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Elementary Discontinuity Formulas and Mueller's Regge Hypothesis*

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We propose to interpret Mueller's recent work on inclusive cross sections in terms of the elementary discontinuity equations for multiparticle scattering amplitudes. The paths of continuations for obtaining these discontinuities are given. We find that, in order to make our identification meaningful, the emergence of anomalous thresholds is necessary to "shield" the effects of box-diagram Landau singularities. We also discuss how Mueller's Regge hypothesis can be derived from the multi-Regge model, provided singularities other than normal thresholds are absent in the asymptotic expansion.

I. INTRODUCTION

Mueller's Regge hypothesis[1] on inclusive reactions has been recognized as a significant breakthrough in the understanding of high-energy particle physics. It has provided an elegant yet simple language for describing various theoretical concepts such as limiting fragmentation, pionization, factorization, etc. Perhaps even more important is its emphasis on the role that unitarity conditions can play in tying together elastic scattering processes with inelastic ones. It is suggested in Ref. 1 that unitarity conditions, when used as discontinuity formulas, relate discontinuities of elastic multiparticle scattering amplitudes to inclusive cross sections which are intrinsically inelastic. How these discontinuities should be taken, however, was not discussed.

We shall discuss in this paper the precise path of continuation for obtaining this discontinuity and also show how the emergence of anomalous thresholds is necessary in order to make our identification meaningful. With these complications in mind, we discuss in what sense one can consider Mueller's Regge hypothesis as a consequence of multi-Regge models.[2]

For an n-particle inclusive reaction

$$a+b\rightarrow x_1+x_2+\cdots+x_n+\text{missing mass}\,, \tag{1}$$

one can show that the differential cross section is proportional to the forward limit of a discontinuity of the $(n+2)$ to $(n+2)$ amplitude describing the process

$$a+b+x_1'+x_2'+\cdots+x_n'\rightarrow a'+b'+x_1+x_2+\cdots+x_n\,. \tag{2}$$

The forward limit corresponds to $p_a=p_{a'}$, $p_b=p_{b'}$, $p_{x_i'}=p_{x_i}$, and the discontinuity is taken in the squared missing-mass variable

$$M^2\equiv\left(p_a+p_b-\sum_{i=1}^{n}p_{x_i}\right)^2 \tag{3}$$

Threshold branch points in this crossed-energy variable can be reached within the physical region of the process (2), so that one does not need to continue each detected particle x_i to its antiparticle $\bar{x}_i$. As one analytically continues the amplitude from one side of the branch cuts in M^2 to the opposite side, certain invariants will have to be held at a definite side of their threshold branch cuts. This can best be summarized by using bubble notations. Let $\alpha=(a,b)$, $\alpha'=(a',b')$, $\beta'=(x_1',\ldots,x_n')$, $\beta=(x_1,x_2,\ldots,x_n)$; we find that the n-particle inclusive cross section is given schematically by Fig. 1. In Fig. 1, the plus (or minus) sign in a small half bubble indicates that all energy variables enclosed by it are to be evaluated above (or below) their cuts. The dashed line indicates that the discontinuity is taken in the variable $(p_\alpha-p_\beta)^2$. Other invariants without "±" specifications are also held fixed; but the discontinuity is independent of their locations.

Figure 1 is a typical example of multiparticle discontinuity formulas[3] proposed by Olive and by Stapp. These general formulas can be shown to follow three technical assumptions: (a) normal-threshold cut structure, (b) extended unitarity, and (c) cross-discontinuity condition (or independence of crossed normal threshold).

Of these three, (a) is an approximation because Landau singularities other than normal thresholds are known to move onto the physical sheet. The correct procedure for handling this problem has been discussed by Hwa.[3] We have found explicitly that the presence of anomalous thresholds plays a very important role so as to make Fig. 1 meaningful. Unlike the 2 to 2 amplitude, a box-diagram double-spectral[4,5] curve can move into the physical region of a multiparticle reaction. We demonstrate, with the help of perturbation theory, that the disappearance of the box-diagram singularities is correlated with the emergence of anomalous thresholds, and, together with our $\pm i\epsilon$ prescriptions for initial and final subenergies, guaran-

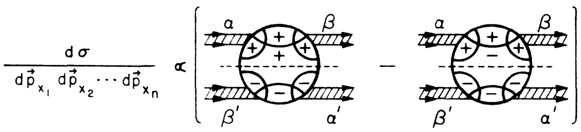

FIG. 1. Path of continuation for obtaining an n-particle inclusive cross section from an $(n+2)$ to $(n+2)$ scattering amplitude.

tees that the discontinuity in Fig. 1 yields a real cross section, as it should. The presence of these higher-order Landau singularities, however, requires a further specification on the paths of continuation in obtaining inclusive cross sections.

To clarify the meaning of a discontinuity formula, let us briefly review the connection between inclusive cross sections and physical unitarity equations. For simplicity, we shall restrict ourselves to the case of a single-particle inclusive reaction of the type

$$a+b \to x + \text{missing mass}\,. \tag{4}$$

The physical unitarity equation

$$S_{\alpha\gamma}(+)S^{\dagger}_{\gamma\beta}(+) = S^{\dagger}_{\alpha\gamma}(+)S_{\gamma\beta}(+) = \delta_{\alpha\beta} \tag{5}$$

can be converted to a discontinuity equation through the use of the Hermitian analyticity property for the connected part[3]:

$$S^{c*}_{\alpha\beta}(+) = S^{\dagger c}_{\beta\alpha}(+) = -S^{c}_{\beta\alpha}(-)\,. \tag{6}$$

Let the initial state be (abx') and the final state be $(a'b'x)$; we find that the connected part T_{33} for the multiparticle process,

$$a+b+x' \to a'+b'+x\,, \tag{7}$$

satisfies a total discontinuity equation. This equation,[6] in bubble notation, is represented by Fig. 2. A bubble with a plus sign refers to an amplitude evaluated in its physical region, and a minus sign refers to its counterclockwise continuation around branch points of all variables. The shaded portion represents an open channel, and an integration

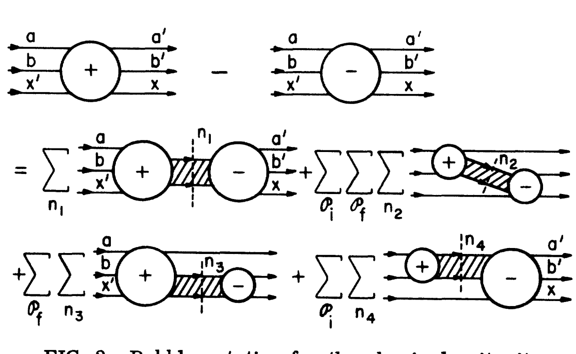

FIG. 2. Bubble notation for the physical unitarity equation for a three-to-three scattering amplitude.

over its phase space is always implied. $\mathcal{P}_i$ and $\mathcal{P}_f$ are permutations of lines for initial and final particles.

In the forward limit, i.e., $p_a = p_{a'}$, $p_b = p_{b'}$, $p_{x'} = p_x$, many of the terms on the right-hand side of Fig. 2 are of the form of a complex-conjugate multiplication [using Eq. (6)]. In particular, one term in the second group (Fig. 3) is simply (up to a flux factor) the inclusive cross section for the process (4). Our objective is to *isolate* this particular term in the total discontinuity by a special path of continuation.

In Sec. II, we discuss the question of "deriving" elementary discontinuity equations, using language and notation which are both suited for our purposes and easy to comprehend. The problem of box-diagram singularities is disposed of in Sec. III. Section IV concerns the derivation of Mueller's Regge hypothesis.

II. ELEMENTARY DISCONTINUITY FORMULAS

For a given multiparticle scattering process, invariant variables can be grouped into four general categories. For the process (7), we have[5]

(i) momentum-transfer variables: $(p_a - p_x)^2$, $(p_b - p_x)^2$, $(p_{a'} - p_a)^2$, etc.,

(ii) total energy: $s = (p_a + p_b + p_{x'})^2$,

(iii) subenergy variables: $s_{ab} = (p_a + p_b)^2$, $s_{a'b'} = (p_{a'} + p_{b'})^2$, $(p_a + p_{x'})^2$, etc.,

(iv) crossed-energy variables: $M^2 = (p_a + p_b - p_x)^2$, $(p_a + p_b - p_{a'})^2$, etc.

With the exception of momentum-transfer variables, singularities in other invariant variables can always be reached within the physical region. An elementary discontinuity formula is one which gives the discontinuity across branch points in one invariant variable only, without encircling threshold branch points of other energy invariants. These formulas have been proposed, and have been shown to follow from three technical assumptions[3]: (a) normal-threshold cut structure, (b) extended unitarity, and (c) cross-discontinuity condition (or independence of crossed normal thresholds). The square of the missing mass $M^2 = (p_a + p_b - p_x)^2$ is a crossed-energy variable for (7), and we shall demonstrate the derivation of the elementary discontinuity formulas in M^2 utilizing the above assumptions.

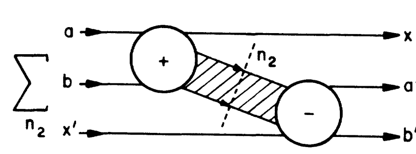

FIG. 3. A single-particle inclusive cross section.

Extended unitarity allows one to write down unitarity-like equations outside the physical region. Let us first continue to a region where all variables except M^2 are below their lowest thresholds.[6] The discontinuity encircling branch points of all communicating channels in M^2 is given by Fig. 4(a). The lack of plus sign designations for s_{ab} and $s_{a'b'}$ in Fig. 4(a) is a reminder that *all* other energy-like invariants are to be held below their cuts. Because of the normal-threshold cut-structure assumption, we can first analytically continue the subenergy s_{ab} $(s_{a'b'})$ above (below) its cuts and arrive at the following configuration [Fig. 4(b)]. The right-hand side of Fig. 4(b) now looks identical to Fig. 3, and we are tempted to identify them immediately. However, before doing so, we have to continue all other energy invariants back to their respective physical values. It is conceivable that this continuation may force a distortion in the intermediate-state phase-space integrations. This has been proved not to be the case if we invoke the cross-discontinuity rule, which asserts that the discontinuity in one invariant does not contain normal-threshold singularities in crossed invariants. An invariant is "crossed" with respect to M^2 if the external lines that this invariant is made of are neither contained in that of M^2, nor of its complement. Since all invariants left for further continuations are crossed invariants of M^2, it follows that the right-hand side of Fig. 4(b), when considered as an analytic function, is independent of normal thresholds of these variables. It is thus immaterial how we continue them back to their physical values, as long as the same path is taken for each term in Fig. 4(b). We note, in particular, that the total energy s can be evaluated at either $s+i\epsilon$ or $s-i\epsilon$ for both terms on the left, without affecting the right-hand side. In contrast, the total discontinuity equation[7] requires one term with $s+i\epsilon$ and the other with $s-i\epsilon$.

We have completed the demonstration where the inclusive cross section for the process (4) can be

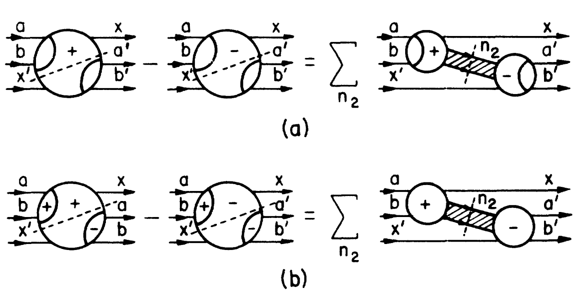

FIG. 4. (a) Extended unitarity, with M^2 above its normal thresholds. (b) Elementary discontinuity formula, before continuing those variables which are "crossed" with respect to M^2 back to their physical values.

identified with the forward limit of a certain discontinuity of T_{33}. The generalization to an n-particle inclusive cross section is then straightforward, and one arrives at the result depicted in Fig. 1.

It is conceptually more appealing to write the discontinuity equation, Fig. 4(b), as a sum of simpler elementary discontinuities. It is important to note that the real axis of M^2 is divided into sectors by the n-particle thresholds. Within each sector, we have a corresponding discontinuity formula like Fig. 4(b). The analytic continuation of one discontinuity formula passing the next normal threshold in M^2 will *not* lead to the corresponding one for the next sector. This indicates that the inclusive cross section is only a piecewise analytic function of M^2. A similar phenomenon also occurs for two-particle total cross sections. One can show that the difference between the appropriate discontinuity is one sector of M^2 and that continued from immediately below, i.e., the discontinuity of T_{33} across only this n-particle-threshold branch point, can be calculated in terms of the values of T_{3n} on both sides of this cut. This is the so-called "basic" discontinuity formula. Once this is done, it follows that the discontinuity, Fig. 4(b), can be written as a sum of such basic discontinuities across each open channel in M^2. As we continue to higher values of M^2, a new term will be added whenever a new threshold opens. However, each one of these basic discontinuities is not to be identified with a term in the sum $\sum_{n2}$ on the right-hand side of Fig. 4(b) except in the case of single- or two-particle states.

III. BOX-DIAGRAM SINGULARITIES AND ANOMALOUS THRESHOLDS

The existence of Landau singularities other than normal-threshold branch points has long been recognized as a major obstacle[3] for a complete specification of discontinuity formulas of multiparticle scattering amplitudes. The justification for neglecting them in the discussion of Sec. II is partially justified by the notion that S-matrix-theory singularity structure is "built up"[5] from the normal thresholds. Since Landau singularities do exist, it might happen that the path of continuation which we need is, in fact, impossible. Short of this disaster, one may still wonder if the existence of simple Landau curves, such as "box-diagram" Landau curves, may spoil the "reality" condition of our discontinuity. Since these box-diagram Landau curves can be brought into the physical region, they are not allowed to be singular there if our basic premise is correct.

This unwarranted concern can be put to rest by using the Steinman relations, which assert that double discontinuities in two crossed variables vanish in the physical region.[3] These relations hold not only at the thresholds but also at all physical points, including the box-diagram Landau curves. However, the presence of these higher-order Landau singularities does require a further specification of the paths of continuation for obtaining inclusive cross sections. The "correct" specification is the one which preserves the undistorted contours for the intermediate-state phase-space integrations in Figs. 2 and 4(b). This kind of analysis has been performed by Hwa[3] in the case of five-point amplitudes; and his technique can be readily applied to our case. In order to gain further insight, we have carried out a study in the ϕ^3 theory to see how the "disappearance" of box-diagram singularities actually takes place.

The simplest diagram which may possibly cause us trouble is shown in Fig. 5(a). Removing the pole factors for initial and final subenergies, we are left with a four-point box diagram, whose momenta are labeled in Fig. 5(b). For the sake of simplicity, we shall restrict ourselves to a case of degenerate masses:

$$q_i^2 \equiv m_i^2, \quad m_1^2 = m_3^2, \quad m_2^2 = m_4^2, \quad p_{i,\,i+1}^2 \equiv m_{i,\,i+1}^2,$$
$$m_{12}^2 = m_{34}^2 = s_{ab} = s_{a'b'}, \quad m_{23}^2 = m_{41}^2 = m_{45}^2 = m_x^2 = m_{x'}^2. \tag{8}$$

Using a standard Feynman parametrization, the possible singular surfaces of this four-point amplitude are given by the solutions of the Landau equations.[4] The restrictions of these Landau curves on the $\mathrm{Re}(s)\times\mathrm{Re}(M^2)$ plane have been discussed in the past for cases where not *both* s_{ab} and $s_{a'b'}$ are large.[8-10] The case of interest to us requires both squared "external" masses s_{ab} and $s_{a'b'}$ to be above their normal thresholds; and we can reach this configuration by a continuation from the standard small-mass limit.

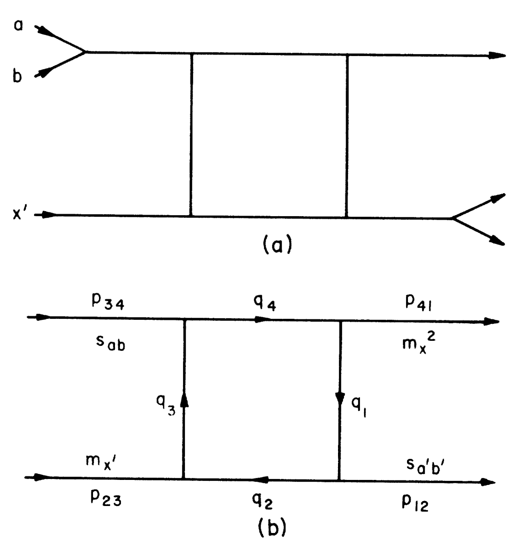

FIG. 5. (a) The simplest possible diagram containing M^2-s double-spectral singularity. (b) The relevant four-point box-diagram and its momenta labeling.

The box-diagram Landau curves can be labeled by Γ, $L_i^\pm$, and $N_{i,\,i+2}^\pm$. They correspond to the solutions where the Feynman parameters α_i satisfy $\alpha_i \neq 0 \forall i$, $\alpha_i = 0$, and $\alpha_i = \alpha_{i+2} = 0$. In a typical configuration shown in Fig. 6, N_{24}^+ and N_{24}^- correspond to the M^2-channel normal threshold and antithreshold, respectively. $L_2^\pm$, $L_4^\pm$ are triangle singularities which may move onto the physical sheet to become anomalous thresholds. Similar interpretations for $N_{13}^\pm$, $L_1^\pm$, $L_3^\pm$ apply to the s channel. We note that Γ, the leading Landau curve, has several branches. The branch Γ_{AB}, in the upper right corner, is the famous Mandelstam double-spectral curve. Figure 6 corresponds to our case of mass degeneracy (8); and the locations of various curves can best be described in terms of the angle variables:

$$\cos\theta_{ij} = -y_{ij} = \frac{q_i \cdot q_j}{2m_i m_j} = \frac{m_i^2 + m_j^2 - (q_i - q_j)^2}{2m_i m_j}. \tag{9}$$

We find that for $L_1^\pm = L_3^\pm$ and $L_2^\pm = L_4^\pm$,

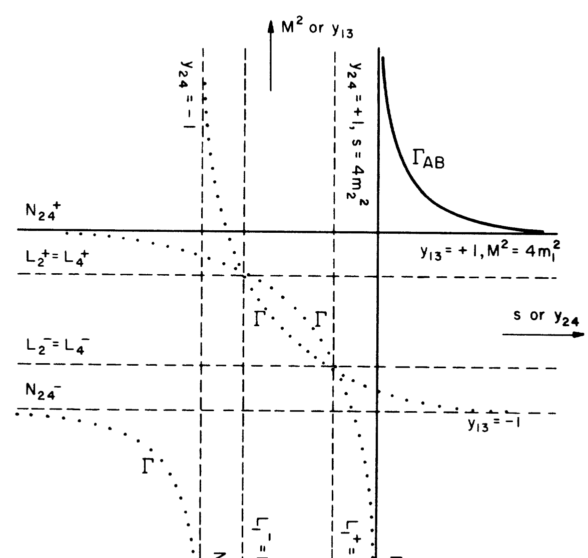

FIG. 6. $\mathrm{Re}(s)\times\mathrm{Re}(M^2)$ plane singularity structure of a box diagram. It corresponds to a normal situation where the external masses are comparable to the internal masses. Singularities on the physical sheet are the normal thresholds N_{13}^+ and N_{24}^+, and the Mandelstam double-spectral curve Γ_{AB}. Masses are degenerate as given by (8). Both s and M^2 are $+i\epsilon$ above their branch points.

$$y_{24}=y_{13}=-\cos(\theta_{12}\pm\theta_{23})\,;$$

for $N_{13}^{\pm}$,

$$y_{24}=\pm 1\,;$$

for $N_{24}^{\pm}$,

$$y_{13}=\pm 1\,;$$

and for Γ,

$$(y_{13}-1)(y_{24}-1)=[\cos(\theta_{12}+\theta_{23})+1]\,[\cos(\theta_{12}-\theta_{23})+1]\,, \tag{10}$$

$$(y_{13}+1)(y_{24}+1)=[\cos(\theta_{12}+\theta_{23})-1][\cos(\theta_{12}-\theta_{23})-1]\,,$$

where y_{13} and y_{24} are directly related to s and M^2. For this reason, it is sometimes simpler to consider plots such as Fig. 6 as $\mathrm{Re}(y_{13})\times\mathrm{Re}(y_{24})$ plots. In writing (10), we have also used the degeneracy conditions $\theta_{12}=\theta_{34}$ and $\theta_{23}=\theta_{41}$.

When external masses satisfy a set of inequalities

$$0<\theta_{i,\,i+1}<\pi,\quad 0<\theta_{i-1,\,i}+\theta_{i,\,i+1}<\pi,\quad i=1,2,3,4, \tag{11}$$

where subscripts are defined as modulus 4, only those solid curves in Fig. 6 are singular on the physical sheet, e.g., the Mandelstam spectral curve Γ_{AB}. The above conditions are usually satisfied if external and internal masses are comparable. However, we are interested in the limit where $s_{ab}, s_{a'b'}\geq(m_1+m_2)^2$, which by (9) is characterized by

$$\theta_{12},\ \theta_{34}=\pi+i\lambda,\quad \lambda \text{ real}\,. \tag{12}$$

We shall do so by first increasing these angles along the real axis, and then move them up or down the line (12).

Before we can reach (12), we will have to first cross the point $\theta_{12}+\theta_{23}=\theta_{23}+\theta_{34}=\pi$, beyond which inequalities (11) are violated. At this point, we find that both sets of Landau curves $L_2^+=L_4^+$ and $L_1^+=L_3^+$ have moved up to their normal thresholds at $y_{13}=1$ and $y_{24}=1$. The leading Landau curves have also become degenerate, such that portions of the curves Γ coincide with $y_{13}=1$ and $y_{24}=1$. This is illustrated in Fig. 7. As we further increase s_{ab} and $s_{a'b'}$, two interesting phenomena occur. First, $L_2^+=L_4^+$ and $L_1^+=L_3^+$ have moved up to the physical sheet. Second, the branch Γ_{AB} has circled around these anomalous thresholds and returned to the upper right-hand corner. In so doing, it is no longer singular in the physical region; and a part of Γ in the central region has, instead, become singular. This is indicated in Fig. 8. If we continue to increase $s_{ab}, s_{a'b'}$ passing their thresholds at $(m_1+m_2)^2$, we find that Landau curves $L_1^{\pm}$, $L_2^{\pm}$, $L_3^{\pm}$, and $L_4^{\pm}$ have gone complex. The direction of their movements depends on the $i\epsilon$ prescription given to the subenergies s_{ab} and $s_{a'b'}$. With our choice, Fig. 1, we find that, on the physical sheet

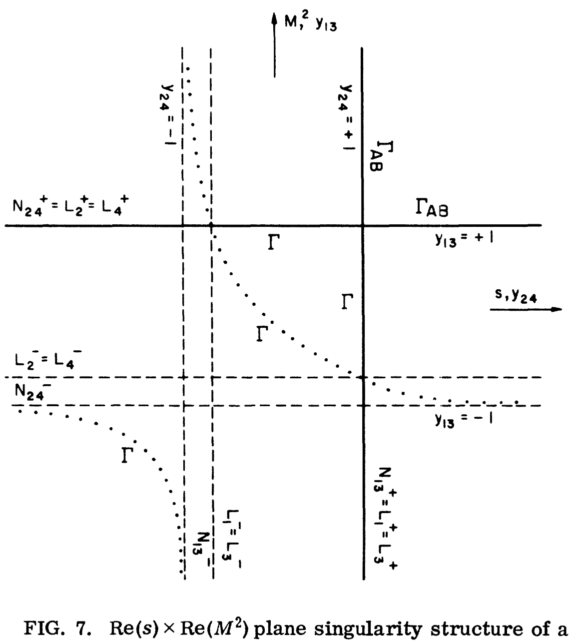

FIG. 7. $\mathrm{Re}(s)\times\mathrm{Re}(M^2)$ plane singularity structure of a box diagram, at the point of passing the stability conditions, e.g., $\theta_{12}+\theta_{23}=\pi$, as we increase s_{ab} and $s_{a'b'}$. Mass degeneracies are maintained throughout. Note in particular the shape of Γ_{AB} and the position of $L_2^+=L_4^+$ and $L_1^+=L_3^+$.

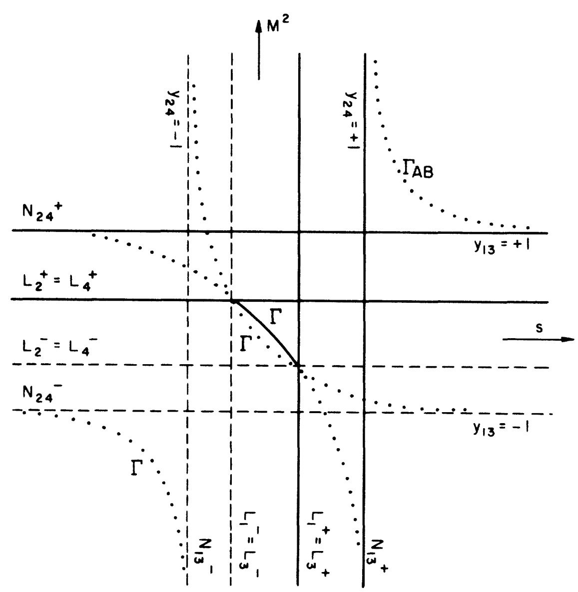

FIG. 8. $\mathrm{Re}(s)\times\mathrm{Re}(M^2)$ plane singularity structure of a box diagram, after passing the stability point and before reaching Eq. (12). Anomalous thresholds are now on the physical sheet.

of the M^2 complex plane, L_2^+ has moved to the lower half plane, and L_4^+ has moved to a conjugate position in the upper half-plane. These conjugate movements of anomalous thresholds L_2^+ and L_4^+, which depend crucially on our $\pm i\epsilon$ prescriptions, is important in insuring the reality of the discontinuity in M^2.

The sheet structure in the M^2 plane, with s held above its anomalous and normal thresholds reached by $+i\epsilon$ prescriptions, is illustrated in Fig. 9. In the case we are interested in, the physical region in bounded by $M^2 = 4m_1^2$ from below and $M^2 = (\sqrt{s_{ab}} - m_x)^2$ from above. This is because M^2 is a crossed-energy variable. With the cuts drawn according to the prescription given by Hwa,[3] we find that the required path of continuation is simply from plus to minus, as in Fig. 9. The double-spectral curve Γ_{AB} is singular in the limit by circling around L_2^+, but not in the region where the desired discontinuity is taken.

The above analysis serves as a prototype for handling general diagrams of the "box" type. In general, we have to consider the case where $y_{23} = y_{41}$ is less than -1. In that case, anomalous thresholds may lie on the real axis, and the modification for the path of continuation has been given by Hwa. We shall not repeat it here. It suffices to say that higher-order Landau singularities can cause great difficulties in practice, but not in principle. Our analysis indicates that an inclusive cross section can always be identified as a forward elementary discontinuity with a special path of continuation.

The conclusions of this section are also expected to hold true from a purely S-matrix viewpoint. This is because the graph analysis is actually of more general validity, and is not restricted to perturbation theories only. Guided by the notion that the S-matrix singularity is built up from the normal thresholds, one can presumably rephrase our discussions here using only the language of a unitary analytic S-matrix theory.

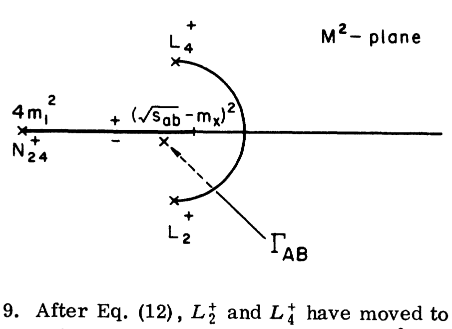

FIG. 9. After Eq. (12), L_2^+ and L_4^+ have moved to complex-conjugate positions in the complex M^2 plane. Cuts are drawn according to the prescription of Ref. 3. Discontinuity is taken between plus and minus; and Γ_{AB} is located at a point reached by circling L_2^+.

IV. RELATION BETWEEN THE MULTI-REGGE HYPOTHESIS AND THE LIMITING DISTRIBUTIONS

In view of the complications discussed in Sec. III, one would naturally wonder whether treating inclusive cross sections as discontinuities of elastic multiparticle amplitudes actually provides any practical advantage over the conventional approaches. It can be argued that better physical insights can probably be gained by the latter methods where one directly works with approximations for production amplitudes. The chief advantage of the present approach lies in its ability to formulate the concept of limiting distributions (or Mueller's Regge hypothesis) in general terms, without having to commit oneself to a specific dynamic approximation from the outset.[11] Furthermore, it suggests that the hypothesis of limiting distribution is probably not independent of the multi-Regge hypothesis.[2]

The connection between these two hypotheses cannot be established immediately since the original Bali-Chew-Pignotti multi-Regge hypothesis[2] lacks specific assumptions about the analyticity structure of multiparticle amplitudes. However, it is generally accepted that Regge-pole dominance is tantamount to the assumption that only ladder graphs are important at high energies. We shall adopt this point of view and shall, therefore, interpret the analyticity structure of a multi-Regge expansion to be that of a properly symmetrized ladder sum. More generally, we can also adopt the dual-resonance model as a starting point.[12,13] In either case, we are confronted with approximations of multiparticle scattering amplitudes, which contain only normal thresholds and poles. These approximated amplitudes not only possess the desired multi-Regge behavior, but also satisfy the analyticity postulate necessary for our discussions in Sec. II. Therefore, in the asymptotic region of interests, we do not have to concern ourselves with the complications of higher-order Landau singularities, and the method outlined in Sec. II can be used as a *constructive* procedure for obtaining inclusive cross sections. Mueller's Regge hypothesis will then emerge as a consequence of the multi-Regge behavior of the elastic multiparticle amplitudes. In the case of the dual-resonance model, this result has been demonstrated in Ref. 12. Since the analysis there depends only on the analyticity and multi-Regge properties of the dual-resonance amplitudes, the result is then of general validity.[14]

To elaborate this further, we note that under our approximation, a multiparticle amplitude is given by a sum of terms, each having the analyticity structure of a B_n function for a given permutation of external lines. Each term contains normal thresholds only in Mandelstam invariants made out of lines that are neighboring each other.[15] Since there are more Mandelstam invariants than there are independent variables, care must be taken in obtaining discontinuities. The correct procedure is to treat all Mandelstam invariants as "independent" as far as the specification of singularities is concerned. Any constraints among them should be enforced only after the discontinuity is taken.

The crossed-discontinuity property guarantees that we need only to worry about those invariants which are not "crossed" with respect to M^2. In the case of T_{33}, they include s_{ab}, $s_{\bar{a}'\bar{b}'}$, $t_{a\bar{x}}$, $t_{b\bar{x}}$, $t_{\bar{a}'x'}$, and $t_{\bar{b}'x'}$, of which only s_{ab} and $s_{\bar{a}'\bar{b}'}$ are energy-like in the physical limit of interests. Squared momentum-transfer variables, t's, are either finite or approaching $-\infty$. Following the procedure outlined in Sec. II, we first obtain the discontinuity in M^2 by keeping all other invariants at $-\infty$ or finite negative values. After this is done, we need only to continue s_{ab} and $s_{\bar{a}'\bar{b}'}$ to $+\infty$ along the infinite contours in the upper and lower complex planes, respectively, as given by Fig. 1. Other invariants which are crossed with respect to M^2 can be continued back to their respective physical values by arbitrary paths, since the discontinuity is free of normal thresholds in these variables. For general n-particle inclusive cross sections, a similar procedure can also be applied.[16] In those cases, it becomes crucial that nonlinear constraints among Mandelstam variables should not be enforced until after the discontinuity in M^2 is taken.

We have discussed in this paper the precise path of continuation for obtaining the inclusive cross sections as discontinuities of elastic multiparticle amplitudes. In particular, the role of anomalous thresholds and the complication of higher-order Landau singularities are studied. We have also pointed out that, with a suitable analyticity assumption, the hypothesis of multi-Regge behavior will lead naturally to the concept of limiting distributions. This not only provides conceptual unity among various theoretical concepts, but also gives us some practical methods for relating parameters of inclusive processes to that of exclusive processes.[17]

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High-Energy Production Processes in the Eikonal Approximation*

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For high-energy production processes, the factorization properties of inelastic effects on the elastic amplitude are studied, first by the eikonal approximation method in a perturbative approach (without assuming the momentum of a produced particle is small), and second by functional methods for many- but soft-particle production. The latter corresponds to the situation in the Bloch-Nordsieck model.

I. INTRODUCTION

Many authors[1-4] have derived relativistic generalizations of the eikonal approximation[5] to describe high-energy elastic scattering. Physically, the approximation is based on the observation that at high energies, scattering is most likely to occur near the forward direction via virtual particles, each of which carries a small momentum. For the sake of definiteness we shall discuss the case of quantum electrodynamics, although our results are applicable to any theory with Yukawa coupling. As shown in Refs. 1 and 3, the most important diagrams in this approximation are the generalized ladder diagrams – those in which a number of photons are emitted by one charged particle and are absorbed by another, in any order. We neglect diagrams in which a photon is emitted and subsequently reabsorbed by the same electron and diagrams with closed fermion loops. These authors[1-4] found an explicit expression for the sum of all such diagrams in the approximation that the momentum carried by any of the virtual photons is small compared to that carried by the scattered electrons.

Our aim in this work is to extend this approximation to describe production processes. In Sec. II, we calculate the amplitude for production of a single photon, using the perturbation-theory approach of Lévy and Sucher.[2] Our main result is that the amplitude factors in a very simple way. More specifically, the amplitude for production of a photon of momentum k in the collision of two electrons having initial momenta p_1, p_2 and final momenta p_3, p_4, respectively, is given by

$$\begin{aligned} M = e^3\bar{u}(p_3)\left[\not{\epsilon}(k)\frac{1}{\not{p}_3+\not{k}-m}\gamma^\mu + \gamma^\mu\frac{1}{\not{p}_1-\not{k}-m}\not{\epsilon}(k)\right]u(p_1) \\ \times\bar{u}(p_4)\gamma_\mu u(p_2)M_{\mathrm{sc}}(p_3+k, p_4; p_1, p_2) \\ +(\text{terms interchanging } p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4), \end{aligned} \tag{1.1}$$

where the elastic scattering amplitude is just

$$\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)M_{\mathrm{sc}}(p_3, p_4; p_1, p_2).$$

This corresponds to a result of Cheng and Wu[6] that, in their language, a produced particle cannot come out of a black dot in an impact diagram.

In Sec. III, we extend the discussion to the production of many photons, with the further restriction that all the produced photons should also be soft. The combinatorial problems of summing