# Coulomb Corrections to One- and Two-Channel Strong Scattering Processes

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We generalize our previous considerations of Coulomb corrections to single-channel processes to the two-channel case. Using a potential model, we derive an extremely simple formula for the corrections which does not necessitate using Coulomb wave functions. This formula is correct to lowest order in $\alpha$ but to all orders of the strong scattering. We discuss the application of our results to recent data on pion-nucleon elastic and charge-exchange scattering in the vicinity of the $N^*(1236)$ resonance. The choice of an appropriate wave equation and the question of mass-difference corrections are also discussed in an appendix.

## I. INTRODUCTION

During the past several years a number of papers have treated the subject of Coulomb corrections to strong scattering processes.[1] The motivation behind many of these was the attempt to determine the charge radius of the pion by a careful analysis of pion-helium scattering data.[2]

One attempt was made to do this calculation in a completely rigorous, field-theoretic approach,[3] but so many approximations are needed that it is difficult to evaluate the validity of the results. It may be possible to approach the calculation by means of the Dashen-Frautschi perturbation theory,[4] but a completely rigorous treatment has not yet been made.[5] Thus, the results of this approach are also questionable. Most authors have preferred to work with a potential-scattering model. Although it is not clear that such a model adequately approximates quantum field theory, one can at least make unambiguous calculations within the context of the model.

In a previous paper[6] we developed a new technique for doing the potential-scattering calculations. This led to an elegant and extremely simple formula for finding the Coulomb-corrected amplitude. Although this method is completely equivalent to other techniques which employ the optical model or Coulomb wave functions, it is much simpler to use in actual calculations.

Motivated by the pion-helium data, we restricted our previous treatment to the case of a single-channel scattering process. Now, however, very accurate new measurements of the pion-nucleon cross sections have been made.[7] In addition, accurate kaon-nucleon measurements are being planned. In both cases we must consider the charge-exchange channel simultaneously with the elastic channel. In this paper we extend our previous treatment to the two-channel case. Results are presented both for the case where the strong interaction has isospin symmetry and the case where no symmetry exists.

Two other questions are of some interest and are discussed in an appendix. The first is whether the Schrödinger or the Klein-Gordon wave equation is more relevant for such a calculation. The second is how one should take into account the fact that particle masses in the two channels are slightly different, which is itself an electromagnetic effect. We compare in detail the Schrödinger and Klein-Gordon equations to see how the usual Coulomb forces and the mass difference enter. On the basis of this comparison, we suggest a possible compromise between the two equations which might be suitable at relativistic energies. To facilitate comparisons, in the main sections of the paper we neglect the mass-difference corrections and use a wave equation which contains the Coulomb potential in the manner employed by previous authors. However, in the Appendix we point out how these results may easily be modified to represent Coulomb forces as they actually appear in either wave equation. We do not treat the mass-difference problem in this paper, but we point out that it can be calculated using techniques analogous to those used in the Coulomb-force problem.

In Sec. II we present the wave equation to be used and review the definition of the $S$ matrix and cross sections in the many-channel problem. In Sec. III we review the results of the single-channel calculation and the method employed. Although we previously found results to higher order, in this paper we shall restrict our work to the lowest order, $\alpha$, Coulomb corrections. Sections IV and V extend this result to the two-channel problem for a strong potential with and without isospin symmetry, respectively. In Sec. VI we discuss the application of our results in general and in particular refer to pion-nucleon scattering near the $N^*(1236)$ resonance. The Appendix is devoted to a consideration of which wave equation is more appropriate for

such calculations.

## II. WAVE EQUATION, $S$ MATRIX, AND CROSS SECTIONS

In analogy with other authors, we shall consider the wave equation

$$[-\hat{\nabla}^2 - \hat{k}^2 + \hat{V}_s]\hat{\psi}(\vec{r}) = -2\hat{E}\hat{V}_c\hat{\psi}(\vec{r}), \tag{1}$$

where $\hat{\nabla}^2$, $\hat{k}^2$, and $\hat{E}$ are multiples of the unit matrix. $\hat{\nabla}^2$ is the gradient operator, $\hat{k}^2$ is the square of the laboratory momentum of the incident particle, and $\hat{E}$ is its total relativistic energy in the laboratory. $\hat{V}_s$ and $\hat{V}_c$ are matrices (in general, nondiagonal) which represent the strong and Coulomb forces, respectively. $\hat{\psi}(\vec{r})$ is the wave function and is a column vector.

Since we shall consider only spherically symmetric potentials, we expand $\hat{\psi}(\vec{r})$ as

$$\psi_p(\vec{r}) = \sum_l \frac{U_{pl}(r)}{r} P_l(\cos\theta), \tag{2}$$

where $p$ labels the element of the vector $\hat{\psi}$, i.e., the scattering channel. In terms of $\hat{U}_l$, the wave equation becomes

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]\hat{U}_l + (\hat{k}^2 - \hat{V}_s)\hat{U}_l = 2\hat{E}\hat{V}_c\hat{U}_l . \tag{3}$$

We define the $S$ matrix in terms of the asymptotic behavior of $U_{pl}(r)$. If for large $r$

$$U_{pl}(r) \to A_{pl}e^{-i(kr-l\pi/2)} - B_{pl}e^{+i(kr-l\pi/2)}, \tag{4}$$

then the $S$ matrix (for the $l$th partial wave) is defined by

$$B_{pl} = \sum_{p'} S^l_{pp'}A_{p'l}. \tag{5}$$

With this definition $\hat{S}^l$ is a symmetric, unitary matrix.

In terms of the $S$ matrix, the elastic cross section in the $p$th channel is given by

$$\sigma_p = \frac{\pi}{k^2}\sum_l (2l+1)|S^l_{pp} - 1|^2 \tag{6}$$

and the partial cross section from channel $p$ to $p'$ by

$$\sigma_{pp'} = \frac{\pi}{k^2}\sum_l (2l+1)|S^l_{pp'}|^2 .$$

If we define

$$A_{pp'} = \frac{1}{2ik}\sum_l (2l+1)(S^l_{pp'} - \delta_{pp'})P_l(\cos\theta), \tag{7}$$

then the differential cross section from channel $p$ to $p'$ is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{pp'} = |A_{pp'}|^2 . \tag{8}$$

Because $\hat{S}^l$ is a unitary, symmetric matrix, its most general form in the two-channel case is

$$\hat{S}^l = \begin{pmatrix} \eta_l e^{2i\delta_l} & i(1-\eta_l^2)^{1/2}e^{i(\delta_l+\Delta_l)} \\ i(1-\eta_l^2)^{1/2}e^{i(\delta_l+\Delta_l)} & \eta_l e^{2i\Delta_l} \end{pmatrix}, \tag{9}$$

where $\eta_l$, $\delta_l$, and $\Delta_l$ are real numbers. If we use an absorptive potential to represent additional inelasticity, then $\delta_l$ and $\Delta_l$ acquire positive imaginary parts.

## III. REVIEW OF THE SINGLE-CHANNEL CALCULATION

Recall that for large $r$ the Coulomb potential is given by

$$v_c(r) \equiv \frac{2E}{k}V_c(r) = \frac{2E}{k}\frac{Z_1Z_2e^2}{r} \equiv \frac{2\xi}{r}, \tag{10}$$

where $\xi = Z_1Z_2\alpha E/k$. For values of $r$ smaller then the sum of the charge radii of the colliding particles, we can expect $V_c$ to be modified. In what follows $V_c$ will always represent this modified Coulomb potential, which behaves asymptotically like Eq. (10).

In the case of pure Coulomb scattering alone when the only potential acting is that of Eq. (10), the wave function $U_l$ behaves asymptotically like

$$U_l(r) \to e^{\pm i[kr - l\pi/2 + p(r)]}, \tag{11}$$

where $p(r) = \delta_l^c - \xi\ln(2kr)$ and $\delta_l^c$ (the pure Coulomb phase shift) is given by

$$\delta_l^c = \arg[\Gamma(l+1+i\xi)]. \tag{12}$$

This should be contrasted with Eq. (4), which holds for potentials which decrease asymptotically faster than $1/r$. Thus, we must be careful to define the total phase shift relative to the $\ln(2kr)$ factor. In the problem which includes a strong potential plus the modified Coulomb potential $V_c$, the total phase shift will be of the form

$$\delta_l = \delta_l^c + \delta_l^s + \delta_l^{cs} \tag{13}$$

relative to the $\ln(2kr)$ term. $\delta_l^s$ is the pure strong phase shift which would result if we set $V_c$ equal to zero.

To find an expression for $\delta_l^{cs}$, let us assume that the solution to the strong-scattering problem is known. The two independent solutions to the wave equation for strong scattering, i.e., Eq. (3) with $V_c = 0$, we call $R_l(r)$, and $I_l(r)$. We choose a normalization such that asymptotically

$$\begin{aligned} R_l(r) &\to \sin(kr - \tfrac{1}{2}l\pi + \delta_l^s), \\ I_l(r) &\to -\cos(kr - \tfrac{1}{2}l\pi + \delta_l^s). \end{aligned} \tag{14}$$

Note that $R_l$ and $I_l$ are analogous to the solutions

of the free wave equation, $krj_l(kr)$ and $krn_l(kr)$, respectively. In terms of $R_l$ and $I_l$ we define the Green's function

$$G_l(r,r') = R_l(r_<)[I_l(r_>) - iR_l(r_>)]. \tag{15}$$

The formal solution to our wave equation, Eq. (3), can be written as

$$U_l(r) = R_l(r) + \int_0^\infty G_l(r,r')v_c(r')U_l(r')dr'. \tag{16}$$

Without modification, Eq. (16) is divergent because of the extra $\ln(2kr)$ factor in the Coulomb phase. As a formal artifice, we remove this divergence by replacing the upper limit of the integration by a cutoff, $R_c$. However, as we shall see, we shall be able to pass to the limit $R_c \to \infty$ in our final answer by removing the $\ln(2kR_c)$ term.

To lowest order in $V_c$, Eq. (16) yields

$$U_l(r) = R_l(r) + \int_0^{R_c} G_l(r,r')v_c(r')R_l(r')dr'. \tag{17}$$

Using Eq. (16), we now let $r$ become very large. In terms of our definition, Eq. (4), we find

$$\begin{aligned} A &= -\frac{1}{2i}\, e^{-i\delta_l^s}, \\ B &= \left[-\frac{1}{2i} + \int_0^{R_c} R_l^2(r')v_c(r')dr'\right] e^{i\delta_l^s}, \end{aligned} \tag{18}$$

which implies that

$$S_l = e^{2i\delta_l^s}\left[1 - 2i\int_0^{R_c} R_l^2(r')v_c(r')dr'\right]. \tag{19}$$

To interpret Eq. (19), we rewrite the integration as

$$\int_0^{R_c}\left[R_l^2(r')v_c(r') - (kr')^2 j_l^2(kr')\frac{2\xi}{r'}\right]dr' + \int_0^{R_c}(kr')^2 j_l^2(kr')\frac{2\xi}{r'}\,dr'. \tag{20}$$

As was shown in a previous paper,[6] for large $R_c$ and to lowest order in $\xi$, the second term in Eq. (20) behaves like

$$\int_0^{R_c}(kr')^2 j_l^2(kr')\frac{2\xi}{r'}\,dr' \to -\delta_l^c + \xi\ln(2kR_c). \tag{21}$$

Thus, we can identify (to lowest order in $\xi$)

$$\delta_l^{cs} = -\int_0^\infty\left[R_l^2(r')v_c(r') - (kr')^2 j_l^2(kr')\,\frac{2\xi}{r'}\right]dr', \tag{22}$$

where we have let $R_c \to \infty$ in Eq. (22) because the integral converges. Note that

$$S_l = e^{2i(\delta_l^c + \delta_l^s + \delta_l^{cs})}e^{-2i\xi\ln(2kR_c)} \tag{23}$$

but as is conventional, we shall redefine $S_l$ by dropping the over-all infinite phase factor.

Equation (22) is our principal single-channel result; a simple expression for the Coulomb strong phase, $\delta_l^{cs}$. In actual data analysis we first find an approximate $\delta_l^s$ by assuming $\delta_l^{cs} = 0$. This allows us to find $R_l(r)$ to within corrections of order $\alpha$. $\delta_l^{cs}$ can now be found to order $\alpha$ by using Eq. (22) and the data reanalyzed with $\delta_l^{cs}$ included, which yields $\delta_l^s$ to within order $\alpha^2$ as desired. To do the analysis it is usually convenient to write the $T$ matrix, $T = S - 1$, as

$$\begin{aligned} T_l &= (e^{2i\delta_l^c} - 1) + e^{2i\delta_l^c}(e^{2i\delta_l^s} - 1) \\ &\quad + e^{2i\delta_l^c}e^{2i\delta_l^s}(e^{2i\delta_l^{cs}} - 1). \end{aligned}$$

Because the Coulomb phases $\delta_l^c$ increase like $\xi\ln(l+1)$ for large $l$, we must include all terms in the partial-wave expansion of the first term in $T_l$. In our normalization

$$\frac{1}{2ik}\sum_l (2l+1)(e^{2i\delta_l^c} - 1)P_l(\cos\theta) = \frac{-\xi}{2k\sin^2(\frac{1}{2}\theta)}\, e^{i\{2\delta_0^c - \xi\ln[\sin^2(\frac{1}{2}\theta)]\}}.$$

The other terms in $T_l$ are easily handled since $\delta_l^s$ and $\delta_l^{cs}$ go to zero rapidly as $l$ increases.

In order to find $\delta_l^{cs}$ we have introduced a cutoff, $R_c$, into our analysis. It is not obvious that the limit $R_c \to \infty$ can be taken uniformly. However, this has been discussed in Ref. 6, where it is shown that this technique can be extended to calculate the higher-order corrections to $\delta_l^{cs}$. We emphasize that the result in Eq. (22) is correct to first order in $V_c$ but to all orders in $V_s$.

## IV. TWO CHANNELS WITH ISOSPIN SYMMETRY

The advantage of having the strong force $\hat{V}_s(r)$ be symmetric under isospin is that we can diagonalize the matrix $\hat{V}_s(r)$ by a rotation which is independent of $r$. For example, in the case of $\pi$-$p$ and charge-exchange pion-nucleon scattering, we can label the matrices as

$$\begin{pmatrix} \pi^-p \to \pi^-p & \pi^-p \to \pi^0n \\ \pi^0n \to \pi^-p & \pi^0n \to \pi^0n \end{pmatrix}. \tag{24}$$

If we assume isospin symmetry of $\hat{V}_s$, then we know it can be written

$$\hat{V}_s = \begin{pmatrix} \frac{1}{3}V_3 + \frac{2}{3}V_1 & \frac{1}{3}\sqrt{2}\,(V_3 - V_1) \\ \frac{1}{3}\sqrt{2}\,(V_3 - V_1) & \frac{2}{3}V_3 + \frac{1}{3}V_1 \end{pmatrix}, \tag{25}$$

which is diagonalized by the matrix

$$U = \begin{pmatrix} -(\frac{2}{3})^{1/2} & (\frac{1}{3})^{1/2} \\ (\frac{1}{3})^{1/2} & (\frac{2}{3})^{1/2} \end{pmatrix}, \tag{26}$$

i.e.,

$$\hat{V}_s^u \equiv U\hat{V}_s U^\dagger = \begin{pmatrix} V_1 & 0 \\ 0 & V_3 \end{pmatrix} \tag{27}$$

and in this representation the isospin $-\frac{1}{2}$ and $-\frac{3}{2}$ states do not mix. Hence, we shall be able to find a simple expression for the Green's function necessary for our calculation.

The Coulomb potential $\hat{V}_c$ is in general a diagonal matrix. For the example just discussed it takes the form

$$\hat{V}_c(r) = \begin{pmatrix} V_c(r) & 0 \\ 0 & 0 \end{pmatrix}, \tag{28}$$

with $Z_1 = +1$, $Z_2 = -1$.

Because it is the most relevant to current experimental data, we shall not work with the completely general case in this section but shall specialize to the example just mentioned. Changes necessary for kaon-nucleon scattering are easily handled.

Since results are always quoted for the isospin $-\frac{1}{2}$ and $-\frac{3}{2}$ phase shifts, it is more convenient to work directly in the representation, where $\hat{V}_s^u$ is diagonal. Thus, when we transform the whole equation with $U$, $\hat{V}_c$ becomes [from Eq. (28)]

$$\hat{V}_c^u = U\hat{V}_c U^\dagger = \begin{pmatrix} \frac{2}{3}V_c & -\frac{1}{3}\sqrt{2}\,V_c \\ -\frac{1}{3}\sqrt{2}\,V_c & \frac{1}{3}V_c \end{pmatrix}. \tag{29}$$

We now proceed to find the Green's function for the strong-scattering problem. Since $\hat{V}_s^u$ is diagonal, the strong equation is completely diagonal. Thus, the strong Green's function takes the simple form

$$\hat{G}_l(r,r') = \begin{pmatrix} G_l^{(1)}(r,r') & 0 \\ 0 & G_l^{(3)}(r,r') \end{pmatrix}, \tag{30}$$

where $G_l^{(i)}(r,r')$ is defined in analogy to $G_l(r,r')$ of Eq. (15), i.e., in terms of $R_l^{(i)}$ and $I_l^{(i)}$, which behave asymptotically like

$$\begin{aligned} R_l^{(i)}(r) &\to \sin(kr - \tfrac{1}{2}l\pi + \delta_l^{s(i)}), \\ I_l^{(i)}(r) &\to -\cos(kr - \tfrac{1}{2}l\pi + \delta_l^{s(i)}). \end{aligned} \tag{31}$$

The solution to the full wave equation is now written

$$\hat{U}_l^u(r) = \hat{U}_l^0(r) + \int_0^{R_c} \hat{G}_l(r,r')\hat{v}_c^u(r')\hat{U}_l^u(r')dr'. \tag{32}$$

In order to find $\hat{S}_l^u$, we choose both

$$\hat{U}_l^0(r) = \begin{pmatrix} R_l^{(1)}(r) \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ R_l^{(3)}(r) \end{pmatrix}. \tag{33}$$

Thus, we find two pairs of vectors: $\hat{A}^{(1)}, \hat{B}^{(1)}$ and $\hat{A}^{(3)}, \hat{B}^{(3)}$. These are sufficient to determine $\hat{S}_l^u$ completely by using Eq. (5). If we define

$$C_{ij}^l = \int_0^{R_c} R_l^{(i)}(r')R_l^{(j)}(r')v_c(r')dr', \tag{34}$$

we find, after some straightforward algebra,

$$\begin{aligned} S_{11}^u &= (1 - \tfrac{4}{3}iC_{11}^l)\exp(2i\delta_l^{s(1)}), \\ S_{13}^u &= S_{31}^u = \tfrac{2}{3}\sqrt{2}\;iC_{13}^l \exp[i(\delta_l^{s(1)} + \delta_l^{s(3)})], \\ S_{33}^u &= (1 - \tfrac{2}{3}iC_{33}^l)\exp(2i\delta_l^{s(3)}). \end{aligned} \tag{35}$$

In order to remove the infinite phases from the $S$ matrix, it is more convenient to transform back to the charge channels by $\hat{S}_l = U\hat{S}_l^u U^\dagger$. In this representation we expect the infinite phases to be distributed in $\hat{S}_l$ as follows:

$$\hat{S}_l \sim \begin{pmatrix} e^{-2i\xi\,\ln(2kR_c)} & e^{-i\xi\,\ln(2kR_c)} \\ e^{-i\xi\,\ln(2kR_c)} & 1 \end{pmatrix}, \tag{36}$$

We remove these using the same trick employed in Sec. III by defining the finite quantities

$$\begin{aligned} d_{11}^l &= -\frac{2}{3}\int_0^\infty \left\{ [R_l^{(1)}(r)]^2 v_c(r) - (kr)^2[j_l(kr)]^2 \frac{2\xi}{r} \right\} dr, \\ d_{33}^l &= -\frac{1}{3}\int_0^\infty \left\{ [R_l^{(3)}(r)]^2 v_c(r) - (kr)^2[j_l(kr)]^2 \frac{2\xi}{r} \right\} dr, \\ d_{13}^l &= \int_0^\infty \left\{ R_l^{(1)}(r)R_l^{(3)}(r)v_c(r) \right. \\ &\qquad \left. -\cos(\delta_l^{s(1)} - \delta_l^{s(3)})(kr)^2[j_l(kr)]^2 \frac{2\xi}{r} \right\} dr. \end{aligned} \tag{37}$$

Finally, we can write the $S$ matrix as

$$\begin{aligned} S_{--}^l &= e^{2i\delta_l^c}\{\tfrac{2}{3}\exp[2i(\delta_l^{s(1)} + d_{11}^l)] + \tfrac{1}{3}\exp[2i(\delta_l^{s(3)} + d_{33}^l)] \\ &\qquad - \tfrac{8}{9}id_{13}^l \exp[i(\delta_l^{s(1)} + \delta_l^{s(3)})]\}, \\ S_{00}^l &= \tfrac{1}{3}\exp[2i(\delta_l^{s(1)} + d_{11}^l)] + \tfrac{2}{3}\exp[2i(\delta_l^{s(3)} + d_{33}^l)] \\ &\qquad + \tfrac{8}{9}id_{13}^l \exp[i(\delta_l^{s(1)} + \delta_l^{s(3)})], \\ S_{-0}^l &= e^{i\delta_l^c}\{-\tfrac{1}{3}\sqrt{2}\exp[2i(\delta_l^{s(1)} + d_{11}^l)] \\ &\qquad + \tfrac{1}{3}\sqrt{2}\exp[2i(\delta_l^{s(3)} + d_{33}^l)] \\ &\qquad - \tfrac{2}{9}\sqrt{2}\,id_{13}^l \exp[i(\delta_l^{s(1)} + \delta_l^{s(3)})]\}. \end{aligned} \tag{38}$$

Note that in this representation not all corrections appear as corrections to the phases directly. This is because we have not written $\hat{S}_l$ in its canonical form, Eq. (9). If we rewrite $\hat{S}_l$ in the form of Eq. (9), it is easily seen that corrections are present not only for the phases $\delta$ and $\Delta$, but also for the elasticity factor $\eta$. This change in $\eta$ appears as the $d_{13}^l$ terms in Eq. (38).

## V. TWO CHANNELS WITH NO STRONG-FORCE SYMMETRY

The basic difficulty in treating the Coulomb cor-

rections in the situation where the strong force satisfies no symmetry is that the strong Green's function is difficult to find. Its form is rather complicated, which makes the details of the derivation complicated. Thus, in this section we shall merely quote results rather than attempt to present the derivation. It should be noted, however, that the method is basically straightforward and follows exactly the principles discussed in Sec. IV.

We begin with a general strong-force matrix $\hat{V}_s$ but quote results for the case when $\hat{V}_c$ has the form

$$\hat{V}_c = \begin{pmatrix} V_c & 0 \\ 0 & 0 \end{pmatrix} \tag{39}$$

as in Sec. IV. With the Coulomb forces turned off, we define the strong $S$ matrix as

$$\hat{S}_l = \begin{pmatrix} \eta_l e^{2i\delta_l} & i(1-\eta_l^2)^{1/2} e^{i(\delta_l+\Delta_l)} \\ i(1-\eta_l^2)^{1/2} e^{i(\delta_l+\Delta_l)} & \eta_l e^{2i\Delta_l} \end{pmatrix}$$

$$\equiv \begin{pmatrix} e^{2i\tilde{\delta}_l} & \dfrac{i(1-\eta_l^2)^{1/2}}{\eta_l} e^{i(\tilde{\delta}_l+\tilde{\Delta}_l)} \\ \dfrac{i(1-\eta_l^2)^{1/2}}{\eta_l} e^{i(\tilde{\delta}_l+\tilde{\Delta}_l)} & e^{2i\tilde{\Delta}_l} \end{pmatrix}, \tag{40}$$

where $\tilde{\delta}_l$ and $\tilde{\Delta}_l$ are complex quantities.

There are four independent solutions to the strong-scattering problem, and we shall need the two regular solutions which behave asymptotically as

$$\hat{R}_l(r) \to \begin{pmatrix} \sin(kr - \frac{1}{2}l\pi + \tilde{\delta}_l) \\ \dfrac{(1-\eta_l^2)^{1/2}}{2\eta_l} e^{i(kr - \frac{1}{2}l\pi + \tilde{\Delta}_l)} \end{pmatrix},$$

$$\hat{W}_l(r) \to \begin{pmatrix} \dfrac{(1-\eta_l^2)^{1/2}}{2\eta_l} e^{i(kr - \frac{1}{2}l\pi + \tilde{\delta}_l)} \\ \sin(kr - \frac{1}{2}l\pi + \tilde{\Delta}_l) \end{pmatrix}. \tag{41}$$

That solutions with these asymptotic forms exist follows directly from Eqs. (40) and (5) by making two independent choices of the components of $A_p$.

We shall refer to the upper and lower components of $\hat{R}_l$ and $\hat{W}_l$ by the superscripts (1) and (2), respectively. In terms of these wave functions we define the following finite quantities:

$$\alpha_l^R = \int_0^\infty \left[ [R_l^{(1)}(r)]^2 v_c(r) - (kr)^2 [j_l(kr)]^2 \frac{2\xi}{r} \right] dr,$$

$$\alpha_l^W = \int_0^\infty [W_l^{(1)}(r)]^2 v_c(r) dr, \tag{42}$$

$$\beta_l = \int_0^\infty \left[ R_l^{(1)}(r) W_l^{(1)}(r) v_c(r) - \frac{i(1-\eta_l^2)^{1/2}}{2\eta_l} (kr)^2 [j_l(kr)]^2 \frac{2\xi}{r} \right] dr.$$

These quantities satisfy the following identities among themselves:

$$\mathrm{Im}(\alpha_l^W) = \mathrm{Im}(\alpha_l^R),$$

$$\mathrm{Re}(\beta_l) = \frac{\eta_l}{(1-\eta_l^2)^{1/2}} \mathrm{Im}(\alpha_l^R), \tag{43}$$

$$\mathrm{Im}(\beta_l) = \frac{(1-\eta_l^2)^{1/2}}{2\eta_l} \mathrm{Re}(\alpha_l^R + \alpha_l^W).$$

We can now specify how the $S$ matrix is modified by the Coulomb interaction. Equation (40) is modified as follows:

$$\begin{aligned} &\eta_l \text{ becomes } \eta_l e^{2\mathrm{Im}(\alpha_l^R)}, \\ &\delta_l \text{ becomes } \delta_l - \mathrm{Re}(\alpha_l^R) + \delta_l^c, \\ &\Delta_l \text{ becomes } \Delta_l - \mathrm{Re}(\alpha_l^W). \end{aligned} \tag{44}$$

Although these results are for the simplified Coulomb force in Eq. (39), more complicated situations can be handled by making symmetric replacements in Eq. (44). For example, if $\hat{V}_c$ takes the form

$$\hat{V}_c = \begin{pmatrix} 0 & 0 \\ 0 & V_c \end{pmatrix} \tag{45}$$

then the lower components of $\hat{R}_l$ and $\hat{W}_l$ are relevant. Addition of the results from Eq. (39) and Eq. (45) then provide us with the most general possible case. Two things should be noted. First, we can add such results because we are working to lowest order in $\alpha$, and second, $Z_1$ and $Z_2$ may be different in Eqs. (39) and (45).

Equations (42) and (44) allow us to find the Coulomb corrections to strong scattering without isospin symmetry. However, in actual practice the wave functions $\hat{R}_l$ and $\hat{W}_l$ may be difficult to calculate. In most physical situations the strong force is approximately symmetric. If we can treat the symmetry violation in $\hat{V}_s$ as a perturbation, then the results of Sec. IV can be used. That is, we can find the corrections due to $\hat{V}_c$ using the symmetric part of $\hat{V}_s$ as a perturbation. Since $\hat{V}_s$ is a short-range force, ordinary first-order perturbation theory can be applied to its symmetry-breaking part. Because we are working only to lowest order, the resulting phase-shift correction can simply be added to the result calculated by using Sec. IV.

## VI. CONCLUSION

The main results of the paper are Eq. (22) for single-channel scattering and Eq. (37) for the two-channel case with isospin symmetry. To use these equations in actual data analysis, one must choose a form for the strong potential (adjusted to give the correct strong phase shifts) and specify the form of the Coulomb potential inside the charge distribution. For example, one might decide to use a

square well of radius $R$ for both the strong force and the charge distribution. The depth of the well is adjusted (for each partial wave) to reproduce the strong phases, and the Coulomb potential takes the form

$$V_c(r) = 2\xi/r, \quad r \geq R$$
$$= (\xi/R)(3 - r^2/R^2), \quad r \leq R. \tag{46}$$

The procedure to be used in fitting was discussed at the end of Sec. III. Possible modifications of Eqs. (22) and (37) to the case where relativistic effects are important are discussed in detail in the Appendix.

Although Eqs. (22) and (37) are unambiguous and simple to apply, one can certainly question their relevance to relativistic strong scattering processes. Two questions present themselves immediately. The first is whether calculations are sensitive to the choice of a form for the strong potential and the inner Coulomb potential. The second is whether the equations represent correctly the most important Coulomb effects in energy regions of experimental interest.

Actual numerical evaluation of these equations seems to indicate that they are rather insensitive to the form of the strong and inner Coulomb potentials. Results obtained by using a square well, Gaussian well, and exponential well are almost identical.[8] Typically, differences of the order of 10% are found in the Coulomb strong phases, $\delta_l^{cs}$. This is well within limits dictated by experimental accuracy. Thus, we can conclude that the first question does not inhibit applications of our results.

No definite conclusion can be reached on the second question. However, applications to actual data have been very encouraging. Inclusion of these Coulomb corrections definitely improves fits to pion-helium data.[9] More recently, Bugg *et al.*[7] have applied our Eqs. (22) and (37) in analyzing their very accurate pion-nucleon cross-section data. This data is a measurement of the total and charge-exchange cross sections in the neighborhood of the $N^*(1236)$ resonance. The data cover the energy range from 70 to 290 MeV with an accuracy of $\pm\frac{1}{2}\%$. Before Coulomb corrections are applied ($\delta^c$ included but not $\delta^{cs}$) they find

$$M(N^{*0}) - M(N^{*++}) = 1.3 \pm 0.8 \text{ MeV},$$
$$\Gamma(N^{*0}) - \Gamma(N^{*++}) = 5 \pm 4 \text{ MeV}.$$

Including $\delta^{cs}$ from Eqs. (22) and (37), these results become

$$M(N^{*0}) - M(N^{*++}) = 2.9 \text{ MeV},$$
$$\Gamma(N^{*0}) - \Gamma(N^{*++}) = 1.9 \text{ MeV},$$

with similar errors. These corrected results are in good agreement with our expectation that $\Gamma(N^{*0}) = \Gamma(N^{*++})$ and with the theoretical prediction by Socolow[10] that $M(N^{*0}) - M(N^{*++}) = 2.4$ MeV.

In conclusion it seems that our results predict Coulomb corrections which are in good agreement with experimental data. It is hoped that experimentalists will find our results of interest for further data analysis. Certainly the simplicity of Eqs. (22) and (37) recommends their application.

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## APPENDIX: COMPARISON OF COULOMB EFFECTS IN THE SCHRÖDINGER AND KLEIN-GORDON EQUATIONS

Both the Schrödinger and the Klein-Gordon equations have serious difficulties associated with them in a calculation such as ours. The Schrödinger equation can be used to describe the motion of the beam and the target simultaneously but gives the nonrelativistic form to the Born approximation to pure Coulomb scattering. On the other hand, the Klein-Gordon equation correctly reproduces the full relativistic Coulomb Born approximation, but it cannot be formulated in other than the static approximation for the target motion. It would presumably be better to work with the Bethe-Salpeter equation, but this presents technical difficulties which have not been overcome.

Let us compare the two equations in the case when we have a strong potential which is isospin invariant and a Coulomb potential. The Schrödinger equation is

$$\begin{pmatrix} -\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + m_1 + m_2 + U_{11} & U_{12} \\ U_{21} & -\frac{\nabla_1^2}{2m_3} - \frac{\nabla_2^2}{2m_4} + m_3 + m_4 + U_{22} \end{pmatrix} \begin{pmatrix} \psi_{12} \\ \psi_{34} \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \psi_{12} \\ \psi_{34} \end{pmatrix}, \tag{A1}$$

where we have explicitly kept the mass contributions to the total energy because they are necessary to conserve energy when the masses of the two channels are different. Introducing relative coordinates (we assume that $\hat{U}$ is a function of $x_1 - x_2$) and assuming that the center of mass is at rest, this becomes

$$\begin{pmatrix} -\frac{\nabla^2}{2\mu_{12}} + m_1 + m_2 + U_{11} & U_{12} \\ U_{21} & -\frac{\nabla^2}{2\mu_{34}} + m_3 + m_4 + U_{22} \end{pmatrix} \begin{pmatrix} \psi_{12} \\ \psi_{34} \end{pmatrix} = E \begin{pmatrix} \psi_{12} \\ \psi_{34} \end{pmatrix}, \tag{A2}$$

where $\mu_{12}$ and $\mu_{34}$ are the reduced masses of the two channels. We now define $k_{12}^2$ and $k_{34}^2$ as

$$E = m_1 + m_2 + \frac{k_{12}^2}{2\mu_{12}} = m_3 + m_4 + \frac{k_{34}^2}{2\mu_{34}}. \tag{A3}$$

Now one can easily see that Eq. (A1) is not in general Galilean-invariant unless certain conditions are satisfied.

We are interested in the case where $m_1 - m_3$ and $m_2 - m_4$ are small. If we refer to this order of magnitude as $\Delta m$, then one can show that Galilean invariance is maintained if we drop terms in our equation which are of order $(\Delta m)^2/m$ and $(\Delta m/m)(k^2/m)$ but keeps terms of order $m$, $k^2/m$, and $\Delta m$.[11] Expanding Eqs. (A2) and (A3) to this order, we secure

$$\begin{pmatrix} -\frac{\nabla^2}{2\mu} + m_1 + m_2 + U_{11} & U_{12} \\ U_{21} & -\frac{\nabla^2}{2\mu} + m_3 + m_4 + U_{22} \end{pmatrix} \begin{pmatrix} \psi_{12} \\ \psi_{13} \end{pmatrix} = E \begin{pmatrix} \psi_{12} \\ \psi_{34} \end{pmatrix} \tag{A4}$$

and

$$E = m_1 + m_2 + \frac{k_{12}^2}{2\mu} = m_3 + m_4 + \frac{k_{34}^2}{2\mu}, \tag{A5}$$

where $\mu = \frac{1}{2}(\mu_{12} + \mu_{34})$. Thus, in matrix notation,

$$(-\hat{\nabla}^2 - \hat{k}^2)\hat{\psi} = -2\mu\hat{U}_s\hat{\psi} - 2\mu\hat{U}_c\hat{\psi}, \tag{A6}$$

where $\hat{U}_s$ and $\hat{U}_c$ are the strong and Coulomb potentials, respectively, and $\hat{k}^2$ is determined by Eq. (A5).

The Klein-Gordon equation is

$$(-\hat{\nabla}^2 + \hat{m}^2)\hat{\varphi} = (W - \hat{V})^2\hat{\varphi}. \tag{A7}$$

Since this equation is a static approximation, the target is assumed to be very massive and at rest, i.e., $m_2 = m_4 \gg m_1$ or $m_3$. The diagonal matrix, $\hat{m}$, refers to the masses $m_1$ and $m_3$, and we define

$$\begin{pmatrix} W^2 & 0 \\ 0 & W^2 \end{pmatrix} = \hat{K}^2 + \hat{m}^2 \tag{A8}$$

so that $\hat{K}$ is the diagonal matrix of the laboratory momenta of particles 1 and 3. Thus

$$(W - \hat{V})^2 = \hat{K}^2 + \hat{m}^2 - 2W\hat{V}_s + \hat{V}_s^2 - 2W\hat{V}_c + \hat{V}_s\hat{V}_c + \hat{V}_c\hat{V}_s.$$

We have neglected $\hat{V}_c^2$ which is of order $\alpha^2$. In actual calculations, $\hat{V}_s$ is chosen at each energy to fit the strong phase shifts. Without loss then, we define an effective strong potential

$$\hat{V}_E = \hat{V}_s - \hat{V}_s^2/2W. \tag{A9}$$

The Klein-Gordon equation becomes

$$(-\hat{\nabla}^2 - \hat{K}^2)\varphi = -2W\hat{V}_E\hat{\varphi} - [(W^2 - 2W\hat{V}_E)^{1/2}\hat{V}_c + \hat{V}_c(W^2 - 2W\hat{V}_E)^{1/2}]\hat{\varphi}. \tag{A10}$$

Note that since $\hat{V}_s$ is isospin symmetric, so is $\hat{V}_E$. Therefore, the square-root matrix in Eq. (A10) is well defined.

In order to compare Eqs. (A10) and (A6) we must take the static approximation, $m_2 = m_4 \gg m_1$ and $m_3$, in Eq. (A6) and redefine the energy with respect to the new level, $m_2 = m_4$. It then becomes

$$(-\hat{\nabla}^2 - \hat{k}_s^2)\hat{\psi} = -2\mu_s\hat{U}_s\hat{\psi} - 2\mu_s\hat{U}_c\hat{\psi},$$

where

$$m_1 + \frac{k_{s_1}^2}{2\mu_s} = m_3 + \frac{k_{s_2}^2}{2\mu_s} \tag{A11}$$

and

$$\mu_s = \tfrac{1}{2}(m_1 + m_3).$$

For comparison we neglect $W\hat{V}_E$ with respect to $W^2$ in Eq. (A10), which then becomes

$$(-\hat{\nabla}^2 - \hat{K}^2)\hat{\varphi} = -2W\hat{V}_E\hat{\varphi} - 2W\hat{V}_c\hat{\varphi}. \tag{A12}$$

Recalling that we are to neglect terms of order $(k^2/m)(\Delta m/m)$, we note that for nonrelativistic energies $\hat{K}^2 = \hat{k}^2$. Therefore, Eqs. (A11) and (A12) differ only in the fact that $\mu_s$ in Eq. (A11) becomes $W$ in Eq. (A12). This is exactly what distinguishes the Coulomb Born approximation in the two equations; Eq. (A12) agrees with the field-theory result.

In our calculations of Secs. II–V we have used Eq. (A12) with $m_1 = m_3$ in order to facilitate com-

parison with other authors who have made this choice.[1] However, we may ask whether a comparison of Eqs. (A11) and (A12) does not suggest another approximation. Certainly for nonrelativistic momenta, we should prefer the Schrödinger equation, Eq. (A6), because it takes into account the recoil of the target. Unfortunately, for larger $k^2$ it does not adequately represent the relativistic dependence of the Coulomb Born term. By comparing Eqs. (A11) and (A12) we saw that this could be traced to the factor $2\mu$ in Eq. (A6). This suggests that changing this factor to a more suitable relativistic form might be a sensible way to extend the nonstatic Schrödinger equation to the relativistic domain. If $E_i$ is the total energy of the $i$th particle in the center-of-mass system, e.g., $E_1^2 = k_{12}^2 + M_1^2$, it is easy to see that we can implement the above idea by replacing, in Eq. (A6), $2\mu$ by the matrix

$$\begin{pmatrix} \dfrac{2(E_1E_2 + k_{12}^2)}{(E_1+E_2)} & 0 \\ 0 & \dfrac{2(E_3E_4 + k_{34}^2)}{(E_3+E_4)} \end{pmatrix}. \qquad \text{(A13)}$$

This yields the correct form for the Coulomb Born term in the center-of-mass system and makes Eq. (A6) reduce to the Klein-Gordon result, Eq. (A12), in the static limit, $E_2 = E_4 >> E_1$ and $E_3$.

This new equation also has one other interesting property. In Eq. (A6) the mass differences enter only through the definition of $\hat{k}^2$. However, our replacement of $2\mu$ by the matrix in Eq. (A13) introduces new effects. If $\sqrt{s} = E_1 + E_2 = E_3 + E_4$ is the total energy, then

$$\frac{2(E_1E_2 + k_{12}^2)}{(E_1+E_2)} = \frac{s - m_1^2 - m_2^2}{\sqrt{s}}$$

and

$$\frac{2(E_3E_4 + k_{34}^2)}{(E_3+E_4)} = \frac{s - m_3^2 - m_4^2}{\sqrt{s}}.$$

We can write the matrix, Eq. (A13), as

$$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix},$$

where

$$D = \frac{1}{\sqrt{s}}\left[s - \tfrac{1}{2}(m_1^2 + m_2^2 + m_3^2 + m_4^2)\right],$$

$$\Delta = \frac{1}{2\sqrt{s}}(m_3^2 - m_1^2 + m_4^2 - m_2^2).$$

Since $\Delta$ is of order $\alpha$, we can neglect $\Delta \hat{U}_c$ and our new equation becomes

$$(-\hat{\nabla}^2 - \hat{k}^2)\hat{\psi} = -\hat{D}\hat{U}_s\hat{\psi} - \hat{\Delta}\hat{U}_s\hat{\psi} - \hat{D}\hat{U}_c\hat{\psi}, \qquad \text{(A14)}$$

in an obvious matrix notation. The term $\hat{\Delta}\hat{U}_s\hat{\psi}$ also contributes to the mass-breaking effect.

Although we do not explicitly treat mass differences in this paper, the other consequences of Eq. (A14) are easily incorporated into our results of Secs. II–V. To do this we merely interpret $k$ as the center-of-mass momentum and replace $\xi$ by

$$\frac{Z_1Z_2\alpha}{k}\frac{1}{2\sqrt{s}}\left[s - \tfrac{1}{2}(m_1^2 + m_2^2 + m_3^2 + m_4^2)\right]$$

instead of the definition given after Eq. (10). We note finally that such a change will not affect the data analysis referred to in Sec. VI because the laboratory and center-of-mass momenta differ by only 15% in that energy region.

Using Eq. (A14) it appears that we may be able to take into account all possible Coulomb effects in strong scattering except for the intrinsic breaking of isospin invariance of the strong potential. That one expects such effects is clear from studying vertex corrections in field theory, but we do not know how to measure this breaking either in terms of $U_c$ or $\Delta M$ in our potential model. We must, therefore, either hope that these effects are small or content ourselves with introducing them phenomenologically into Eq. (A14).

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