^*\}$, $\{6\}$, $\{6^*\}$, $\{8\}$. From I we know that $\{8\}$ can work in conjunction with $\{1\}$. The combination $\{6\} \oplus \{6^*\}$ is self-adjoint but has dimensionality 12, so $\{6\}$ and $\{6^*\}$ can be eliminated.

Consider $\{3\} \otimes \{3^*\}$. This $\underline{R}$ contains a part that is $\underline{3} + \underline{3} \to \underline{3} + \underline{3}$ in one channel (s), and $\underline{3} + \underline{3}^* \to \underline{3} + \underline{3}^*$ in the other two channels (t, u). In the s channel the intermediate states are $\{3\} \otimes \{3\} = \{6\} \oplus \{3^*\}$, while in the t and u channels we have $\{1\} \oplus \{8\}$, neither of which is in $\{3\} \oplus \{3^*\}$. The possibility $\{3\} \oplus \{3^*\}$ is therefore unsatisfactory, leaving only $\{3\} \oplus \{3^*\} \oplus \{1\}$.

In fact, $\underline{R} = \{3\} \oplus \{3^*\} \oplus \{1\}$ satisfies conditions (I)–(IV). Clearly $\underline{3} + \underline{1} \to \underline{3} + \underline{1}$ and $\underline{1} + \underline{1} \to \underline{1} + \underline{1}$ work trivially. The only problem could arise in the $\underline{3} + \underline{3} \to \underline{3} + \underline{3}$ process mentioned above, and either by direct calculation of the SU(3) crossing matrices or by using $\epsilon_{abs} \epsilon_{cds} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}$ it is easy to see the algebraic constraints are satisfied.[5] However, the global analyticity requirement above cannot be satisfied in this dynamical framework. The SU(2) content of $\{3\} \oplus \{3^*\} \oplus \{1\}$ is only $\frac{1}{2}$ and 0, so that $\underline{3} + \underline{3} \to \underline{3} + \underline{3}$ contains an analog of $KK \to KK$, with only isospin-zero intermediate states in all channels KK and $K\overline{K}$.

A consideration of the SU(2) crossing matrices for $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ reveals[3] that if isospin-zero saturation is required, the amplitude must be constructed out of functions $F(s,u)$ having the properties $D_t F(s,u) = D_s[F(s,u) + F(s,t)] = D_u F(s,t) = 0$.[6] There is no way to construct such a function in the pole approximation considered here, without introducing essential singularities at infinity.[7]

B_2

The smallest possible representations are $\{1\}$, $\{4\}$, and $\{5\}$, each of which is self-adjoint. The relevant Kronecker products are $\{4\}\otimes\{4\}=\{10\}\oplus\{5\}\oplus\{1\}$ and $\{5\}\otimes\{5\}=\{14\}\oplus\{10\}\oplus\{1\}$. Clearly $\{1\}$, $\{4\}$, $\{5\}$, $\{4\}\oplus\{1\}$, $\{5\}\oplus\{1\}$, $\{4\}\oplus\{5\}$ are the only possibilities having dim $\leqslant 9$ and none of these can possibly work. For instance, $\{4\}\oplus\{1\}$ contains $\underline{4}$-$\underline{4}$ scattering which contains no $\{4\}$ intermediate state, but only a $\{1\}$, while $\{5\}\oplus\{4\}$ contains $\underline{5}$-$\underline{5}$ scattering which has no $\{4\}$ and no $\{5\}$ as an intermediate state.

G_2

The smallest representations are $\{7\}$ and $\{1\}$, the next one being $\{14\}$, which is too large. Now $\{7\}\otimes\{7\}=\{27\}\oplus\{14\}\oplus\{7\}\oplus\{1\}$, so $\{7\}\oplus\{1\}$ is a possibility. Furthermore, $\{7\}\otimes\{7\}$ couples symmetrically to $\{1\}$ and antisymmetrically to $\{7\}$.[8] This case therefore requires us to check the crossing matrix explicitly, and it turns out that the eigenvalue in Eq. (4) is -1 rather than $+1$, which in physical terms implies that a repulsive force generates bound states,[9] violating condition (IV).

The arguments above prove the assertion for $n=2$. The mathematical details for the remaining portion of the argument will be found in the Appendix.

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APPENDIX

In this Appendix we complete the general proof of our assertion above.

First of all, in order to implement condition (I), we list in Tables I–V all irreducible representations of A_l, B_l, C_l, D_l, and G_2, F_4, E_6, E_7, E_8 with dimensionality $\leqslant (l+1)^2$ for rank $l \geqslant 2$. The case $l=2$ has already been discussed, but is included in the tables for the sake of comparison.

Condition (II) requires that the representation $\underline{R}$ be self-conjugate. Thus if an irreducible representation $\mathfrak{D}$ is not equivalent to its conjugate representation $\mathfrak{D}^*$, then $\mathfrak{D}$ and $\mathfrak{D}^*$ must appear in $\underline{R}$ the same number of times. It was in this way that we ruled out the six-dimensional representations of SU(3). The dimensions of the representations so ruled out

TABLE I. Representations of A_l which have dimensionality $\leq (l+1)^2$. At the top we designate the representations by their Young shapes. If a representation is conjugate to a representation on its left, it is designated by an asterisk next to its dimensionality. Self-representations are indicated by the subscript S, adjoint representations by the subscript A, and pseudoreal representations by the subscript Q.

Algebra	ϵ	[1]	$[1^2]$	$[1^3]$	$[1^4]$	$[1^5]$	$[1^6]$	$[1^7]$	$[1^8]$			[2]	$[2^l]$	$[2,1^{l-1}]$
A_2	$\underline{1}$	3_S	$\underline{3}^*$	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	6	6^*	$\underline{8}_A$
A_3	$\underline{1}$	4_S	$\underline{6}$	$\underline{4}^*$	⋯	⋯	⋯	⋯	⋯	⋯	⋯	10	10^*	$\underline{15}_A$
A_4	$\underline{1}$	5_S	$\underline{10}$	$\underline{10}^*$	$\underline{5}^*$	⋯	⋯	⋯	⋯	⋯	⋯	15	15^*	$\underline{24}_A$
A_5	$\underline{1}$	6_S	$\underline{15}$	$\underline{20}_Q$	$\underline{15}^*$	$\underline{6}^*$	⋯	⋯	⋯	⋯	⋯	21	21^*	$\underline{35}_A$
A_6	$\underline{1}$	7_S	$\underline{21}$	35	35^*	$\underline{21}^*$	$\underline{7}^*$	⋯	⋯	⋯	⋯	28	28^*	$\underline{48}_A$
A_7	$\underline{1}$	8_S	$\underline{28}$	56		56^*	$\underline{28}^*$	$\underline{8}^*$	⋯	⋯	⋯	36	36^*	$\underline{63}_A$
A_8	$\underline{1}$	9_S	$\underline{36}$					$\underline{36}^*$	$\underline{9}^*$	⋯	⋯	45	45^*	$\underline{80}_A$
⋮	⋮	⋮	⋮						⋱	⋱	⋱	⋮	⋮	⋮
A_l	$\underline{1}$	$\underline{(l+1)}_S$	$\underline{\frac{1}{2}l(l+1)}$							$\underline{(\frac{1}{2}l(l+1))^*}$	$\underline{(l+1)^*}$	$\frac{1}{2}(l+2)(l+1)$	$(\frac{1}{2}(l+2)(l+1))^*$	$\underline{l(l+2)}_A$
⋮	⋮	⋮	⋮							⋱	⋱	⋮	⋮	⋮

are shown in Tables I–V without underlines, while the remaining dimensions are shown with underlines. Because of conditions (I) and (II), for large rank l there are very few possibilities.

For the five exceptional algebras, we are left with only two representations apart from the trivial representations, and one of these, the representation $\{7\}$ of G_2, has already been ruled out for violating condition (IV). For B_l, C_l, and D_l with large l we are left with only the trivial representations $\{1\}$ and the self-representations.[10] For A_l with large l we are left with the trivial representations, the adjoint representations, the self-representations and their conjugates, and one other series of representations and their conjugates. All other series of representations grow faster than $(l+1)^2$ when l gets large.

For small l apart from the series just mentioned, we have only the representation $\{20\}$ of A_4, four spinor representations of the B series, two 14-dimensional representations of C_3, and six spinor representations of the D series. In order to apply condition (III), it is necessary to discuss the symmetrized and antisymmetrized Kronecker squares for the remaining representations. Some general results are available. These can be phrased in terms of the classification[11] of irreducible representations of simple compact Lie algebras into three mutually exclusive types called complex, potentially real, and pseudoreal. The classification is as follows:

If in an irreducible representation $\mathfrak{D}$ is *not* equivalent to its complex-conjugate representation $\mathfrak{D}^*$, then $\mathfrak{D}$ is said to be complex. For a complex representation, the negative of the highest weight $\underline{\Lambda}$ is *not* a weight, while for potentially real and pseudoreal representations, $-\underline{\Lambda}$ is a weight. Only the algebras A_l, D_{2n+1}, and E_6 have complex representations.

If an irreducible representation $\mathfrak{D}$ is equivalent to $\mathfrak{D}^*$, then it is either potentially real or pseudoreal. It is defined as potentially real if the matrices representing the algebra in the representation $\mathfrak{D}$ can be transformed by a similarity transformation to matrices of real numbers. Otherwise it is pseudoreal. The characters of both potentially real and pseudoreal representations are real. Knowing the highest weight of an irreducible representation and the root diagram of the algebra enables one to determine whether the representation is potentially real or pseudoreal. Since $-\underline{\Lambda}$ is a weight, and

TABLE III. Representations of C_l with dimensionality $\le (l+1)^2$. The subscripts S and Q mean self-representation and pseudoreal representation, respectively.

Algebra	Representations			
C_3	$\underline{1}$	$\underline{6}_{S,Q}$	$\underline{14}$	$\underline{14}_Q$
C_4	$\underline{1}$	$\underline{8}_{S,Q}$		
.	.	.		
.	.	.		
.	.	.		
C_l	$\underline{1}$	$\underline{(2l)}_{S,Q}$		
.	.	.		
.	.	.		
.	.	.		

TABLE II. Representations of B with dimensionality $\le (l+1)^2$. The subscripts S, Sp, and Q mean self-representation, spinor representation, and pseudoreal representation, respectively.

Algebra	Representations		
B_2	$\underline{1}$	$\underline{5}_S$	$\underline{4}_{Sp,Q}$
B_3	$\underline{1}$	$\underline{7}_S$	$\underline{8}_{Sp}$
B_4	$\underline{1}$	$\underline{9}_S$	$\underline{16}_{Sp}$
B_5	$\underline{1}$	$\underline{11}_S$	$\underline{32}_{Sp,Q}$
B_6	$\underline{1}$	$\underline{13}_S$	
.	.	.	
.	.	.	
.	.	.	
B_l	$\underline{1}$	$\underline{(2l+1)}_S$	
.	.	.	
.	.	.	
.	.	.	

TABLE IV. Representations of D_l with dimensionality $\le (l+1)^2$. The subscripts S, Sp, and Q mean self-representation, spinor representation, and pseudoreal representation, respectively.

Algebra	Representations			
D_4	$\underline{1}$	$\underline{8}_S$	$\underline{8}_{Sp}$	$\underline{8}'_{Sp}$
D_5	$\underline{1}$	$\underline{10}_S$	$\underline{16}_{Sp}$	$\underline{16}^*_{Sp}$
D_6	$\underline{1}$	$\underline{12}_S$	$\underline{32}_{Sp,Q}$	$\underline{32}'_{Sp,Q}$
D_7	$\underline{1}$	$\underline{14}_S$	64_{Sp}	64^*_{Sp}
D_8	$\underline{1}$	$\underline{16}_S$		
.	.	.		
.	.	.		
.	.	.		
D_l	$\underline{1}$	$\underline{(2l)}_S$		
.	.	.		
.	.	.		
.	.	.		

TABLE V. Representations of the exceptional algebras with dimensionality equal to or less than $(l+1)^2$.

Algebra	Representations			$(l+1)^2$	Comment
G_2	$\underline{1}$	$\underline{7}$		9	Fails because eigenvalue of crossing matrix is −1
F_4	$\underline{1}$			25	Fails trivially
E_6	$\underline{1}$	27	27*	49	Fails because 27 + 27 + 1 > 49
E_7	$\underline{1}$	$\underline{56}_Q$		64	{56} is a pseudoreal representation
E_8	$\underline{1}$			81	Fails trivially

since any weight M is of the form $\underline{\Lambda}-\underline{\alpha}_1-\underline{\alpha}_2-\underline{\alpha}_3-\cdots-\underline{\alpha}_r$ where the $\underline{\alpha}_i$ are (not necessarily distinct) simple roots,[12] we have $2\Lambda=\sum_{i=1}\underline{\alpha}_i$. Bose and Patera[13] have shown that $\mathfrak{D}^{(\Lambda)}$ is potentially real if r is even, and pseudoreal if r is odd. We note that this implies that for pseudoreal representations there can be no weight M at the origin. However, some potentially real representations also have no root at the origin. An adjoint representation (whose weight diagram is the same as the root diagram for the algebra) is always potentially real. A complete classification of unitary irreducible representations into the three types was first given by Mehta and Srivastava.[14] In Tables I–V we have marked the pseudoreal representations with the subscript Q because of their close relation to quaternions.[15]

The significance of all of this is the following[14]: If the Clebsch-Gordan series of the direct product of two representations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ which are not complex contains a representation $\mathfrak{D}_3$ which also is not complex, then $\mathfrak{D}_3$ is potentially real if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are of the same type. Otherwise $\mathfrak{D}_3$ is pseudoreal. The Kronecker square of a pseudoreal representation can therefore never contain itself. Thus a representation $\underline{R}$ of the form $\{1\}\oplus\mathfrak{D}$ with $\mathfrak{D}$ irreducible and pseudoreal is ruled out by condition (III). This leaves us with five series of representations of A_l, the self-representations of B_l and D_l, some spinor representations of B_3, B_4, D_4, and D_5, and one of the 14-dimensional representations of C_3.

A direct calculation of the Kronecker squares of the self-representations rules all of them out by condition (III) except for the representations {3} and {3*} of SU(3), which have been discussed above. The calculations can be made by using weight-diagram techniques[16] or in this case very simply by using Littlewood's plethysm.[17]

The results for the squares of the self-representations are, in the notation of Eqs. (B6)–(B10) of I,

$$A_l:\ \langle 1,0,\ldots,0\rangle\otimes\langle 1,0,\ldots,0\rangle=[\langle 2,0,\ldots,0\rangle]_{\text{sym}}\oplus[\langle 0,1,0,\ldots,0\rangle]_{\text{antisym}}, \tag{A1}$$

$$B_l \text{ and } D_l:\ \langle 1,0,\ldots,0\rangle\otimes\langle 1,0,\ldots,0\rangle=[\langle 0,0,\ldots,0\rangle\oplus\langle 2,0,0,\ldots,0\rangle]_{\text{sym}}\oplus[\langle 0,1,0,\ldots,0\rangle]_{\text{antisym}}, \tag{A2}$$

$$C_3:\ \langle 1,0,\ldots,0\rangle\otimes\langle 1,0,\ldots,0\rangle=[\langle 2,0,0,\ldots,0\rangle]_{\text{sym}}\oplus[\langle 0,0,\ldots,0\rangle\oplus\langle 0,1,0,\ldots,0\rangle]_{\text{antisym}}. \tag{A3}$$

The representations $\langle 0,0,\ldots,0\rangle$, $\langle 1,0,\ldots,0\rangle$, $\langle 2,0,0,\ldots,0\rangle$, and $\langle 0,1,0,\ldots,0\rangle$ in this notation are the same as the representations ϵ, [1], [2], and $[1^2]$, respectively, in the Young-shape notation used in the column headings of Table I. For each algebra the number of entries between brackets is l. In the case of A_2, $\langle 0,1\rangle$ is the conjugate of $\langle 1,0\rangle=\{3\}$, which is why condition (III) does not rule out $\{1\}\oplus\{3\}\oplus\{3^*\}$.

For the algebras A_l we now have only two series of representations to rule out. An explicit calculation for the $\langle 0,1,0,\ldots,0\rangle$ series gives

$$A_l(l\geq 4):\ \langle 0,1,0,\ldots,0\rangle\otimes\langle 0,1,0,\ldots,0\rangle=[\langle 0,2,0,\ldots,0\rangle\oplus\langle 0,0,0,1,0,\ldots,0\rangle]_{\text{sym}}\oplus[\langle 1,0,1,0,\ldots,0\rangle]_{\text{antisym}}, \tag{A4}$$

$$A_l(l=3):\ \langle 0,1,0\rangle\otimes\langle 0,1,0\rangle=[\langle 0,2,0\rangle\oplus\langle 0,0,0\rangle]_{\text{sym}}\oplus[\langle 1,0,1\rangle]_{\text{antisym}},$$

which in Young's notation is

$$\begin{aligned}&A_l(l\geq 4):\ [1^2]\otimes[1^2]=[2^2]\oplus[1^4]\oplus[2,1^2],\\&A_l(l=3):\ [1^2]\otimes[1^2]=[2^2]\oplus\{1\}\oplus[2,1^2].\end{aligned} \tag{A5}$$

Note that the representations $\langle 0,1,0,\ldots,0\rangle$ do not appear on the right-hand side. The only difficult case here is the six-dimensional representation, $\{6\}=\langle 0,1,0\rangle$, of (A3). The representation $\{1\}\oplus\{4\}\oplus\{6\}\oplus\{4^*\}$ has dimension one less than the representation adjoint $\oplus$ singlet. However, by using

$$\begin{aligned}&\{6\}\otimes\{6\}=[\{1\}\oplus\{20\}]_{\rm sym}\oplus[\{15\}]_{\rm antisym},\\&\{4\}\otimes\{6\}=\{4^*\}\oplus\{20\},\\&\{4\}\otimes\{4\}=[\{10\}]_{\rm sym}\oplus[\{6\}]_{\rm antisym},\end{aligned}\tag{A6}$$

the reader can easily check, in parallel with our arguments above, that it is not possible to construct self-consistent amplitudes for 4-4, 6-6, and 4-6 scatterings.

For the spinor representations of B_3, B_4, D_4, D_5 we have[18]

$$\begin{aligned}&B_3:\ \{8\}_{\rm Sp}\otimes\{8\}_{\rm Sp}=\{1\}\oplus\{21\}\oplus\{35\},\\&B_4:\ \{16\}_{\rm Sp}\otimes\{16\}_{\rm Sp}=\{1\}\oplus\{9\}\oplus\{36\}\oplus\{84\}\oplus\{126\},\\&D_4:\ \{8\}_{\rm Sp}\otimes\{8\}_{\rm Sp}=\{1\}\oplus\{28\}\oplus\{35\},\\&D_4:\ \{8\}_{\rm Sp}\otimes\{8'\}_{\rm Sp}=\{8\}\oplus\{56\},\\&D_5:\ \{16\}_{\rm Sp}\otimes\{16\}_{\rm Sp}=\{10\}\oplus\{120\}\oplus\{126\},\\&D_5:\ \{16\}_{\rm Sp}\otimes\{16^*\}_{\rm Sp}=\{1\}\oplus\{45\}\oplus\{210\}.\end{aligned}\tag{A7}$$

In none of these cases do we have spinor representations appearing on the right-hand side, so they are ruled out. We conjecture that spinor $\otimes$ spinor never contains spinor representations.

Finally we are left with the interesting case of the 14-dimensional representation of C_3. Here we have

$$\{14\}\otimes\{14\}=[\{1\}\oplus\{14\}\oplus\{90\}]_{\rm sym}\oplus[\{21\}\oplus\{70\}]_{\rm antisym}.\tag{A8}$$

If we try $\underline{R}=\{1\}\oplus\{14\}$ we see that there is no antisymmetric trilinear coupling as neither the $\{1\}$ nor the $\{14\}$ couple to themselves antisymmetrically via $\{1\}$ or $\{14\}$.

This proves our assertion.

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[1]C. M. Andersen and J. Yellin, Phys. Rev. D 3, 846 (1971), referred to in the text below as I. A detailed set of references can be found there, as well as an exhaustive set of definitions of the mathematical and physical terminology used here.

[2]S. Mandelstam, Phys. Rev. 184, 1625 (1969).

[3]We use the crossing conventions of C. Rebbi and R. Slansky, Rev. Mod. Phys. 42, 68 (1970), in which the channels are

$(s)\ a+b\to c+d$,
$(t)\ a+\bar c\to \bar b+d$,
$(u)\ a+\bar d\to \bar b+c$.

The details of the procedure for going from Eq. (4) to a relation between coupling constants of the form of Eq. (5), are given in Sec. I of I, and further in C. Schmid and J. Yellin, Phys. Rev. 182, 1449 (1969), Sec. III and the Appendix; Phys. Rev. D 2, 1354(E) (1970).

[4]This requirement guarantees that our bootstrap structure cannot include completely self-consistent scalar intermediate states. (No Pomeranchon is admissible here.) We will return to this point below.

[5]In the notation of Ref. 3, the crossing matrices are

$$\begin{pmatrix}A_s(3^*)\\A_s(6)\end{pmatrix}=\begin{pmatrix}\frac13&\frac43\\\frac13&-\frac23\end{pmatrix}\begin{pmatrix}A_t(1)\\A_t(8)\end{pmatrix}=\begin{pmatrix}-\frac13&-\frac43\\\frac13&-\frac23\end{pmatrix}\begin{pmatrix}A_u(1)\\A_u(8)\end{pmatrix}$$

and

$$X_{tu}=X_{ut}=\begin{pmatrix}\frac13&-\frac83\\-\frac13&-\frac13\end{pmatrix}.$$

If u_a is a $\underline{3}$, then the antisymmetric combination $\epsilon^{cab}u_au_b=u^c$ is a $\underline{3}^*$, and ϵ^{cab} is exactly the Clebsch-Gordan coefficient we need.

[6]The symbol D_s means "discontinuity in s."

[7]In a more approximate, and perhaps more realistic, treatment of the bootstrap constraints, this difficulty, in which a (Pomeranchon) scalar intermediate state is disallowed, can be avoided. See J. Mandula, J. Weyers, and G. Zweig, Ann. Rev. Nucl. Sci. 20, 289 (1970).

[8]In $\{7\}\otimes\{7\}$ the antisymmetric piece contains $\{7\}\oplus\{14\}$ and the symmetric piece contains $\{27\}\oplus\{1\}$.

[9]H.-M. Chan, P. DeCelles, and J. E. Paton, Phys. Rev. Letters 11, 521 (1963); Y. Ne'eman, Nuovo Cimento 33, 133 (1964).

[10]The *self-representations* of the simple classical Lie algebras are the representations associated with the classical groups themselves, e.g., the n-dimensional representation of $\underline{R}(n)$. Usually the self-representation is a lowest-dimensional nontrivial representation of the algebra, but in the case of B_2 and D_3 there are spinor or "double-valued" representations of lower dimension.

[11]E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959), English edition, Chap. 24. Earlier references are A. Loewy, Trans. Am. Math. Soc. 4, 171 (1903) and G. Frobenius and I. Schur, Sitzber. Preuss. Akad. Wiss., Phys.-Math. Kl. 186 (1906).

[12]The *simple roots* $\underline{\alpha}_1,\underline{\alpha}_2,\dots,\underline{\alpha}_l$ are roots which are chosen such that any positive root $\underline{\alpha}$ satisfies $\underline{\alpha}=k_1\underline{\alpha}_1+k_2\underline{\alpha}_2+\cdots+k_l\underline{\alpha}_l$ with non-negative integers k_i.

[13]A. K. Bose and J. Patera, J. Math. Phys. 11, 2231 (1970).

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[16]Reference 1 and C. M. Andersen, J. Math. Phys. 8, 988 (1967).

[17]D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups* (Oxford Univ. Press, Oxford, England, 1950), 2nd ed., Appendix A and references cited therein. This technique has been recently discussed by B. G. Wybourne, *Symmetry Principles and Atomic Spectroscopy* (Wiley-Interscience, New York, 1970), p. 49.

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Analysis of the "Best" Positivity and Analyticity Inequalities on a General, Crossing-Symmetric π^0-π^0 Amplitude*

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Inequalities derived previously from rigorous positivity and analyticity for the π^0-π^0 double partial-wave amplitudes ($a_{n\,l}$) are reexpressed in terms of Roskies's crossing-symmetric parametrization. Using theorems on moment problems, the "best" inequalities for any value of $\sigma = l + n$ are discussed, and the result is expressed as an allowed region of the otherwise unrestricted parameter space. The $\sigma = 4$ and $\sigma = 5$ "best" inequalities are explicitly calculated and compared with results previously given by Roskies for the case $\sigma = 4$. New inequalities relating the partial-wave amplitudes $f_l(s)$ and their derivatives $f_l'(s)$ are derived, valid for all $l \geq 2$, and all $0 \leq s \leq 1$. They are combined into the general analysis and compared with similar conditions. The moment approach is used to examine the nature of the inequalities resulting from an alternative method due to Roskies and to Piguet and Wanders; we discuss the advantages and disadvantages of this approach. Finally, applications are discussed, to show the constraint that is present in these inequalities. The extension of the method to the case of the π-π amplitudes with isospin is given in an Appendix.

I. INTRODUCTION

In this work we continue the study of the constraint placed on possible π^0-π^0 amplitudes by crossing symmetry, positivity, and analyticity using an approach developed previously.[1] As pointed out by Roskies,[2] these constraints, expressed as inequalities involving only a finite number of the Balachandran and Nuyts[3] double partial waves, a_{nl}, are most economically studied by using a crossing-symmetric parametrization of the amplitude, and expressing all the inequalities in terms of the coefficients of this expansion – this ensures that all results will automatically be consistent with crossing symmetry.

The emphasis in this work is on establishing a method to discuss the "best" inequalities that follow from the positivity of the absorptive part, $A(s, t)$, and of its derivative, $dA(s, t)/ds$, when combined with crossing symmetry. This positivity leads to positive combinations of partial waves and their derivatives [the simplest is the familiar $f_l(s) \geq 0$ for $0 \leq s \leq 4\mu^2$, $l \geq 2$], each of which gives rise to an infinite sequence of inequalities on the a_{nl}'s. Many of these inequalities are redundant, and by using theorems on moment problems, we are able to pick out a "best" set of inequalities, necessary and sufficient for the positivity of each combination. In addition, the method organizes the different inequalities, and clarifies the relationship between them, automatically picking out the "best" set of these. We do not have to examine the many redundant inequalities in detail.[4] The analysis is general and works with equal facility for all $l \geq 2$.

Only constraints on the amplitude within the Mandelstam triangle have been considered; in addition, the consequences of full positivity [$\mathrm{Im} f_l(t) \geq 0$ for $t \geq 4\mu^2$ and all l] have only been partially explored. As a consequence of this, our final results are not the most constraining possible.[5]

In Sec. II we establish our notation, review aspects of our previous work,[1] and collect pertinent results on the Balachandran and Nuyts expansion,[3] on Roskies's parametrization of a crossing-symmetric π^0-π^0 amplitude,[6] and on the relation of our inequalities to moment problems. Section III obtains and examines specific inequalities, utilizing only simple positivity, $A(s, t) \geq 0$, and a compari-