Is SU(3) × SU(3) Realized with Goldstone Bosons?*

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The linear σ model is used to study how full chiral symmetry is realized as the $(3,3^*)+(3^*,3)$ interaction is turned off. The Lagrangian which fits the observed spin-zero mass spectrum chooses either a normal realization with degenerate SU(3)×SU(3) multiplets or a Goldstone realization with an octet of massless pseudoscalar mesons, depending on whether the mass term of the Lagrangian is greater or less than a certain critical value. The value of the mass term depends sensitively on the masses of the $I=0$ scalar resonances and present knowledge of these masses is compatible with either type of realization. However, the values of the nucleon σ commutators calculated from meson-nucleon scattering data can resolve this issue. For the normal solution the π-N σ commutator is predicted to be about 75 MeV, whereas for the Goldstone solution the prediction is 30 MeV. The symmetry-breaking parameters are $a=-0.91$ and $b=-0.16$ independent of how full symmetry is achieved.

I. INTRODUCTION

The linear σ model[1] provides a framework for studying various symmetry-breaking schemes. Generally, the model lends support[2] to the proposal[3] that SU(3)×SU(3) symmetry is violated by an interaction of the form

$$L' = \epsilon_0 u_0 + \epsilon_8 u_8 , \qquad (1)$$

where u_0 and u_8 are the $I=0$ scalar members of a $(3,3^*)+(3^*,3)$ representation of operators. In the tree approximation, the known masses of the pseudoscalar mesons can be used to derive predictions for the masses of the scalar mesons which are compatible with the available (although incomplete) evidence for the existence of these scalar resonances. This procedure also gives an evaluation of the parameters

$$a=\epsilon_8/\sqrt{2}\,\epsilon_0 \quad\text{and}\quad b=\langle 0|u_8|0\rangle/\sqrt{2}\langle 0|u_0|0\rangle$$

which respectively measure the ratio of octet to singlet breaking in the interaction (1) and in the vacuum state. The estimates so derived are in approximate agreement with estimates of these ratios based on other considerations.[3,4]

Since these features of the model agree with some of the currently favored ideas concerning the breaking of SU(3)×SU(3), the present work is motivated by the belief that the model can serve as a useful guide to understanding how full chiral symmetry would be achieved if the symmetry-breaking interactions could be turned off. It is well known that full chiral symmetry can be realized either in the normal way with degenerate SU(3) ×SU(3) multiplets or with a number of different configurations of Goldstone bosons.[5]

It is found that the question of how full symmetry is realized depends sensitively on the detailed structure of the Lagrangian. Depending on the size of the mass term in the Lagrangian that fits the observed spin-zero mass spectrum, full symmetry is realized either in the normal way or with an octet of massless pseudoscalar mesons. Furthermore, the evaluation of this mass term with sufficient accuracy to decide between these two possibilities depends on more accurate knowledge of the masses of the two $I=0$ scalar resonances than is now available. Independently of how the symmetry limit is achieved, the symmetry-breaking parameters are determined to be

$$a=-0.91 \quad\text{and}\quad b=-0.16 , \qquad (2)$$

indicating that the interaction (1) is approximately SU(2)×SU(2)-symmetric [SU(2)×SU(2) is exact if $a=-1$] and that the vacuum is approximately SU(3)-symmetric. These results are somewhat surprising since it appears to be commonly believed that if the interaction (1) is approximately SU(2)×SU(2) symmetric, then full symmetry is realized with an octet of massless pseudoscalar bosons.[6]

Because the $I=0$ scalar resonances are very broad, it may not be possible to determine with any certainty how full chiral symmetry is achieved solely on the basis of the observed spin-zero mass spectrum. This issue, however, can be decided[7] from calculation of the σ commutators from $\pi-N$ and $K-N$ scattering data.[8]

In Sec. II, the σ model is presented and the spin-zero mass spectrum is studied. Although a number of authors[1,2] have made similar studies, the procedure followed here is somewhat different in that the parameters a and b and the coupling constants of the Lagrangian are evaluated so as to be compatible with the measured value of $f_K/f_\pi f^+(0)$

and the baryon octet mass splitting, as well as with the spin-zero mass spectrum.

In Sec. III, the model is studied in the SU(3) × SU(3) limit $\epsilon_0 = \epsilon_8 = 0$. Using the values of the coupling constants determined in Sec. II, it is found that if the mass term in the Lagrangian is greater than a certain critical value, full symmetry is realized in the normal way, and that if the mass term is less than this value symmetry is realized with the vanishing of the pseudoscalar octet masses.

Since it is not possible to decide conclusively which of these possibilities is chosen because of lack of knowledge of the $I=0$ scalar masses, in Sec. IV it is shown how the value of the σ commutators calculated from $\pi - N$ and $K - N$ scattering data can be used to settle the issue. For example, if the normal solution is chosen then the $\pi - N\,\sigma$ commutator is predicted to be about 75 MeV, whereas for the Goldstone solution its value is expected to be 30 MeV.

II. LINEAR σ MODEL

The SU(3)×SU(3)-invariant Lagrangian for the spin-zero mesons is taken to be

$$L_M = \tfrac{1}{4}\,\mathrm{Tr}(\partial_\mu M \partial_\mu M^\dagger) - \tfrac{1}{4}\,\alpha\,\mathrm{Tr}(MM^\dagger) - \tfrac{1}{8}\beta[\mathrm{Tr}(MM^\dagger)]^2 - \tfrac{1}{8}\gamma\,\mathrm{Tr}(MM^\dagger MM^\dagger) - \tfrac{1}{4}\delta(\det M + \det M^\dagger)\,, \qquad (3)$$

where

$$M = (u_i + i v_i)\lambda_i$$

and the fields u_i and v_i $(i = 0 \cdots 8)$ form a $(3,3^*) + (3^*,3)$ representation of the chiral group. Conservation of the SU(3)×SU(3) charges is broken by the additional term (1).

In a theory conserving parity and isospin, u_0 and u_8 are the only fields which can develop nonzero vacuum expectation values. These nonzero expectation values split the masses of the 18 mesons through the tadpole mechanism.[9]

After rewriting $L = L_M + L'$ in terms of scalar-meson fields σ_i, defined by

$$\sigma_i = u_i - \omega_i\,, \qquad (4)$$

where $\omega_i = \langle 0 | u_i | 0 \rangle$ (only ω_0 and ω_8 are nonzero in the model considered here), one must apply the equilibrium conditions

$$\left.\frac{\partial L}{\partial \sigma_0}\right|_{\sigma_i = v_i = 0} = 0, \quad \left.\frac{\partial L}{\partial \sigma_8}\right|_{\sigma_i = v_i = 0} = 0\,. \qquad (5)$$

Equations (5) give ϵ_0 and ϵ_8 as functions of ω_0, ω_8, and the coupling constants α, β, γ, and δ. Once the values of these parameters are determined Eq. (4) will be used to determine the values of ω_0 and ω_8 in the symmetry limit $\epsilon_0 = \epsilon_8 = 0$.

In the tadpole approximation the mass matrix is given by the coefficient of the quadratic terms of $L(\sigma_i, v_i)$,

$$(m_s^2)_{ij} = -\left.\frac{\partial^2 L}{\partial \sigma_i \partial \sigma_j}\right|_{\sigma_i = v_i = 0}\,, \qquad (\mu_p^2)_{ij} = -\left.\frac{\partial^2 L}{\partial v_i \partial v_i}\right|_{\sigma_i = v_i = 0}\,. \qquad (6)$$

Explicit expressions for (5) and (6) are given in Appendix A.

The masses of the π, K, and κ mesons can be put into the particularly simple forms

$$\mu_\pi^2 = \frac{\epsilon_0}{\omega_0}\,\frac{1+a}{1+b}\,, \qquad \mu_K^2 = \frac{\epsilon_0}{\omega_0}\,\frac{1-\frac{1}{2}a}{1-\frac{1}{2}b}\,, \qquad m_\kappa^2 = \frac{\epsilon_0}{\omega_0}\,\frac{a}{b}\,. \qquad (7)$$

In this same approximation the corresponding decay constants are[10]

$$f_\pi = (\tfrac{2}{3})^{1/2}\omega_0(1+b)\,, \qquad f_K = (\tfrac{2}{3})^{1/2}\omega_0(1-\tfrac{1}{2}b)\,, \qquad f_\kappa = (\tfrac{3}{2})^{1/2}\omega_0 b. \qquad (8)$$

A. Determination of a and b

From the Glashow-Weinberg relation[2] for the zero-momentum-transfer value of the K_{l3} form factor,

$$f^+(0) = \frac{f_\pi^2 + f_K^2 - f_\kappa^2}{2 f_\pi f_K}\,, \qquad (9)$$

and in Eqs. (8) one finds the relation between the quantity $f_K/f_\pi f^+(0)$ and the parameter b,

$$f_K/f_\pi f^+(0) = (1 - \tfrac{1}{2}b)/(1+b) \qquad (10)$$

$[f^+(0) = 1$ independent of $b]$. From the value $f_K/f_\pi f^+(0) = 1.28 \pm 0.04$ determined from Cabibbo theory,[11] one finds

$$b = -0.16 \pm 0.02\,. \qquad (11)$$

Equations (7) and the experimental values $\mu_\pi^2 = 0.0191\ \mathrm{BeV}^2$ and $\mu_K^2 = 0.246\ \mathrm{BeV}^2$ give

$$a = -0.911 \pm 0.003 \qquad (12)$$

in close agreement with the Gell-Mann-Oakes-Renner[3] value $a = -0.88$. Equation (7) also gives the prediction $m_\kappa = 1.02 \pm 0.05$ BeV.

Before turning to further analysis of the spin-zero mass spectrum, it is interesting to consider how the baryon octet can be included in the model. These considerations shall give additional support to the value of b determined above.

Since the baryon mass term in a Lagrangian is not chiral-invariant, the stipulation that (1) specifies the symmetry-violating interaction implies that in the tadpole approximation all the baryon mass arises through the chiral-invariant Yukawa coupling of the baryon octet to the $(3,3^*)+(3^*,3)$ spin-zero fields. If the baryon-meson Yukawa coupling constant is G_B, then after applying the translation (4) to the Lagrangian one finds that the average baryon mass M_B is proportional to $G_B\omega_0$ and that the scale of the mass splitting relative to M_B is set by the ratio ω_8/ω_0.

An exact formulation of the baryonic Lagrangian is complicated by the fact that the baryon octet belongs to a mixture of chiral representations.[12] However, a quark model with

$$L_q=\bar{q}\partial\!\!\!/ q-G_q(\bar{q}\lambda_i q u_i+\bar{q}\lambda_i\gamma_5 q v_i) \tag{13}$$

can be used to estimate the value of b which produces the observed baryon mass splittings. The quark mass matrix M_q is given by

$$M_q=G_q\omega_i\lambda_i \tag{14}$$

so that the nonstrange- to strange-quark mass ratio is

$$M_p/M_\lambda=(1+b)/(1-2b)\,. \tag{15}$$

If one assumes that the octet baryons are built of three nonrelativistic quarks, then the observed splitting gives $M_p/M_\lambda\approx 0.62$, implying $b\approx -0.17$ in agreement with (11).

B. Spin - Zero Mass Spectrum

With b given by (11) one can see from Eqs. (A3)–(A7) that the squares of the pseudoscalar masses μ_π^2, μ_K^2, μ_η^2, and μ_χ^2 depend on the three additional parameters, $\alpha+2\beta(\omega_0^2+\omega_8^2)$, $\gamma\omega_0^2$, and $\delta\omega_0$. Although a number of alternatives are available for evaluating these parameters from the experimental data, here the procedure has been adopted which is simplest from the computational point of view. The known values of μ_π^2, μ_K^2, and $\mu_\chi^2+\mu_\eta^2$ have been used to fix the parameters and the resulting predictions for $\mu_\chi^2-\mu_\eta^2$, and the scalar masses $m_{\pi'}$ and m_κ are given in Table I for values of b in the range (11).

For $b=-0.16$, $m_\chi^2-m_\eta^2$ is predicted to within 2% of the observed value, while the predicted values $m_\kappa=1020$ MeV and $m_{\pi'}=960$ MeV are in rough agreement with the existing evidence for these scalar resonances.

For values of b in the range (10) the $\chi-\eta$ mass difference is predicted to be on the order of 10 MeV greater than the observed mass difference. This discrepancy could be removed by including in the symmetry-breaking interaction L' a term which is bilinear in the $(3,3^*)+(3^*,3)$ spin-zero fields.

The masses of the two $I=0$ scalar mesons, m_σ and $m_{\sigma'}$, depend on knowledge of the additional parameter α. Since the masses of the σ and σ' are not accurately known, it is not possible to make an accurate determination of α. In Table II, the values for m_σ and $m_{\sigma'}$ are listed for $\alpha=+0.05$, 0, -0.05 BeV^2 with $b=-0.16$. The value $\alpha=0$ leads to the plausible results $m_\sigma\approx 680$ MeV and $m_{\sigma'}\approx 1170$ MeV.[13]

Finally, it should be noted that Lévy in the first extension of the σ model to SU(3)×SU(3) (Ref. 1) and many other authors[2] have found similar fits to the spin-zero mass spectrum. The procedure followed by many of these authors is to use the four known pseudoscalar masses to determine the coupling constants and the value of b. This approach gives $b=-0.21$ (for example, see Carruthers and Haymaker, Ref. 2) which leads to less satisfactory agreement with the measured value of $f_K/f_\pi f^+(0)$ and the observed baryon mass splitting although it avoids the small discrepancy in the $\chi-\eta$ mass difference found here. None of the discussion in Secs. III and IV, however, would be significantly affected by such a change in the value of b.

III. SU(3) × SU (3) LIMIT

Equations (A1) and (A2) resulting from the equilibrium conditions (5) can be used to study how the meson Lagrangian realizes full SU(3)×SU(3) symmetry in the limit $\epsilon_0=\epsilon_8=0$.[14] For $b=-0.16$ the analysis of Sec. II gives

TABLE I. For values of b in the range (11), the known values of μ_π^2 and μ_K^2 have been used to obtain predictions for a, $f_K/f_\pi f^+(0)$, and m_κ. Using $\mu_\chi^2+\mu_\eta^2$ as additional input, one obtains predictions for $\mu_\chi^2-\mu_\eta^2$ and $m_{\pi'}$.

	b	a	$f_K/f_\pi f^+(0)$	m_κ (BeV)	$\mu_\chi^2-\mu_\eta^2$ (BeV^2)	$m_{\pi'}$ (BeV)
Experiment	···	···	1.28 ± 0.04	1.1 (?)	0.616	0.96–1.00 (?)
Theory	−0.14	−0.908	1.24	1.08	0.636	1.00
	−0.16	−0.911	1.29	1.02	0.630	0.96
	−0.18	−0.914	1.33	0.97	0.625	0.94

TABLE II. With $b = -0.16$ the predicted values of the two $I=0$ scalar resonances are listed for different values of α. The σ has been observed with a mass around 700 MeV and a width greater than 100 MeV.

α (GeV^2)	m_σ (MeV)	$m_{\sigma'}$ (MeV)
0.05	635	1150
0.00	680	1170
−0.05	715	1190

$$\alpha + 2\beta\omega_0^2(1+2b^2) = 0.240\,\text{BeV}^2\,,$$
$$2\gamma\omega_0^2 = 0.495\,\text{BeV}^2\,, \tag{16}$$
$$(\tfrac{2}{3})^{1/2}\delta\omega_0 = -0.256\,\text{BeV}^2\,.$$

Since the value of α cannot be accurately determined without more precise knowledge of the $I=0$ scalar masses, its value will be assumed to lie in the range

$$0.05 \geq \alpha \geq -0.05\,\text{BeV}^2\,. \tag{17}$$

If ω_0^s and ω_8^s are used to designate the value of $\langle 0|\mu_0|0\rangle$ and $\langle 0|\mu_8|0\rangle$, respectively, in the limit $\epsilon_0 = \epsilon_8 = 0$, and χ and b_s are defined by

$$\chi = \omega_0^s/\omega_0\,,$$
$$b_s = \omega_8^s/\sqrt{2}\,\omega_0^s\,, \tag{18}$$

then Eqs. (A1) and (A2) yield several solutions for χ and b_s. However, the requirement that the symmetry limit should correspond to a physically acceptable solution with a non-negative mass spectrum serves to eliminate some of the possibilities.

In Appendix B, the equations determining the solutions for χ and b_s are recorded along with expressions for the masses in each different case. The results can be summarized as follows:

(i) $b_s = 0$, $\chi \neq 0$. χ is determined by Eq. (B2). For $\alpha \leq \alpha_c = 0.047$ BeV^2 the mass spectrum is non-negative for the larger of the two roots of (B2). The vacuum is SU(3)-symmetric, and there is an octet of pseudoscalar Goldstone bosons. For $\alpha > \alpha_c$ the roots of (B2) are complex.

(ii) $b_s = -1$, $\chi \neq 0$. χ is determined by Eq. (B4). The vacuum is SU(2)×SU(2)-symmetric and the mass spectrum breaks up into SU(2)×SU(2) multiplets. With $\gamma > 0$ at least one of the multiplets μ_π^2, m_-^2, or $m_{\pi'}^2$, μ_-^2 has a negative mass.

(iii) $b_s = 2$, $\chi \neq 0$. χ is determined by Eq. (B6). For $\alpha > 0$ there is no acceptable limit, but for $\alpha \leq 0$ there is an octet of massless mesons and a non-negative mass spectrum for the negative value of χ.

(iv) χ and b_s are determined by the pair of Eq. (B8). For $\gamma > 0$, this solution can be rejected since it gives $\mu_+^2 < 0$.

(v) $\omega_0^s = 0$, $\omega_8^s \neq 0$. The value of ω_8^s is determined by Eq. (B10). These equations are consistent for $\alpha \approx 5.2$ BeV^2 which is well outside the range (17). Moreover, μ_-^2 and $m_{\pi'}^2$ are negative for $\gamma > 0$.

(vi) $\omega_0^s = \omega_8^s = 0$. This is the normal realization of symmetry with degenerate SU(3)×SU(3) multiplets. For $\alpha \geq 0$ it possesses a non-negative mass spectrum.

The conclusion is that for $\alpha < 0$ there is an acceptable solution for $b_s = 0$ and another for $b_s = 2$. Both of these are characterized by an SU(3)-symmetric vacuum and an octet of massless mesons, although the SU(3) group is different in each case. For $\alpha \geq 0$ there is normal solution, and for $\alpha_c \geq \alpha \geq 0$ there is also a Goldstone solution with $b_s = 0$. Since for a given value of α the requirement of a non-negative mass spectrum does not lead to a unique solution, it is necessary to use equations (A1) and (A2) to learn which of the possibilities corresponds to the symmetry limit which is reached from the physical values of ϵ_0 and ϵ_8.

Although there are an infinite number of paths between the physical point and the symmetry point in the $\epsilon_0 - \epsilon_8$ plane, the simplifying assumption will be made that the interaction (1) is turned off by keeping $a = -0.91$ and letting ϵ_0 approach zero. One can study how χ and b_s vary as ϵ_0 is decreased to zero by eliminating ϵ_0 and ϵ_8 from (A1) and (A2). The resulting equation is

$$A\chi^2 + B\chi + C = 0 \tag{19}$$

with

$$A = 2\beta\omega_0^2(1+2b_s^2)(b_s - a) + 2\gamma\omega_0^2\left[3b_s(1-b_s+b_s^2) - \tfrac{1}{3}a(1+6b_s^2-2b_s^3)\right]\,,$$
$$B = -(\tfrac{2}{3})^{1/2}\,\delta\omega_0(1+b_s)\left[b_s + a(1-b_s)\right]\,, \tag{20}$$
$$C = \alpha(b_s - a)\,.$$

One finds from (19) and (20) that if $\alpha \leq \alpha_c$, then the Lagrangian traces a path from the physical point $\chi = 1$, $b_s = -0.16$ to the symmetry point with $b_s = 0$ and χ determined by the larger root of (B2). If $\alpha > \alpha_c$, then the path moves from the physical point to the normal solution $\chi = 0$, $b_s = a$.

The final conclusion then is that for $\alpha > 0.047$ BeV^2, the Lagrangian determined in Sec. II chooses the normal symmetry limit with degenerate SU(3) ×SU(3) multiplets and that for $\alpha \leq 0.047$ BeV^2 symmetry is realized with an octet of Goldstone bosons.

Although the experimental data for m_σ slightly favor a value for α corresponding to the Goldstone limit, because of the broadness of this resonance the normal realization of chiral symmetry cannot be excluded. The present considerations indicate that the commonly held belief that the pseudoscalar mesons are Goldstone bosons is only an assumption and not a conclusion that can be deduced from the observed mass spectrum. In Sec. IV it will be shown that the question of how SU(3)×SU(3) is realized can be decided on the basis of σ-commutator matrix elements which can be calculated from meson-nucleon scattering data.

Before proceeding to this subject, a few observations will be noted concerning the size of different parameters which characterize the σ model. From the experimental value $f_\pi = 95$ MeV, one finds that $\omega_0 = 139$ MeV. Equations (16) then indicate that the two dimensionless coupling constants are $\beta \approx 6$ and $\gamma \approx 13$. The trilinear coupling constant δ is about −2.3 BeV. In the SU(3)×SU(3) limit the meson spectrum is characterized by a small constant with the dimension of mass. For the normal realization of symmetry this constant is the degenerate mass of the spin-zero mesons and is on the order of 200–300 MeV. In the case of the Goldstone realization, the characteristic constant is ω_0^s. Its magnitude is set by the ratio δ/γ and it is calculated to be about 100 MeV.

IV. NUCLEON σ COMMUTATORS

From π-N and K-N scattering data one can derive estimates for the σ-commutator matrix elements. These are defined by

$$\sigma_{\pi N} = \langle N | [Q_3^5, [Q_3^5, H']] | N\rangle$$

and (21)

$$\sigma_{KN} = \langle N | [Q_4^5, [Q_4^5, H']] | N\rangle ,$$

where H' is the interaction which violates SU(3) ×SU(3) symmetry invariance. If H' is determined by the interaction (1), then these double commutators are given by the matrix elements

$$\sigma_{\pi N} = -\left(\frac{\sqrt{2}\,\epsilon_0 + \epsilon_8}{\sqrt{3}}\right) \left\langle N \left| \frac{\sqrt{2}\,u_0 + u_8}{\sqrt{3}} \right| N\right\rangle$$

and (22)

$$\sigma_{KN} = -\left(\frac{2\sqrt{2}\,\epsilon_0 - \epsilon_8}{2\sqrt{3}}\right) \left\langle N \left| \frac{2\sqrt{2}\,u_0 - u_8}{2\sqrt{3}} \right| N\right\rangle .$$

In order to derive a theoretical prediction for the σ terms it is necessary to make estimates of the matrix elements $\epsilon_0\langle N | u_0 | N\rangle$ and $\epsilon_8\langle N | u_8 | N\rangle$. From the model of the baryon mass outlined in Sec. II, it follows that the observed octet splitting of the baryons is due to the term $\epsilon_8\mu_8$ in (1) and hence one has

$$-\epsilon_8\langle N | u_8 | N\rangle \approx -200 \text{ MeV}. \qquad (23)$$

To determine the contribution of the singlet term $\epsilon_0\mu_0$ of (1) to the average baryon mass M_B, one must know the value of the baryon mass in the SU(3)×SU(3) limit. In this limit M_B is proportional to $G_B\omega_0^s$. If symmetry is realized in the normal manner, then $\omega_0^s = 0$ and the baryon mass vanishes while for the Goldstone realization ω_0^s can be determined from (B2) to be $0.64\omega_0$ (for $\alpha = 0$).

Taking 1150 MeV as the average mass of the physical baryons then gives

$$-\epsilon_0\langle N | u_0 | N\rangle \approx 1150 \text{ MeV} \qquad (24a)$$

if symmetry is realized in the normal way, and

$$-\epsilon_0\langle N | u_0 | N\rangle \approx 400 \text{ MeV} \qquad (24b)$$

if symmetry is realized with Goldstone bosons.

With $a = -0.91$ one then has the predictions

$$\sigma_{\pi N} \approx 75 \text{ MeV}, \quad \sigma_{KN} \approx 1065 \text{ MeV}, \qquad (25a)$$

for the normal realization, and

$$\sigma_{\pi N} \approx 30 \text{ MeV}, \quad \sigma_{KN} \approx 345 \text{ MeV}, \qquad (25b)$$

for the Goldstone realization. The prediction for $\sigma_{\pi N}$ is sensitive to the quantity $1 + a$ so that the 25% difference between the results quoted here and those of Ref. (7) is due to the use of the Gell-Mann–Oakes–Renner value $a = -0.88$ in the latter case. In addition, uncertainties in the estimates (23) and (24) can introduce uncertainties in the predictions for $\sigma_{\pi N}$ and σ_{KN}. Qualitatively, however, it is clear that if symmetry is realized in the normal way, then the σ commutator matrix elements are expected to be two or three times larger than if symmetry is achieved with an octet of Goldstone mesons.

The situation concerning the determination of $\sigma_{\pi N}$ and σ_{KN} from scattering data is unclear at the present time.[8] von Hippel and Kim have used K-N scattering data to calculate $-\epsilon_0\langle N | u_0 | N\rangle \approx 215$ MeV (implying $\sigma_{\pi N} \approx 20$ MeV) and their result, therefore, favors the Goldstone solution. Using a different calculational method Cheng and Dashen have analyzed π-N data and found $\sigma_{\pi N} \approx 100$ MeV which favors the normal realization of symmetry. A calculation by Höhler *et al.* using π-N data gives $\sigma_{\pi N} \approx 40$ MeV while Ericson and Rho found $\sigma_{\pi N} \approx 34$ MeV using π-nuclear scattering data.

V. SUMMARY

In the tree approximation the linear σ model with a $(3, 3^*) + (3^*, 3)$ symmetry-breaking interaction gives a reasonable picture of the hadrons. If the parameters a and b are determined from the measured values of μ_π^2, μ_K^2, and $f_K/f_\pi f^+(0)$, good predictions result for the baryon mass splitting and the κ meson mass. Using $\mu_\chi^2 + \mu_\eta^2$ as additional input gives good predictions for $\mu_\chi^2 - \mu_\eta^2$ and for the scalar π' mass. If one is willing to ignore the unknown effect of all the higher-order terms, these results are fairly impressive.

Turning to the question of how the σ-model Lagrangian realizes SU(3)×SU(3), it was found that either a normal realization or a Goldstone realization with an octet of massless pseudoscalar mesons is compatible with present knowledge of the spin-zero mass spectrum. The choice between these two possibilities was found to depend primarily on more accurate knowledge of the $I=0$ scalar-meson masses. This information will be extremely difficult to obtain because of the very large widths of these resonances.

Many authors have favored the Goldstone realization on the basis of a more intuitive evaluation of the hadronic mass spectrum. Since the results of this study show that the presence of a small positive bare-mass term has little effect on the hadron masses, but can make the Lagrangian choose the normal realization of chiral symmetry, the issue cannot be resolved solely on the basis of the observed masses.

In Sec. IV, it was shown that the nucleon σ commutators provide a more clear-cut and less model-dependent method of deciding between these two possiblilities. The normal solution predicts matrix elements two or three times larger than one would expect if the Goldstone solution prevails.

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APPENDIX A

Equations (5) have the explicit form

$$\epsilon_0 = \alpha\omega_0 + 2\beta\omega_0^3(1+2b^2) + \tfrac{2}{3}\gamma\omega_0^3(1+6b^2-2b^3) + (\tfrac{2}{3})^{1/2}\delta\omega_0^2(1-b^2), \quad \text{(A1)}$$

$$\epsilon_8 = \alpha\omega_8 + 2\beta\omega_0^2\omega_8(1+2b^2) + 2\gamma\omega_0^2\omega_8(1-b+b^2) - (\tfrac{2}{3})^{1/2}\delta\omega_0\omega_8(1+b). \quad \text{(A2)}$$

The pseudoscalar masses have the form

$$\mu_\pi^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + \tfrac{2}{3}\gamma\omega_0^2(1+b)^2 + (\tfrac{2}{3})^{1/2}\delta\omega_0(1-2b), \quad \text{(A3)}$$

$$\mu_K^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + \tfrac{2}{3}\gamma\omega_0^2(1-b+7b^2) + (\tfrac{2}{3})^{1/2}\delta\omega_0(1+b), \quad \text{(A4)}$$

$$\mu_8^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + \tfrac{2}{3}\gamma\omega_0^2(1-2b+3b^2) + (\tfrac{2}{3})^{1/2}\delta\omega_0(1+2b), \quad \text{(A5)}$$

$$\mu_0^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + \tfrac{2}{3}\gamma\omega_0^2(1+2b^2) - 2(\tfrac{2}{3})^{1/2}\delta\omega_0, \quad \text{(A6)}$$

$$\mu_{08}^2 = \sqrt{2}\left[\tfrac{2}{3}\gamma\omega_0^2(2b-b^2) + \tfrac{2}{3}\delta\omega_0 b\right]. \quad \text{(A7)}$$

The scalar masses have the form

$$m_{\pi'}^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + 2\gamma\omega_0^2(1+b)^2 - (\tfrac{2}{3})^{1/2}\delta\omega_0(1-2b), \quad \text{(A8)}$$

$$m_\kappa^2 = \alpha + 2\beta\omega_0^2(1+2b^2) + 2\gamma\omega_0^2(1-b+b^2) - (\tfrac{2}{3})^{1/2}\delta\omega_0(1+b), \quad \text{(A9)}$$

$$m_8^2 = \alpha + 2\beta\omega_0^2(1+6b^2) + 2\gamma\omega_0^2(1-2b+3b^2) - (\tfrac{2}{3})^{1/2}\delta\omega_0(1+2b), \quad \text{(A10)}$$

$$m_0^2 = \alpha + 2\beta\omega_0^2(3+2b^2) + 2\gamma\omega_0^2(1+2b^2) + 2(\tfrac{2}{3})^{1/2}\delta\omega_0, \quad \text{(A11)}$$

$$m_{08}^2 = \sqrt{2}\left[4\beta\omega_0^2 b + 2\gamma\omega_0^2(2b-b^2) - \tfrac{2}{3}\delta\omega_0 b\right]. \quad \text{(A12)}$$

APPENDIX B

The solutions of (A1) and (A2) are discussed in the SU(3)×SU(3) limit $\epsilon_0=\epsilon_8=0$. For each of the different solutions for χ and b_s the resulting mass spectrum is determined from (A3)–(A12). In the cases where the $I=0$ masses can be diagonalized in terms of rational expressions, these masses are determined from the equations

$$\mu_\pm^2 = \{\mu_0^2 + \mu_8^2 \pm [(\mu_0^2-\mu_8^2)^2 + 4\mu_{08}^4]^{1/2}\}/2, \qquad m_\pm^2 = \{m_0^2 + m_8^2 \pm [(m_0^2-m_8^2)^2 + 4m_{08}^4]^{1/2}\}/2. \quad \text{(B1)}$$

Otherwise the nondiagonal matrix elements are recorded.

(i) $b_s=0$, $\chi\neq 0$. χ is determined by

$$\alpha + (2\beta + \tfrac{2}{3}\gamma)\omega_0^2\chi^2 + (\tfrac{2}{3})^{1/2}\delta\omega_0\chi = 0. \quad \text{(B2)}$$

The masses are then given by

$$\mu_\pi^2 = \mu_K^2 = \mu_8^2 = 0, \quad m_{\pi'}^2 = m_\kappa^2 = m_8^2 = \tfrac{4}{3}\gamma\omega_0^2\chi^2 - 2(\tfrac{2}{3})^{1/2}\delta\omega_0\chi, \quad \mu_0^2 = -3(\tfrac{2}{3})^{1/2}\delta\omega_0\chi, \quad m_0^2 = -2\alpha - (\tfrac{2}{3})^{1/2}\delta\omega_0\chi. \quad \text{(B3)}$$

Since $\delta\omega_0<0$, $\gamma>0$, and $\beta>0$, the scalar octet masses and μ_0^2 are non-negative if $\chi \geq 0$. $m_0^2 \geq 0$ for the larger of the two roots of (B2).

The solutions to (B2) are complex if $\alpha > \alpha_c = 0.047$ BeV^2. Therefore, for $\alpha \leq \alpha_c$ the larger of the two roots of (B2) is the only acceptable solution.

(ii) $b_s=-1$, $\chi\neq 0$, χ is determined by

$$\alpha + (6\beta + 6\gamma)\omega_0^2\chi^2 = 0 \quad \text{(B4)}$$

and the mass spectrum is

$$\begin{aligned}
&\mu_K{}^2 = m_\kappa{}^2 = 0\,,\\
&\mu_\pi{}^2 = m_-{}^2 = 3(\tfrac{2}{3})^{1/2}\delta\omega_0\chi - 6\gamma\omega_0{}^2\chi^2\,,\\
&m_{\pi'}{}^2 = \mu_+{}^2 = -3(\tfrac{2}{3})^{1/2}\delta\omega_0\chi - 6\gamma\omega_0{}^2\chi^2\,,\\
&\mu_-{}^2 = 0\,,\\
&m_+{}^2 = -2\alpha\,.
\end{aligned} \tag{B5}$$

SU(2)×SU(2) is an exact symmetry of the vacuum and the mass spectrum breaks up into SU(2)×SU(2) multiplets. Since $\gamma > 0$ at least one of the pairs $\mu_\pi{}^2$, $m_-{}^2$ or $m_{\pi'}{}^2$, $\mu_+{}^2$ has a negative mass.

(iii) $b_s = 2$, $\chi \neq 0$. χ is determined by

$$\alpha + (18\beta + 6\gamma)\omega_0{}^2\chi^2 - 3(\tfrac{2}{3})^{1/2}\delta\omega_0\chi = 0 \tag{B6}$$

and the mass spectrum is

$$\begin{aligned}
&\mu_\pi{}^2 = m_\kappa{}^2 = \mu_-{}^2 = 0\,,\\
&m_{\pi'}{}^2 = \mu_K{}^2 = m_-{}^2 = 12\gamma\omega_0{}^2\chi^2 + 6(\tfrac{2}{3})^{1/2}\delta\omega_0\chi\,,\\
&\mu_+{}^2 = 9(\tfrac{2}{3})^{1/2}\delta\omega_0\,,\\
&m_+{}^2 = -2\alpha\,.
\end{aligned} \tag{B7}$$

For $\alpha \leq 0$ the negative solution of (B6) gives an allowable spectrum. The vacuum is symmetric under the SU(3) group generated by $Q_{1,2,3}$, $Q^5_{4,5,6,7}$, and Q_8 [called "chimeral" SU(3) by Mathur *et al.* in Ref. 4] and there is an octet of massless bosons.

(iv) χ and b_s are determined by

$$\begin{aligned}
&\tfrac{2}{3}\gamma\omega_0{}^2\chi^2(1-2b_s) - (\tfrac{2}{3})^{1/2}\delta\omega_0\chi = 0\,,\\
&\alpha + [2\beta(1+2b_s{}^2) + \tfrac{2}{3}\gamma(2 - 2b_s + 5b_s{}^2)]\omega_0{}^2\chi^2 = 0\,.
\end{aligned} \tag{B8}$$

The mass spectrum is

$$\begin{aligned}
&\mu_-{}^2 = \mu_\pi{}^2 = \mu_K{}^2 = m_\kappa{}^2 = 0\,,\\
&m_{\pi'}{}^2 = 4\gamma\omega_0{}^2\chi^2(2b_s - b_s{}^2)\,,\\
&\mu_+{}^2 = -2\gamma\omega_0{}^2\chi^2[(1-b_s)^2 + 2b_s{}^2)]\,,\\
&m_8{}^2 = 8\beta\omega_0{}^2\chi^2 b_s{}^2 + \tfrac{2}{3}\gamma\omega_0{}^2\chi^2(-4b_s + 8b_s{}^2)\,,\\
&m_0{}^2 = 4\beta\omega_0{}^2\chi^2 + \tfrac{2}{3}\gamma\omega_0{}^2\chi^2(3 - 2b_s + b_s{}^2)\,,\\
&m_{08}{}^2 = \sqrt{2}\,[4\beta\omega_0{}^2\chi^2 b_s + \tfrac{2}{3}\gamma\omega_0{}^2\chi^2(5b_s + b_s{}^2)]\,.
\end{aligned} \tag{B9}$$

This solution can be ruled out since $\mu_+{}^2$ is negative for $\gamma > 0$.

(v) $\chi = 0$, $\omega_8^s \neq 0$. If y is defined by $y = \omega_8{}^s/\sqrt{2}\,\omega_0$, then y is determined by the equations

$$\begin{aligned}
&\alpha + (4\beta + \tfrac{10}{3}\gamma)\omega_0{}^2 y^2 = 0\,,\\
&\tfrac{4}{3}\gamma\omega_0{}^2 y^2 + (\tfrac{2}{3})^{1/2}\delta\omega_0 y = 0.
\end{aligned} \tag{B10}$$

The mass spectrum is

$$\begin{aligned}
&\mu_+{}^2 = \mu_\pi{}^2 = \mu_K{}^2 = m_\kappa{}^2 = 0\,,\\
&\mu_-{}^2 = -6\gamma\,\omega_0{}^2 y^2\,,\\
&m_{\pi'}{}^2 = -4\gamma\omega_0{}^2 y^2\,,\\
&m_8{}^2 = (8\beta + \tfrac{16}{3}\gamma)\omega_0{}^2 y^2\,,\\
&m_0{}^2 = \tfrac{2}{3}\gamma\omega_0{}^2 y^2\,,\\
&m_{08}{}^2 = -\tfrac{1}{3}\sqrt{2}\,(2\gamma\omega_0{}^2 y^2)\,.
\end{aligned} \tag{B11}$$

Equations (B10) are consistent only for $\alpha \approx 5.2$ BeV^2. $\mu_-{}^2$ and $m_{\pi'}{}^2$ are negative for $\gamma > 0$.

(vi) $\omega_0^s = \omega_8^s = 0$. All the masses assume the value α. This is the normal realization of symmetry with an SU(3)×SU(3)-symmetric vacuum.

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[1]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960); M. Lévy, *ibid.* 52A, 23 (1967).

[2]For a review of chiral Lagrangians, see S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969). Some more recent studies are: A. K. Bhargava and T. Dass, Phys. Rev. D 1, 649 (1970); L. M. Brown and H. Munczek, *ibid.* 1, 2595 (1970); Y. P. Lam and Y. Y. Lee, *ibid.* 2, 2976 (1970); J. Schechter and Y. Ueda, *ibid.* 3, 168 (1971); P. Carruthers and R. W. Haymaker, Cornell Report No. CLNS-158, 1971 (unpublished).

[3]M. Gell-Mann, Phys. Rev. 125, 1067 (1962); S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[4]P. R. Auvil and N. G. Deshpande, Phys. Rev. 185, 2043 (1969); S. Okubo, *ibid.* 188, 2300 (1969); V. S. Mathur, S. Okubo, and J. Subba Rao, Phys. Rev. D 1, 2058 (1970).

[5]J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[6]For example, see M. Gell-Mann, R. J. Oakes, and B. Renner (Ref. 3) or R. Dashen, Phys. Rev. 183, 1245 (1969).

[7]R. Olshansky, Phys. Rev. D 4, 1244 (1971).

[8]F. von Hippel and J. Kim, Phys. Rev. D 1, 151 (1970); T. P. Cheng and R. Dashen, Phys. Rev. Letters 26, 594 (1971); G. Höhler, H. P. Jakob, and R. Strauss, University of Karlsruhe report, 1971 (unpublished); M. Ericson and M. Rho, CERN Report No. CERN-Th-1350, 1971 (unpublished).

[9]S. Coleman and S. L. Glashow, Phys. Rev. 134, B671 (1964).

[10]$f_\pi = 95$ MeV.

[11]N. Cabibbo, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (California Univ. Press, Berkeley, Calif., 1967); N. Brene, M. Roos, and A. Sirlin, Nucl. Phys. B6, 255 (1968).

[12]H. Harari, Phys. Rev. Letters 16, 964 (1966).

[13]If $\alpha = 0$ then both scale invariance and the conservation of the ninth axial charge, Q_0^5, are violated only by the trilinear term in the SU(3)× SU(3) limit.

[14]P. Carruthers and R. W. Haymaker (Ref. 2) have carried out a similar analysis for the case $\alpha = 0$.