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Gibbs Paradox in Particle Physics

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An attempt to understand the structure of particles by means of statistical mechanics is presented. This is accomplished by introducing the Gibbs paradox into particle physics. An interacting-particle model results with level density $\rho(m)$ in the range determined by $[\rho(m)]^{-1}=\phi(e^{-\beta m})$, $\beta>0$, and $\rho(m)\le e^{\gamma m \ln m}$, $\gamma>0$, as $m\to\infty$. This result is to be contrasted to existing free-particle models which give an exponentially increasing mass spectrum.

I. INTRODUCTION

In present theories of interacting fields the structure of particles does not take a primary role. It usually arises from these theories by beginning with point particles, and then the effect of the interaction gives the particle a structure through functions like the form factors.

Recently, there has been a great deal of interest in physical processes at very high energies. In fact, some rather puzzling experimental results have given rise to a large number of theoretical ideas as possible explanations of these high-energy processes.[1] It is clear that at these energies the fundamental structure of particles becomes a central issue and is directly linked to these very energetic processes.

The structure of matter has also great implications for understanding the early stages of the universe. Hagedorn,[2] for instance, obtains a highest possible temperature of the universe ($T_0 = 160$ MeV) during the earliest moments of the "big bang."

In the present work a statistical mechanics model for the structure of particles is developed. It resembles somewhat results obtained by Hagedorn in his statistical thermodynamics theory of strong interactions at high energies.[3] These results of Hagedorn recently have received considerable attention since one can obtain the same result from dual-resonance models.[4]

II. THE GIBBS PARADOX

Let us review the Gibbs paradox in statistical mechanics and, thus, show why it is relevant to the structure of a particle.

Classical statistical mechanics is based on the notion that particles are truly distinguishable. This basic assumption determines the number of microstates which correspond to one specific macrostate (given total energy, total number of particles, etc.). For instance, if there exist N particles in cells, with the $\vec{p}$th cell having $g(\vec{p})$ levels, then the total number of ways to obtain the macrostate with $n(\vec{p})$ particles in the $\vec{p}$th cell is given by

$$\Omega\{n(\vec{p})\} = N!\prod_{\vec{p}} \frac{[g(\vec{p})]^{n(\vec{p})}}{n(\vec{p})!} . \tag{1}$$

The entropy of the system is defined by

$$S \equiv k \ln\Omega\{n(\vec{p})\} . \tag{2}$$

If one maximizes the entropy for given N and E [with $\sum_{\vec{p}} n(\vec{p})=N$ and $\sum_{\vec{p}} n(\vec{p})E(\vec{p})=E$], one obtains

$$S = Nk\ln V + \tfrac{3}{2}Nk\ln T + Nk\ln[(2\pi mk/h^2)^{3/2}] + \tfrac{3}{2}Nk. \tag{3}$$

The above expression for S does not satisfy the third law of thermodynamics ($S\to 0$ as $T\to 0$) since it is only valid for higher temperatures (and, also, finite-mass particles). However, the truly unacceptable feature of this result for S is that the entropy is not an extensive parameter, that is, if $N\to\alpha N$ and $V\to\alpha V$, with $\alpha>0$, then S does not become αS at it should. Nevertheless, if $N\to\alpha N$, with V fixed, then $S\to\alpha S$. This peculiar feature of (3), strange for ordinary thermodynamic systems, is what makes it applicable to the constitution of a particle as a gas of quasiparticles (in equilibrium). This may be seen by considering the vapor-pressure formula. If λ is the heat of evaporation per particle then

$$N\lambda/T = Nk\ln V + \tfrac{3}{2}Nk\ln T + Nk\ln[(2\pi mk/h^2)^{3/2}] + \tfrac{3}{2}Nk. \tag{4}$$

One gets the amazing result, by canceling N on both sides of (4), that the heat of evaporation per particle is independent of the number of particles in the system and depends *only* on the volume of the system. In the words of Schrödinger,[5] "what is then determined (given the temperature) is not the vapor *pressure*, but the vapor *volume*, the absolute volume of the vapor, independent of the number N of particles it contains. Given this "correct" volume any amount of liquid could evaporate into it, or vice versa, without disturbing the equilibrium!" This behavior fits the description of the structure of a particle. First, there exists a *unique* volume for which there is an equilibrium between the liquid and the gas phases. Secondly, the number of quasiparticles in this unique volume is arbitrary, that is, the constituents of the particle may be *any* number of quasiparticles. Finally, if the temperature T is bounded from above by T_0 (the boiling point of hadronic matter), as seems to be required by the bootstrap condition of Hagedorn, the volume V cannot be arbitrarily small, so that a maximum hadronic temperature seems to imply the absence of point particles.

The above discussion of the Gibbs paradox makes it clear that in describing the structure of a particle by statistical mechanics, one must abandon the fundamental contribution of Gibbs to statistical mechanics, that is, one must not divide (1) by $N!$ Hence, the paradox which arises in the description of an ordinary gas ceases to be a paradox when describing a gas of quasiparticles constituting a physical particle.

III. PARTICLE STRUCTURE

The model one wishes to propose to study the structure of particles is that suggested in the preceding section. A physical particle is composed of a gas cloud (in equilibrium) of all other particles in interaction. Scattering between physical particles is described by the gas clouds going away from equilibrium. In this way, particle physics is reduced to the study of the statistical mechanics of localizable equilibrium and nonequilibrium states. The question, of course, arises as how to describe this complicated interacting gas of particles. The notion of quasiparticles will be used to describe some of the collective interactions of the particles in the system. Hopefully, this will give most of the important features of a particle.

Traditionally, quasiparticles have been used in many-body theories for systems of infinite extent. However, one very important feature of the interacting system describing a particle is that it should be confined (localized in ordinary space). Therefore, the question arises if the notion of a quasiparticle is sufficiently general to describe a localized system. The surprising result is that if the Gibbs paradox is introduced quasiparticles can describe successfully a localized system.

Consequently, the postulates we make are the following: First, a physical particle is composed of an *ideal* gas of quasiparticles. Second, for every physical particle with mass m there exists a quasiparticle also of mass m. The quasiparticles have momentum $\vec{p}$ and energy $\epsilon(\vec{p}) = (p^2+m^2)^{1/2}$. Third, the quasiparticles are distinguishable.

The last assumption can be made since, in general, quasiparticles need not satisfy the same statistics as the particles. Note also that although for each type of particle there exists a quasiparticle, the total number of quasiparticles is certainly not the same as the total number of particles in the system. In fact, the number of quasiparticles is not even conserved.

In what follows it will be shown that this model for the structure of a particle gives the exponentially rising hadron spectrum of Hagedorn with the added feature that the size of a hadron is determined. (Recall that in existing works this value is assumed to be $\frac{4}{3}\pi m_\pi^{-3}$ and forms a separate assumption.) Therefore, this model has succeeded in taking part, and hopefully a large part, of the interaction which gives rise to the localization of a particle and placed it in the statistics of the quasiparticles.

This conception of a particle is new. The usual picture of a particle does not distinguish it from that of a room full of particles.[6] Therefore, confinement or localization cannot be obtained.[7]

A. Classical Statistics

Let us consider first the case of strictly classical statistics. The density of states of the system at the energy E is

$$\omega^{M-B}(E) = \sum_n \prod_{\vec{p}} \prod_i \frac{N_i!}{n_i(\vec{p})!}[g_i(\vec{p})]^{n_i(\vec{p})} \times \delta\left(\sum_{i,\vec{p}} E_i(\vec{p}) n_i(\vec{p}) - E\right), \tag{5}$$

where the subscript i refers to the ith type of quasiparticle. Note the presence of the $N_i!$. By considering the Laplace transform of (5), one has

$$\int_0^\infty e^{-E\alpha}\omega^{M-B}(E)dE = \prod_i \left[1 - \sum_{\vec{p}} e^{-E_i(\vec{p})\alpha} g_i(\vec{p})\right]^{-1}. \tag{6}$$

The quantity $[1 - \sum_{\vec{p}} e^{-E_i(\vec{p})\alpha} g_i(\vec{p})]^{-1}$ has a simple pole at $\alpha = \alpha_0$ with $\sum_{\vec{p}} e^{-E_i(\vec{p})\alpha_0} g_i(\vec{p}) = 1$ and $0 < \alpha_0 < \infty$. Consequently, for α sufficiently large

$$\int_0^\infty e^{-E\alpha}\omega^{M-B}(E)dE < \infty. \tag{7}$$

As α decreases, the first pole,[8] the farthest to the right with $\alpha_0 = \beta$, will be approached. Therefore, the high-energy behavior of the density of states is given by

$$\omega^{M-B}(E) \xrightarrow[E\to\infty]{} E^{z-1} e^{\beta E}, \tag{8}$$

where $z = (2J+1)(2I+1)2^\gamma$, $\gamma = 1$ if particle $\neq$ antiparticle, $\gamma = 0$ if particle = antiparticle, is the multiplicity factor of the lowest mass state. Result (8) is *independent* of the level density and even for a finite number of levels, with

$$\beta \leq \alpha < \infty \ . \tag{9}$$

The mathematical step of taking the Laplace transform has the physical meaning of going from the microcanonical ensemble to the canonical ensemble of Gibbs. Hence, the parameter α denotes $(kT)^{-1}$ with T the temperature of the system. Result (9) states that the temperature of the system is bounded by[9]

$$0 \leq kT \leq 1/\beta \equiv kT_0. \tag{10}$$

This indeed is a peculiar result from the point of view of ordinary thermodynamics. One has that the temperature of this cloud of quasiparticles cannot attain an arbitrarily large value. This result already emphasizes the fact that classical statistics is not applicable to real particles, but is suitable to the quasiparticles constituting the physical particle. This conclusion will be reinforced when one shows that the "size" of the particle is determined by the highest temperature attainable by the system and is of the right order of magnitude.

The sum $\sum_{\vec{p}}$ in (6) can be approximated by the integral

$$\sum_{\vec{p}} e^{-E_i(\vec{p})\alpha} g_i(\vec{p}) = \frac{V}{h^3}\int e^{-E_i(\vec{p})\alpha} d^3p, \tag{11}$$

where V is the volume containing the quasiparticles.

The parameter β in (8) is determined by the *lowest* mass m_0 of the system. The mass m_0 gives rise to the first (and only) pole in (6) as $\alpha \to \beta$. The value of β is given by

$$1 = \frac{4\pi V}{(hc)^3}(m_0c^2)^2 \frac{K_2(m_0c^2\beta)}{\beta} \tag{12}$$

for given values of m_0 and V. (The function K_2 is the modified Bessel function of the second kind with subscript 2.) Note that

$$\frac{d}{dx}[x^2K_2(x)] = -x^2K_1(x) < 0,$$

which proves that the lowest mass[10] determines β.

We shall adopt the self-consistency condition (bootstrap condition) of Hagedorn,[11] which is consistent with our second postulate,

$$\ln\rho(m) \to \ln\omega(m) \text{ for } m \to \infty, \tag{13}$$

where $\rho(m)$ denotes the number of hadron states in the interval $\{m, dm\}$. Expression (6) gives rise to three possible results:

(i) The divergence arises *only* from the lowest mass state. This leads to $\rho^{M-B}(m) \to m^a e^{m\beta}$ for $m \to \infty$ with $a < -\frac{5}{2}$.

(ii) The divergence is due to *both* the lowest mass state and the very heavy-mass states (the infinite product diverges and all its factors are finite except the one due to the lowest mass state which is divergent). Thus, $\rho^{M-B}(m) \to m^{-5/2}e^{m\beta}$ as $m \to \infty$.

(iii) The divergence is due *only* to the very heavy-mass states (the infinite product diverges and all its factors are finite). Hence, $\rho^{M-B}(m) \to m^{-5/2}e^{m\beta}$ as $m \to \infty$.

Of these three possibilities the first two are preferable since one would expect that an approximate bootstrapped theory should emerge even if one has a *finite* but very large number of particles in the system. The acceptance of (iii) would imply that the low-mass states do not enter at all in the establishment of bootstrap. Therefore, this possibility must be rejected, or better, it is reduced to either (i) or (ii). That is, the divergence of the infinite product, with all its factors finite, requires the volume of the system to be *less* than the volume determined by (12). Thus, what occurs is that the volume increases so that the lowest mass state also produces the divergence. Experimentally $\rho(m)$ seems to grow exponentially with the mass m with $\beta^{-1} = 160$ MeV. If one takes this experimental value of β and the mass of the pion, $m_\pi \approx 140$ MeV, as the lowest mass of the system then the "size" of a hadron is determined by (12). One obtains

$$V = 21.6 \times 10^{-39} \text{ cm}^3, \tag{14}$$

a result in rather good agreement with the effective radius of a proton. Note that if one allows for a zero-mass particle in the theory, the value of V remains *finite*. In fact, one obtains for the volume the value

$$V = 18.4 \times 10^{-39} \text{ cm}^3. \tag{15}$$

B. Quantum Statistics

In Sec. II arguments were presented in favor of accepting the Gibbs paradox into particle physics as a means of describing the structure of particles. This was done in the context of classical statistics. One can extend the description to quantum statistics; however, one must *multiply* the results of the ordinary quantum statistics by a factor $N!$ This is so that the quantum results lead to the classical result in the limit of high temperatures and low densities.

Consider first the case of "Fermi-Dirac" statistics. The density of states of the system at the energy E is given by

$$\omega^{F-D}(E)=\sum_n \prod_{\vec{p}} \prod_i N_i! \frac{g_i(\vec{p})!}{n_i(\vec{p})![g_i(\vec{p})-n_i(\vec{p})]!} \times \delta\left(\sum_{i,\vec{p}} n_i(\vec{p})E_i(\vec{p})-E\right). \quad (16)$$

The sum over $n_i(\vec{p})$ goes from 0 to $g_i(\vec{p})$. Note the presence of the $N_i!$. From (16) one obtains

$$\int_0^\infty e^{-E\alpha}\omega^{F-D}(E)dE = \prod_i \int_0^\infty e^{-t}\exp\left\{\sum_{\vec{p}} g_i(\vec{p})\ln[1+te^{-E_i(\vec{p})\alpha}]\right\}dt. \quad (17)$$

Now,

$$\sum_{\vec{p}} g_i(\vec{p})\ln[1+te^{-E_i(\vec{p})\alpha}] \leq \sum_{\vec{p}} g_i(\vec{p})\ln[1+te^{-p\alpha}] \approx (\ln t)^4 \text{ as } t\to\infty. \quad (18)$$

Therefore,

$$\int_0^\infty e^{-t}\exp\left\{\sum_{\vec{p}} g_i(\vec{p})\ln[1+te^{-E_i(\vec{p})\alpha}]\right\} < \infty \quad (19)$$

for all values of the volume V and $\alpha>0$. Consequently, there exist no critical volume for the particle and the volume can be arbitrarily large. If the cloud of quasiparticles consists of "fermions" only, then the "size" of the particle can be arbitrarily large.

The integral in (17) is bounded by

$$\int_0^\infty e^{-E\alpha}\omega^{F-D}(E)dE \leq \prod_i \int_0^\infty e^{-t}\exp\left[t\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}\right]dt = \prod_i \frac{1}{1-\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}. \quad (20)$$

Therefore,

$$\omega^{F-D}(E) \leq \omega^{M-B}(E) \approx E^{z-1}e^{\beta E}, \text{ as } E\to\infty. \quad (21)$$

From (17) we have that an exponentially increasing level density is a possible boostrapped solution, but by *no means* the only type of bootstrapped solution which is allowed, with

$$\rho^{F-D}(m)\to m^a e^{m\beta} \text{ as } m\to\infty \text{ with } a\leq -\tfrac{5}{2} \quad (22)$$

so that the equality sign holds in (21). One has then that the result for fermions can be the same as that for classical particles except that the "size" of the particle is not restricted as in the classical case.

Next consider the case of "Bose-Einstein" statistics. The density of states is given by

$$\omega^{B-E}(E)=\sum_n \prod_i \prod_{\vec{p}} N_i! \frac{[n_i(\vec{p})+g_i(\vec{p})-1]!}{n_i(\vec{p})![g_i(\vec{p})-1]!} \times \delta\left(\sum_{i,\vec{p}} n_i(\vec{p})E_i(\vec{p})-E\right). \quad (23)$$

Suppose $E_i(\vec{p})=\epsilon l_i(\vec{p})$, where ϵ is the smallest unit of energy and $l_i(\vec{p})$ is a positive integer. Then,

$$E=\epsilon\sum_{i,\vec{p}} n_i(\vec{p})l_i(\vec{p}) \geq \epsilon\sum_{i,\vec{p}} n_i(\vec{p}) = \epsilon\sum_i N_i. \quad (24)$$

Therefore,

$$(E/\epsilon)! \geq \left(\sum_i N_i\right)! \geq \prod_i N_i!\ . \quad (25)$$

Consequently,

$$f(E) < \omega^{B-E}(E) \leq \left(\frac{E}{\epsilon}\right)! f(E)\ , \quad (26)$$

where

$$f(E)=\sum_n \prod_{\vec{p}} \prod_i \frac{[n_i(\vec{p})+g_i(\vec{p})-1]!}{n_i(\vec{p})!(g_i(\vec{p})-1)!} \times \delta\left(\sum_{i,\vec{p}} n_i(\vec{p})E_i(\vec{p})-E\right). \quad (27)$$

The equality sign does not appear in the left-hand side of inequality (26) since it would lead to the result of Hagedorn, $\rho(m)\to e^{m\beta}/m^{5/2}$, which contradicts the result derived below. [See (29).] Now $\lim_{E\to\infty}[\omega^{B-E}(E)e^{-E\beta}]=\infty$, for $\beta\geq 0$; that is, $\omega^{B-E}(E)$ grows *faster* than exponentially. Since, for one type of particle, with $\sum_p n(\vec{p})=N$,

$$M(E)\equiv\sum_n \prod_{\vec{p}} N!\frac{[n(\vec{p})+g(\vec{p})-1]!}{n(\vec{p})![g(\vec{p})-1]!}\,\delta\left(\sum_{\vec{p}} n(\vec{p})E(\vec{p})-E\right), \quad (28)$$

so that

$$\int_0^\infty M(E)e^{-E\alpha}dE = \sum_n \prod_{\vec{p}} N!\frac{[n(\vec{p})+g(\vec{p})-1]!}{n(\vec{p})![g(\vec{p})-1]!}e^{-n(\vec{p})E(\vec{p})\alpha} \geq \prod_{\vec{p}} \sum_{n(\vec{p})=0}^{\infty} n(\vec{p})!\frac{[n(\vec{p})+g(\vec{p})-1]!}{n(\vec{p})![g(\vec{p})-1]!}e^{-n(\vec{p})E(\vec{p})\alpha} = \infty. \quad (29)$$

The last equality follows since the terms in the series do not approach zero as $n(\vec{p})\to\infty$. [In fact, they approach infinity.] Hence, from (26) and the above, one has that

$$[\omega^{B-E}(E)]^{-1}=o(e^{-\beta E})$$

and (30)

$$\omega^{B-E}(E)\leq e^{E\ln E}f(E) \text{ as } E\to\infty.$$

Let us find an upper bound to the growth of $\omega^{B-E}(E)$. Now

$$\int_0^\infty e^{-E\alpha} f(E)dE = \exp\left\{-\sum_{i,\vec{p}} g_i(\vec{p}) \ln[1 - e^{-E_i(\vec{p})\alpha}]\right\}$$
$$= \exp\left\{-\int \rho^{B-E}(m)dm \frac{V}{h^3} \int d^3p \ln[1 - e^{-\alpha(p^2+m^2)^{1/2}}]\right\}. \quad (31)$$

From (30) and the self-consistency condition (13), one has that $\rho^{B-E}(m)$ grows faster than exponentially; therefore, $\int_0^\infty e^{-E\alpha} f(E)dE = \infty$, that is, $f(E)$ also grows faster than exponentially as $E \to \infty$. There exist then two possibilities,

$$[f(E)]^{-1} = o(e^{-E \ln E}) \quad (32a)$$

or

$$f(E) = o(e^{E \ln E}) \text{ (but still } [f(E)]^{-1} = o(e^{-\beta E})). \quad (32b)$$

If (32a) is true, then from (26)

$$[\omega^{B-E}(E)]^{-1} = [f(E)]^{-1}$$
$$= o(e^{-E \ln E}) \text{ as } E \to \infty, \quad (33)$$

which violates the inequality (26).

Therefore, (32b) is the correct case, and one has from (30)

$$[\omega^{B-E}(E)]^{-1} = o(e^{-E\beta})$$

and (34)

$$\omega^{B-E}(E) \leq e^{\gamma E \ln E},$$

which gives for the mass spectrum

$$[\rho^{B-E}(m)]^{-1} = o(e^{-\beta m})$$

and

$$\rho^{B-E}(m) \leq e^{\gamma m \ln m} \text{ as } m \to \infty. \quad (35)$$

Result (35) is rather different from previously derived results[3,4,7] where the density of hadrons grows exponentially with their mass.[12] Our result for the classical case is as in previous work, that is, there exists a highest possible temperature, a sort of boiling point of hadronic matter, so that as more energy is added to the system, particle creation of massive particles prevents the temperature from increasing beyond this highest possible temperature. The bounds on result (35) imply, if one can still use the language of ordinary thermodynamics since the canonical ensemble does not exist, that as more energy is added to the system, the creation of the massive particles has the effect of cooling the system so that in the limit of infinite energy the matter cools to absolute zero.[13] Therefore, the creation of massive particles is partly done at the expense of the kinetic motion of the system. However, for a gas of real hadrons (finite total energy) thermodynamically stable states may exist. One must distinguish between what happens inside a real particle and what can occur inside a room full of hadrons. These two cases are quite *distinct*. The temperature associated with a physical particle is determined by the asymptotic behavior of $\rho(m)$. In a room full of hadrons with finite total energy not all hadrons can exist (the massive ones cannot be present by energy conservation), therefore, a thermodynamic system may exist and a different temperature will emerge.

IV. INTERACTING-PARTICLE MODEL

In Sec. III the quantum statistics of noninteracting quasiparticles (with strange statistics) was developed. In what follows it will be shown that this peculiar system is identical to an interacting gas of *real* particles. Consequently, our model for the structure of particles is not a free-particle model as is the case in other works.[14]

Consider first the case of fermion quasiparticles. The distribution function is easily obtained from the canonical form (17) and is

$$\bar{n}_i(\vec{p}) = -\frac{1}{\alpha}\frac{\partial}{\partial E_i(\vec{p})} \ln\left[\int_0^\infty e^{-E\alpha}\omega^{F-D}(E)dE\right]. \quad (36)$$

On substituting (17) in (36) one gets

$$\bar{n}_i(\vec{p}) = g_i(\vec{p})\int_0^\infty \frac{dt}{t^{-1}e^{E_i(\vec{p})\alpha}+1} e^{[Y(t;\alpha)-t]}$$
$$\times\left(\int_0^\infty dt\, e^{[Y(t;\alpha)-t]}\right)^{-1}$$
$$\leq g_i(\vec{p}), \quad (37)$$

where

$$Y(t;\alpha) \equiv \sum_{\vec{p}} g_i(\vec{p}) \ln[1 + t e^{-E_i(\vec{p})\alpha}] . \quad (38)$$

The upper bound in (37) is as it should be for real fermions enclosed in a box.

Result (37) is rather interesting since it is a specific case of a general integral representation already derived by the author[15] and applies for a system of interacting (real) fermions in equilibrium. Thus, the ideal gas of fermion quasiparticles is equivalent to an interacting gas of real fermions.

A similar integral representation can be derived for the boson quasiparticles but it is lengthy and, thus, will be omitted.

For a system composed both of boson and fermion quasiparticles

$$\int_0^\infty e^{-E\alpha}\omega(E)dE=\left[\int_0^\infty e^{-E\alpha}\omega^{B-E}(E)dE\right]\times\left[\int_0^\infty e^{-E\alpha}\omega^{F-D}(E)dE\right]. \quad (39)$$

Hence,

$$[\rho(m)]^{-1}=o(e^{-\beta m})$$

and (40)

$$\rho(m)\leqslant e^{\gamma m\ln m} \text{ as } m\to\infty.$$

The results due to Hagedorn and the multiparticle Veneziano model are in agreement. (See, however, Huang and Weinberg in Ref. 14.) Both have the feature of being free real-particle models, in contradiction to our model describing interacting *real* particles. One may wonder as to the specific form of this interaction. This, unfortunately, cannot be determined. The choice which gives our specific results was based on the extension of the Gibbs paradox to quantum statistical mechanics.

Finally, with the many attempts of unitarizing the Veneziano model, one wonders if the result $\rho(m)\to e^{m\beta}$ would survive such unitarization attempts.

V. CHARGED QUASIPARTICLE GAS

In the preceding sections the structure of particles was developed along the lines of a gas of quasiparticles. In the composition of the particle all that was considered was energy conservation. Of course one must put further constraints to take into account the quantities, other than mass, which describe a particle, e.g., charge, spin, isotopic spin, strangeness, etc. In order to show how this can be done let us consider the case of the charge of the particles. This will be done for the simpler case of the (strictly) classical statistics.

One has for the density of states in this case

$$\omega_Q(E)=\sum_n\prod_{\vec{p}}\prod_i\frac{N_i![g_i(\vec{p})]^{n_i(p)}}{n_i(\vec{p})!}\times\delta\left(\sum_{i,\vec{p}}n_i(\vec{p})E_i(\vec{p})-E\right)\delta\left(\sum_i N_iQ_i-Q\right), \quad (41)$$

where Q_i denotes the charge of the ith type of quasiparticle and Q the charge of the particle under consideration. Since Q can be either positive or negative,

$$\int_{-\infty}^\infty e^{-Q\beta}dQ\int_0^\infty e^{-E\alpha}\omega_Q(E)dE=\prod_i\frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}, \quad (42)$$

with the requirement that

$$e^{-Q_i\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}\leqslant 1. \quad (43)$$

Note that (43) demands the charge of a quasiparticle to be finite, that is $|Q_i|<\infty$.

The right-hand side of (42) is an analytic function of β for

$$e^{-Q_i\,\mathrm{Re}\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}\leqslant 1 \text{ for all } i. \quad (44)$$

That is, (42) is analytic in the strip $-l\leqslant\mathrm{Re}\beta\leqslant l$, with $l>0$, and represents a two-sided Laplace integral whose inverse is

$$\int_0^\infty e^{-E\alpha}\omega_Q(E)dE=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\beta Q}d\beta\prod_i\frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}, \quad (45)$$

where $-l<c<l$.

In the integral (45) the quasiparticles with zero charge ($Q_i=0$) do not contribute to the integral. This gives

$$\int_0^\infty e^{-E\alpha}\omega_Q(E)dE=\left[\prod_i\frac{1}{1-\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}\right]\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\beta Q}d\beta\prod_i\frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}, \quad (46)$$

where the integral contains nonzero-charge quasiparticles. From (46) one may again conclude that the "volume" of the particle is determined by the lowest mass state, the photon, and one gets result (15). [For the case where there is no electromagnetic interaction the "volume" is given by (14).]

One can derive a rather interesting result from (46). Suppose the mass of a particle and the mass of its corresponding antiparticle are equal. Therefore,

$$H(\beta)=\frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}}g_i(\vec{p})e^{-E_i(\vec{p})\alpha}}\times\frac{1}{1-e^{-Q_{\bar{i}}\beta}\sum_{\vec{p}}g_{\bar{i}}(\vec{p})e^{-E_{\bar{i}}(\vec{p})\alpha}}, \quad (47)$$

where the first term refers to the particle and the second to the antiparticle. Since $Q_i = -Q_{\bar{i}}$, we have

$$H(\beta) = H(-\beta) \text{ if } m_i = m_{\bar{i}} . \tag{48}$$

Therefore,

$$S(Q) \equiv \int_{c-i\infty}^{c+i\infty} e^{\beta Q} d\beta \prod_i \frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}} g_i(\vec{p}) e^{-E_i(\vec{p})\alpha}} = S(-Q) . \tag{49}$$

Consequently, if the temperature α is the same for the system with charge Q or $-Q$ then the following result follows from (49):

$$\int_{-\infty}^{\infty} e^{-E\alpha}\omega_Q(E)dE = \int_0^{\infty} e^{-E\alpha}\omega_{-Q}(E)dE, \tag{50}$$

or, what is the same, the "volume" of a positively charged particle is the same as the negatively charged particle (even in the absence of electromagnetic interactions).

Finally, let us see if one can study the question of whether a charged particle is larger or smaller than its neutral component. The integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\beta Q}d\beta \prod_i \frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyQ}dy \prod_i \frac{1}{1-e^{-iQ_iy}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} , \tag{51}$$

since the integral is independent of c, and c has been set equal to zero. The product in the integrand is an even function of y if the masses of the particles and their antiparticles are equal. Thus,

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\beta Q}d\beta \prod_i \frac{1}{1-e^{-Q_i\beta}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \cos yQ \prod_i \frac{1}{1-e^{-iQ_iy}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} \\ &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \prod_i \frac{1}{1-e^{-iQ_iy}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \prod_i \frac{1}{1-e^{-\beta Q_i}\sum_{\vec{p}} g_i(\vec{p})e^{-E_i(\vec{p})\alpha}} . \end{aligned} \tag{52}$$

Therefore,

$$\int_0^{\infty} e^{-E\alpha}\omega_Q(E)dE \leq \int_0^{\infty} e^{-E\alpha}\omega_{Q=0}(E)dE, \tag{53}$$

or, what is the same, the "volume" of the zero-charge component is less than or equal to that of the charged component, that is, $V_{Q=0} \leq V_Q$. (This result holds even in the absence of electromagnetic interactions.)

VI. CONCLUSION

The fundamental idea presented in this work is the use of the Gibbs paradox in quantum statistical mechanics as a means of generating the strange statistics needed to describe the cloud of particles composing a real particle. The mathematical statement of this idea is carried through and leads to results quite different from presently considered free-particle models. One finds that the hadronic level density is not exponentially increasing as in free-particle models. Instead, it grows faster than exponentially, but less than or equal to $e^{\gamma m \ln m}$. This difference in result is shown to be a consequence of the interacting-particle nature of the present model.

The level density found from the Veneziano model (without satellites) is exponentially increasing with mass. It is then interesting to see what level density will be found in a unitarized Veneziano model. Although this question remains unanswered now, one may guess that the exponentially increasing result will not survive correct unitarization attempts. Presently the most appealing unitarization of the four-particle Veneziano amplitude is that due to Martin.[16] By smearing the Veneziano amplitude, Martin obtains at high energies a behavior peculiar not to Regge poles, but to Regge cuts in the J plane. (Also, of course, the Regge trajectories are no longer straight lines.) At high energies the amplitude contains some power of $\ln s$ in the denominator. It is interesting that the present model gives rise also to a logarithmic function in the level density. Unfortunately, from a possible nonlinear behavior of Regge trajectories one cannot conclude anything definite about the level density.

In closing it should also be noted that our present model establishes a relationship between the "size" of a particle and the hadron spectrum. If the "size" of a particle is fixed, as in the classical-statistics case, then the hadron spectrum rises exponentially. If, however, the "size" of the particle is not fixed, as it happens in the quantum-statistics case, then the hadron spectrum rises faster than exponentially. Note also that even in the case of classical statistics one must assume that the volume is fixed, that is, the volume cannot be arbitrary, otherwise the geometric sum (6) would diverge and $\omega^{M-B}(E)$ would grow faster than exponentially. It would be interesting to relate these conclusions to the Froissart bound, the Pomeranchuk theorem, or the "shrinkage of the diffraction peak."

A final comment concerning the distinguishability of the quasiparticles constituting a physical particle seems to be appropriate. It is known that the tree-graph approximation is all that survives of the S matrix in quantum field theory in the classical limit, $\hbar \to 0$.[17] The tree diagrams represent a semiclassical approximation to the S matrix which is an analog in quantum field theory of the WKB approximation to the Schrödinger equation.[18] This result may be an indication that in the tree-graph approximation particles may behave in a many-body system as distinguishable particles. Since virtual particles satisfy the same statistics as physical particles, one has that the quasiparticles considered in the text may, in the "tree" approximation, be the virtual particles constituting a physical particle. This connection is presently being investigated.

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[1]S. D. Drell, invited paper presented at the International Conference on Expectations for Particle Reactions at the New Accelerators, University of Wisconsin, 1970 (SLAC Report No. SLAC-PUB-745) (unpublished).

[2]R. Hagedorn, Astron. Astrophys. 5, 184 (1970); K. Huang and S. Weinberg, Phys. Rev. Letters 25, 895 (1970).

[3]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965); 6, 311 (1968); Nuovo Cimento 52A, 1336 (1967); 56A, 1027 (1968); R. Hagedorn and J. Ranft, Nuovo Cimento Suppl. 6, 169 (1968); R. Hagedorn, Nucl. Phys. B24, 93 (1970).

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[5]E. Schrödinger, *Statistical Thermodynamics* (Cambridge Univ. Press, New York, 1967), p. 60.

[6]When describing a room full of gas, one eliminates the Gibbs paradox by insisting that the entropy of the gas in the room is independent of the history of the gas. That is, suppose a partition divides the room in halves, if the partition is removed no change in entropy must occur. It is clear that this type of thinking cannot (and should not) be applicable to a particle.

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[8]The right-hand side of (6) may be written as

$$\exp\left\{-\int_0^\infty \rho(m)dm \ln\left[1-\frac{V}{h^3}\int e^{-\alpha(p^2+m^2)^{1/2}}d^3p\right]\right\},$$

where $\rho(m)$ denotes the single-particle hadron-mass spectrum and includes the different states of spin, charge, strangeness, etc. For instance, the pion is counted as $(2I+1)=3$ states, the photon twice, and so on.

[9]This result was already noticed by Hagedorn in the first article in Ref. 3. However, the connection to the Gibbs paradox and its extension to quantum statistics was not mentioned or pursued by Hagedorn. Hagedorn (private communication), however, was aware of the finite volume required by the distinguishability of the particles.

[10]See Ref. 8 for the proper way to go to the continuous case.

[11]See the first article of Ref. 3.

[12]If in the Veneziano formula satellite terms are included a spectrum $\sim e^{cm^2}$ is obtained. [P. Olesen, Nucl. Phys. B18, 459 (1970).] This result is not in agreement with the claim that such behavior should be $\sim e^{cm^{4/3}}$ [D. K. Sinclair, Phys. Rev. D 3, 384 (1971).]

[13]The limiting-temperature feature of Hagedorn's model is consistent with the cosmological theories of a hot beginning for the universe. [R. Hagedorn, Astron. Astrophys. 5, 184 (1970).] Our present result implies a cold model of the early universe. Such a model was once preferred by Zeldovich. (Ya. B. Zeldovich, Zh. Eksperim. i Teor. Fiz. 43, 1561 (1962) [Soviet Phys. JETP 16, 1102 (1962)].) See also D. Layzer, Astrophys. Letters 1, 99 (1968).

[14]See Refs. 3, 4, 7, and also K. Huang and S. Weinberg, Phys. Rev. Letters 25, 895 (1970).

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