The Two Faces of a Dual Pion-Quark Model

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One of the new pion models recently constructed by the author is identical to Bardakci's pion sector of the dual quark model.

I. INTRODUCTION

At present, the literature contains three dual pion models, each of which is free of tachyons, possesses adequate spin gauges, and is extendable to fermions. These are: (1) the sector of the dual quark model[1] studied by Bardakci,[2] (2) a model by the author,[3] couched in terms of Neveu-Schwarz[4]-type (NS) operators, and (3) a "Pomeranchon-pion" model[3] with "internal isospin." It is the purpose of this paper to show that, although they are formulated on inequivalent Hilbert spaces, models (1) and (2) are identical. [Model (3), however, is not a quark model of the Bardakci-Halpern type.] Thus, we have two entirely different languages to describe the same meson universe. Each language carries its own distinctive approach to fermions,[5] of course, and the fascinating open question is whether or not the fermionic worlds are also identical. We make no attempt to answer this question here. It is further shown that the amplitudes for $n\,\rho$ mesons (even G-parity sector) of the two models are the same as in the NS model.

The plan of the paper is as follows. In Sec. II, we give a simple proof of the equivalence of the two models, using the z-dependent formalism.[6] The central observation is that the quark-model algebra of currents [the closed algebra of $A^\mu(z)$ (axial vector) and $T^{\mu\nu}(z)$ (tensor)] has an inequivalent representation in terms of NS-type operators. This allows us to write a single unified expression for the vertex and amplitudes of both models. In Sec. III, we push the identification further, finding a unified expression for the integrated form of both models. In this section, we use the vehicle of certain spin-spin interactions in the quark model. Though we use them only as a matter of technique here, these interactions may well be of interest in their own right for future models. Section IV is reserved for briefer remarks, including some related structure of the "Pomeranchon-pion" model (3).

II. EQUIVALENCE IN z FORMALISM

We begin by briefly summarizing the two models in the z-dependent formalism. For the dual quark model, the pion vertex[2] is

$$\Gamma^Q(k,z)=\Gamma^0(k,z)[\pi^5(z)-(i/\sqrt{2}\,)k\cdot A(z)]\,, \tag{2.1}$$

where $\Gamma^0(k,z)$ is the orbital vertex, π^5 is a fifth operator (without zero mode), and $A^\mu(z)$ is the axial-vector current defined as

$$\begin{aligned}A^\mu(z)&\equiv\sum_{n=-\infty}^{+\infty}A_n^\mu z^n\\&=:\bar\psi(z)\gamma_5\gamma^\mu\psi(z):\,.\end{aligned} \tag{2.2}$$

The n-pion function involves the usual z integration of the integrand

$$\langle 0|\Gamma^Q(k_1,z_1)\cdots\Gamma^Q(k_n,z_n)|0\rangle \times\langle 0|\Gamma^0(k_1,z_1)\cdots\Gamma^0(k_n,z_n)|0\rangle\,. \tag{2.3}$$

In evaluating the first factor above, which we will call the "spin factor," it is sufficient to forget the quarks and use instead the current algebra,

$$\begin{aligned}&[A_l^\mu,A_m^\nu]=2iT^{\mu\nu}_{n+m}-4g^{\mu\nu}l\delta_{l,-m}\,,\\&[A_l^\mu,T_m^{\lambda\eta}]=2i(g^{\mu\lambda}A^\eta_{l+m}-g^{\mu\eta}A^\lambda_{l+m})\,,\\&[T_l^{\mu\nu},T_m^{\lambda\eta}]=2i(g^{\nu\lambda}T^{\mu\eta}_{l+m}-g^{\nu\eta}T^{\mu\lambda}_{l+m}+g^{\mu\eta}T^{\nu\lambda}_{l+m}-g^{\mu\lambda}T^{\nu\eta}_{l+m})\\&\qquad+4l(g^{\mu\lambda}g^{\nu\eta}-g^{\mu\eta}g^{\nu\lambda})\delta_{l,-m}\,,\end{aligned} \tag{2.4}$$

together with the mode structure

$$A_l^\mu|0\rangle=T_l^{\mu\nu}|0\rangle=0,\quad l\geq 0\,. \tag{2.5}$$

Of course, $T_l^{\mu\nu}$ are the modes of the antisymmetric tensor current

$$\begin{aligned}T^{\mu\nu}(z)&=\sum_{n=-\infty}^{+\infty}T_n^{\mu\nu}z^n\\&=:\bar\psi(z)\sigma^{\mu\nu}\psi(z):\,.\end{aligned} \tag{2.6}$$

Now we state model (2). In this case, the vertex is

$$\Gamma^H(k,z)=\Gamma^0(k,z)[\pi^5(z)+\sqrt{2}\,k\cdot H(z)H^5(z)]\,, \tag{2.7}$$

and so on. In particular, the n-pion function involves the *same* z integration of the product of the *same* "orbital factor" now times the spin factor

$$\langle 0|\Gamma^H(k_1,z_1)\cdots\Gamma^H(k_n,z_n)|0\rangle\,. \tag{2.8}$$

Our next step is to prove that the two spin factors are equal.

We have already given the quark representation of the current algebra (2.4) above. Now we remark that the algebra has an inequivalent[7] representation in terms of the NS-type operators of model (2); that is,

$$\overline{A}_l^\mu \equiv 2i \sum_{n=-\infty}^{+\infty} b_{-n}^\mu b_{n+l}^5 \text{ (n summed over half-integers)}, \tag{2.9}$$

$$\overline{T}_l^{\mu\nu} \equiv -i \sum_{n=-\infty}^{+\infty} [b_{-n}^\mu, b_{n+l}^\nu],$$

also satisfy the current algebra (2.4) and preserve the mode structure (2.5). Reexpressing then

$$\Gamma^H(k,z) = \Gamma^0(k,z)[\pi^5(z) - (i/\sqrt{2})k\cdot\overline{A}(z)], \tag{2.10}$$
$$\overline{A}^\mu(z) \equiv \sum_{n=-\infty}^{+\infty} \overline{A}_n^\mu z^n = 2iH^\mu(z)H^5(z),$$

we see immediately that the two spin factors are algebraically identical; hence, the n-point functions are as well.

Understanding now that we are free to choose either representation, it is more elegant to drop the bar and summarize this section by writing a simple unified vertex,

$$\Gamma^Q(k,z) = \Gamma^H(k,z) = \Gamma(k,z)$$
$$= \Gamma^0(k,z)[\pi^5(z) - (i/\sqrt{2})k\cdot A(z)], \tag{2.11}$$

good in either language. Although the equivalence of the two models is now completely established, it will be instructive to see the equivalence over again in the integrated formalism. We now turn to this.

III. UNIFICATION IN THE INTEGRATED FORM: A HIERARCHY OF SPIN-SPIN INTERACTIONS

We begin with the integrated form of the quark model, namely (with momentum conservation understood),

$$B_n^Q = \langle 0|[\pi_{+1}^5 - (i/\sqrt{2})k_1\cdot A_{+1}]\Gamma(k_2,1)\Delta_{2,3}\cdots\Delta_{n-2,n-1}\Gamma(k_{n-1},1)[\pi_{-1}^5 - (i/\sqrt{2})k_n\cdot A_{-1}]|0\rangle, \tag{3.1}$$

where Δ_{ij} is the propagator between vertex i and j,

$$\Delta = (J_0^Q - 1)^{-1}; \tag{3.2}$$

and J_0^Q is the zero mode (Hamiltonian) of the dual quark-model conformal algebra. We recall

$$J_l^Q \equiv \hat{L}_l + N_l^Q, \tag{3.3}$$

where $\hat{L}_l$ is the five-dimensional orbital group (with $\pi_0^5 = 0$) and

$$N_l^Q \equiv -\frac{i}{4\pi}\int_0^{2\pi} e^{-il\theta} :\overline{\psi}\,\overleftrightarrow{\partial}_\theta\,\psi: . \tag{3.4}$$

We know from Sec. II that the vertices of this model have the algebraic structure of model (2), so we will focus our attention now on the propagators. At the moment, the propagators look entirely different. With an eye toward Sec. II, we suspect that our path should be to rewrite N_l^Q in terms of a sum of current-current (or spin-spin) interactions. As in Ref. 1, we have the alternative form,

$$N_l^Q = \tfrac{1}{40}(V^2 - A^2 - P^2)_l + \tfrac{1}{80}(T^2)_l + \tfrac{1}{8}(S^2)_l, \tag{3.5}$$

where

$$V^\mu(z) = :\overline{\psi}(z)\gamma^\mu\psi(z):,$$
$$P(z) = :\overline{\psi}(z)i\gamma_5\psi(z):,$$
$$S(z) = :\overline{\psi}(z)\psi(z):,$$

and we follow the notation of Ref. 1. For simplicity of notation, we will in general suppress mode-number subscripts except when needed, thus writing expressions like (3.5) as

$$N^Q = \tfrac{1}{40}(V^2 - A^2 - P^2) + \tfrac{1}{80}T^2 + \tfrac{1}{8}S^2. \tag{3.6}$$

N^Q is one of 31 spin-spin conformal groups that can be constructed out of these quartics (in the fashion of the spin-orbit models of Ref. 1). Each of these 31 models (except for N^Q) is paired with a (commuting) K-conjugate theory, as also discussed in Ref. 1. Moreover, each theory has linear trajectories with integral and/or half-integral spacing. In the discussion below, we shall encounter a number of these theories, and shall return in Sec. IV to make some remarks about their possible usefulness for new models. For this section, however, we shall be concerned with their use in explicitly eliminating certain degrees of freedom from (3.6).

We are now going to make a number of breakups of N^Q into commuting conjugate pairs. As a first breakup, we write

$$N^Q \equiv \overline{N}^Q + \tfrac{1}{8}S^2 \tag{3.7}$$

and note that each of the two terms form a conformal group, and the two groups commute. Of course, all bilinears except S are vectors under $\overline{N}^Q$, while S is a vector under $\tfrac{1}{8}S^2$. Thinking now of N^Q inserted *inside* the n-point functions (3.1), we note that we can simply *drop* the $\tfrac{1}{8}S^2$ term because it commutes with the vertices, and annihilates on the pion. Thus we have the quark model in the form (3.1) now with

$$\Delta \equiv \Delta(\overline{N}^Q) \equiv (\hat{L}_0 + \overline{N}_0^Q - 1)^{-1}. \tag{3.8}$$

In that the model is now formulated on the conformal group $\overline{N}^Q$ we have simply shown that, for the pion vertex, the S_l degrees of freedom can be eliminated. We have carefully explained this procedure of dropping the K-conjugate theory, as we intend doing it more rapidly several times further.

As our next breakup, we write

$$\begin{aligned} &\overline{N}^Q = N^H + K^H, \\ &N^H \equiv -\tfrac{1}{32}A^2 + \tfrac{1}{64}T^2, \\ &K^H \equiv \tfrac{1}{40}V^2 + \tfrac{1}{160}A^2 - \tfrac{1}{40}P^2 - \tfrac{1}{320}T^2, \end{aligned} \tag{3.9}$$

which again forms a conjugate pair of conformal groups. These groups have the following interesting properties: $A^\mu(z)$ and $T^{\mu\nu}(z)$ transform as vectors under N^H and commute with K^H. $V^\mu(z)$ and $P(z)$ transform nonlinearly under both N^H and K^H. For example,

$$K_0^H P_{-1}|0\rangle = \tfrac{1}{2}P_{-1}|0\rangle, \tag{3.10}$$

but $P(z)$ is not a spinor under K^H. Thinking now of $\overline{N}^Q$ as inserted in the n-point functions, we realize we can drop K^H entirely because it commutes with N^H, and the vertices, and annihilates on the pion. Thus we have eliminated the V and P degrees of freedom from the conformal group, leaving

$$\Delta(N^H) \equiv (J_0^H - 1) \equiv (\hat{L}_0 + N_0^H - 1)^{-1}. \tag{3.11}$$

Although this is not on our primary path, we remark parenthetically that this last breakup allows us to write the quark model in a "$\frac{1}{2}$ unit-shifted" form, for which, of course, vacuum decoupling is obvious. We back up momentarily to the n-point functions with $\Delta(\overline{N}^Q)$. In this form, Bardakci's gauges[8] go through

$$\begin{aligned} &G_0 P_{-1}|0\rangle = -4\,|\text{pion}\rangle, \\ &G_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}(\pi\cdot V + P\pi^5)\,d\theta, \\ &[G_0, \Delta(\overline{N}^Q)] = 0, \end{aligned} \tag{3.12}$$

yielding

$$B_n^Q \sim \langle 0|P_{+1}\Gamma(k_2,1)\Delta(\overline{N}^Q)\cdots\Delta(\overline{N}^Q)\,\Gamma(k_{n-1},1)P_{-1}|0\rangle\,.$$

Now, using our second breakup (3.9), we move K_0^H through to $P_{-1}|0\rangle$. Using (3.10), we achieve

$$\begin{aligned} &B_n^Q \sim \langle 0|P_{+1}\Gamma(k_2,1)\Delta^Q(N^H)\cdots\Delta^Q(N^H)\,\Gamma(k_{n-1},1)P_{-1}|0\rangle, \\ &\Delta^Q(N^H) = (\hat{L}_0 + N_0^H - \tfrac{1}{2})^{-1}. \end{aligned} \tag{3.13}$$

This form though elegant for the quark model is not convenient to get at model (2). This is obvious because it makes explicit reference to $P(z)$, which we do not know how to represent in NS-type operators.

Returning our attention now to the form of B_n using $\Delta(N^H)$, we notice that we have now eliminated reference to all currents but $A^\mu(z)$ and $T^{\mu\nu}(z)$ – just the ones we can represent with NS-type operators. We take our cue from this, and use the representation (2.9) to reexpress N^H in terms of NS-type operators. After a little algebra, one finds that the quartics collapse to bilinears in this representation. Not surprisingly, we find in fact that

$$\begin{aligned} &\tfrac{1}{48}(T^2)_l \equiv N_l^{\rm NS}, \\ &-\tfrac{1}{32}(A^2)_l - \tfrac{1}{192}(T^2)_l \equiv N_l^5, \\ &N_l^{\rm NS} + N_l^5 = N_l^H, \end{aligned} \tag{3.14}$$

where $N_l^{\rm NS}$ is the old NS conformal group bilinear in b_l^μ, and N_l^5 is its fifth-dimensional counterpart. Thus J^H are precisely the generators of model (2).

This completes the equivalence proof in the integrated formalism. To summarize with a uniform notation, good in either language, we write simply

$$\begin{aligned} &\Gamma(k,1) = \Gamma^0(k,1)[\pi^5(1) - (i/\sqrt{2})k\cdot A(1)], \\ &J_l \equiv \hat{L}_l - \tfrac{1}{32}(A^2)_l + \tfrac{1}{64}(T^2)_l = J_l^H, \\ &\Delta = (J_0 - 1)^{-1}. \end{aligned} \tag{3.15}$$

To close this section, we remark that, if we go to the even G-parity sector of the two models, they are both identical with the even G-parity sector of the NS model.[9] We leave as an exercise for the reader to show that the unified notation for *all three* models in the case of n ρ mesons outside is

$$\begin{aligned} \Gamma_\rho(k,1) &= \epsilon_\mu \Gamma_\rho^\mu \\ &= [\epsilon\cdot\pi(1) + (i/\sqrt{2})k_\nu \epsilon_\mu T^{\nu\mu}(1)]\,\Gamma^0(k,1), \\ \overline{J}_l &= L_l + \tfrac{1}{48}(T^2)_l, \\ \Delta &= (\overline{J}_0 - 1)^{-1}. \end{aligned} \tag{3.16}$$

In showing this for the quark model, one uses the fact that $N^{\rm NS}$ and N^5 [see (3.14)] are conjugate spin-spin groups, and that N^5 commutes with Γ_ρ and annihilates on the ρ at the end. This is somewhat tedious to show with quarks, but essentially obvious in the NS-type representation.

IV. LOOSE ENDS

In light of the preceding discussion, we are led to ask whether the "Pomeranchon-pion" model of Ref. 3 is also a dual quark-type model, this time, of course, with nonmultiplicative ("internal") isospin. At first glance, the answer would seem to be no because our model has its first exotic state at

1 GeV, whereas it was expected in Ref. 1 that exotic states would first appear at 2 GeV. We[10] have reexamined the argument of Ref. 1, however, and found that, for quark models with both spin *and* internal isospin, the exotic states would in general start at 1 GeV; thus we must look into the model more closely. In fact, it turns out that this model is not a dual quark model (of the Bardakci-Halpern type), but it is very close, differing only in the "Schwinger terms."

To see this, we go back to the algebraic approach of Sec. II. For reference, we give a sample commutator in the dual quark model with internal isospin,

$$[A_l^{\mu\alpha}, A_m^{\nu\beta}] = -2ig^{\mu\nu}\epsilon^{\alpha\beta\gamma}S_{l+m}^{\gamma} + 2i\delta^{\alpha\beta}T_{l+m}^{\mu\nu} - 8l\delta^{\alpha\beta}g^{\mu\nu}\delta_{l,-m}, \tag{4.1}$$

where

$$A^{\mu\alpha}(z) = :\bar{\psi}(z)\gamma_5\gamma^{\mu}\tau^{\alpha}\psi(z):,$$

$$S^{\alpha}(z) = :\bar{\psi}(z)\tau^{\alpha}\psi(z):,$$

and so on. The 8 in the Schwinger term is the number of quark dimensions – namely, 4 for spin times 2 for isospin. The algebra of $A^{\mu\alpha}$, S^{α}, $T^{\mu\nu}$ is closed, and in each case, the Schwinger terms of the diagonal commutator involve the same 8. Now turning to a representation in terms of NS-type operators, we find that

$$\begin{aligned} A_l^{\mu\alpha} &= 2i\sum_{n=-\infty}^{+\infty} b_{-n}^{\mu} b_{n+l}^{\alpha}, \\ T_l^{\mu\nu} &= -i\sum_{n=-\infty}^{+\infty}[b_{-n}^{\mu}, b_{n+l}^{\nu}], \\ \epsilon^{\alpha\beta\gamma}S_l^{\gamma} &= -i\sum_{n=-\infty}^{+\infty}[b_{-n}^{\alpha}, b_{n+l}^{\beta}] \end{aligned} \tag{4.2}$$

satisfies the algebra including (4.1), *but* with 4 instead of 8 in every Schwinger term. Thus, although we could write a current-algebraic formulation (in terms of $A^{\alpha}A^{\alpha}$, T^2, and $S^{\alpha}S^{\alpha}$) of the "Pomeranchon-pion" model, it does not appear to be a quark model of the Bardakci-Halpern type.

Finally, we want to say a few more words about the quark-model spin-spin interactions introduced in Sec. III. We have used seven of the 31 models in our discussion of the hierarchical connection between models (1), (2), and the NS model and we have used them in a pedantic fashion, resulting in no "new" models. For example, by using these seven Hamiltonians always with this pion (or ρ) vertex, we have decoupled the half-integral mass-squared mesons that abound in their spectra. The Hamiltonian N_0^H, for example, contains half-integrally massed states involving V_l^{μ} and/or P_l. We can find no obvious connection between these and the half-integer states of b_l^{μ}, b_l^5 – so it is an interesting question whether new models can be constructed such that these couple.[11] The problem we encountered in such a search for new vertices is that most of the other states in the various theories have K-degenerate families,[12] as in Ref. 1. Thus, we would first need to construct the "real" states.

Note added. V^{μ} and P can be represented as $V_l^{\mu} = 2\sum_n b_{-n}^{\mu} d_{n+l}^5$, $P_l = 2i\sum_n b_{-n}^5 d_{n+l}^5$, where d_l^5 are new pseudoscalar modes. In this representation, Eq. (3.10) and many of our remarks about spin-spin theories become quite transparent.

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[1]K. Bardakci and M. B. Halpern, Phys. Rev. D 3, 2493 (1971).

[2]K. Bardakci (unpublished).

[3]M. B. Halpern, Phys. Rev. D (to be published).

[4]A. Neveu and J. H. Schwarz (unpublished).

[5]See Ref. 2, and M. B. Halpern and C. B. Thorn Phys. Rev. D (to be published).

[6]S. Fubini and G. Veneziano, Nuovo Cimento 67A, 29 (1970).

[7]We have not been able to construct representations of any other quark currents (e.g., V^{μ}, P, S) in terms of b_l^{μ}, b_l^5 operators, and we doubt that such exist. On the same grounds, we doubt that, say, b_l^{μ} can be constructed from quarks, or vice versa.

[8]See Ref. 2. It is curious that the new gauge algebra closes onto exactly the spin-spin models considered here; for example, $[G_0, G_l] = 4(J_l^H + 5K_l^H)$.

[9]This was first conjectured by S. Mandelstam (private communication).

[10]C. B. Thorn and I noted that (quark)⊗(quark) is pure 3 only for no spin degrees of freedom.

[11]M. B. Halpern and C. B. Thorn, Phys. Letters 35B, 441 (1971). The spin-spin models can, in principle, transcend the no-go theorem of this reference; the mechanism is that of "anomalous dimensions" – currents no longer carry, in general, the sum of the quark dimensions.

[12]Note that, with the pion and/or ρ vertices above, the K-degeneracy problem (of each of the models of Sec. III) is trivial. In fact, all K-degenerate states above the "parent" simply decouple. Thus the K-degeneracy problem is very much "vertex-dependent."