PHYSICAL REVIEW D VOLUME 4, NUMBER 8 15 OCTOBER 1971

Gauge Field Theory of Particles. I. Bosons*

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(Received 16 June 1971)

We suggest that it is a reasonable approximation to consider the bosons and fermions found in nature as the normal modes of an underlying field theory which is invariant under gauge transformations of the second kind. The field theories for the boson and fermion "trajectories" contain an infinite number of fields with bilinear interactions between nearest neighbors in the index space of the finite-dimensional $(\frac{1}{2}k, \frac{1}{2}k)$ representations of the Lorentz group. When we turn on the electromagnetic field, we find that the physical particles have form factors. We discuss the connection between the bare masses of the underlying fields and the masses and form factors of the physical particles. We show how to choose the bare masses so that the physical particles lie on asymptotically linearly rising Regge trajectories. We expand our space to include isospin and trajectory normality, and construct a Lagrangian which contains both the π and ρ trajectories. From this Lagrangian, we then find the vector and axial-vector currents, and show that these currents obey the chiral $SU(2)\otimes SU(2)$ algebra. By postulating that scattering is the absorption and emission of external quanta by the underlying fields, and that the external ρ and π mesons couple to the underlying fields via the vector and axial-vector currents, we are able to give rules for calculating, in the narrow-resonance approximation (only tree diagrams), any N-point function, where all but two of the external legs are π, ρ, or A_1 mesons. We also can calculate any particle + current → particle + current amplitude in that approximation.

I. INTRODUCTION

Recent attempts to understand duality, whether in terms of finite-energy sum rules,[1] or in terms of Veneziano-type models,[2,3] suggest that the gross features of a scattering amplitude are reproduced by exchanging an infinite number of narrow resonances lying on Regge trajectories as long as a large number of satellites is included. Some attempts at understanding νW_2 and W_1, the structure functions for inelastic electroproduction, suggest that this approximation is also reasonable for the off-mass-shell Compton scattering amplitude.[4,5] Given the fact that an infinite number of resonances can approximate amplitudes reasonably well, one would like to get some physical intuition as to how the resonances arise and what the dynamics of scattering is. In an effort to give a physical picture of the Veneziano model (and at the same time "explain" the particle spectrum) several people have proposed that the physical particles are the excitations of a quark-antiquark system (or three-quark system) which is bound by rubber-band forces.[6,7] In this picture, the states are the normal modes of the "rubber band" to first approximation. Scattering is thought of as the absorption of an external quantum, the excitation of all the normal modes, and subsequent emission of quanta with deexcitation. This picture leads directly to the Veneziano model. The main problem with the Veneziano picture has been the absence of good models for scattering processes such as $\pi N \to \pi N$. The problems of spin and isospin were overwhelming, and there were also technical problems such as ghosts.

In 1966, before "duality" was discovered, Nambu investigated the question of whether wave equations exist which describe an infinite number of particles with reasonable spectra and electromagnetic form factors.[8] He considered fields which were infinite-sum fields of the type $\sum_k \oplus(\frac{1}{2}k, \frac{1}{2}k)$, where $(\frac{1}{2}k, \frac{1}{2}k)$ stands for the $(\frac{1}{2}k, \frac{1}{2}k)$ representation of the Lorentz group, i.e., a symmetric traceless tensor of k indices. His fields obeyed a first-order wave equation which implied that the source of the $(\frac{1}{2}k, \frac{1}{2}k)$ field was the fields $(\frac{1}{2}(k+1), \frac{1}{2}(k+1))$ and $(\frac{1}{2}(k-1), \frac{1}{2}(k-1))$. Thus with each point in space-time there was associated a one-dimensional lattice in Lorentz-index space with nearest-neighbor interactions. Nambu restricted his attention to wave equations which could be solved by group-theoretic means. The theories he studied had non-realistic hydrogenlike spectra (accumulation point) and hydrogenlike form factors for the particles. The first-order wave equations also had the problem of not having a positive definite probability density. Nambu's work showed that a rich particle spectrum, with a particle having intrinsic structure, could be obtained by an underlying field theory with nearest-neighbor interactions.

Recently Chodos and Haymaker showed that one could choose non-group-theoretic masses for the underlying fields which would result in the physical particles lying on Regge trajectories which were asymptotically linear.[9] In that paper difference-equation techniques were developed for diagonalizing the wave equation of Nambu.

In a recent paper of the authors,[10] it was shown that the first-order wave equation had certain gauge transformation properties. These properties led us to consider Lagrangian field theories which were invariant under the same transformations as the wave equations. The canonical field-theory aspects of these theories were explored in that paper. These new Lagrangians lead to second-order wave equations which admit solutions which have particles lying on rising Regge trajectories, as well as having a positive definite probability density. Even more important, there is a pressing aesthetic reason to consider these field theories. We will show in Sec. II that if we ask that a theory with nonzero bare-mass fields be gauge invariant of the second kind, we are automatically led to consider field theories which contain an infinite number of fields coupled with nearest-neighbor bilinear interactions. By letting the fields take on an extra spinor index (Rarita-Schwinger fields), we are able to consider similar Lagrangians for fermions as well.

We then go on in Secs. III–V to diagonalize the second-order wave equations for bosons and show the connection between the parameters of the underlying field theory and the masses and form factors of the physical particles. Specifically we give one choice of parameters which result in the physical particle lying on asymptotically linearly rising Regge trajectories. In the Lagrangian framework it is easy to introduce electromagnetic interactions via minimal coupling. We show how the form factors, diagonal and off-diagonal, depend on the input parameters. The diagonal form factors are given in terms of a power series in the $(3+1)$-dimensional Legendre polynomials and their derivatives.

In order to discuss the physical π and ρ trajectories, we generalize our fields in Sec. VI to have two additional indices, i $(=1,2,3)$ corresponding to isospin one and η $(=0,1)$ corresponding to the normality of the ρ,π trajectory $P=(-1)^{j+\eta}$. With these additional indices we look at two rotations in the (i,η) space and determine the vector and axial-vector current. We show that the vector and axial-vector currents generate chiral $SU(2)\otimes SU(2)$ algebra.

Since we are considering the underlying fields as the fundamental entities, we picture scattering as taking place the same way as in Susskind's rubber-band picture of the Veneziano model[6]–it is merely the absorption and emission of external quanta by the underlying fields, the Regge poles being the propagated normal-mode excitations. We obtain the vertices by postulating that the external photon couples to the electromagnetic current, the external ρ couples to the isospin current and the external pion couples to the divergence of the axial-vector current of the underlying field theory.

We then show that we can generate in the narrow-resonance approximation all the tree diagrams for the 4-point processes:

$$j_\mu + A \rightarrow j_\nu + B\,,$$
$$\begin{Bmatrix}\pi\\ \rho\\ A_1\end{Bmatrix} + A \rightarrow \begin{Bmatrix}\pi\\ \rho\\ A_1\end{Bmatrix} + B\,, \tag{1.1}$$

where j_μ is either a vector or an axial-vector current, A and B are arbitrary particles on the π or ρ trajectories. We also give rules for calculating the N-point functions when the external particles are π or ρ or A_1 in the tree-diagram narrow-resonance approximation.

Although we know the vector–and axial-vector-current matrix elements in principle, in practice they depend on the coefficients which are used to expand the underlying fields in terms of physical particle fields. These coefficients are solutions to second-order difference equations, and we lack explicit expressions for them except in particular cases. In general we shall be forced to resort to solving for them numerically. Thus we will never get simple formulas for scattering, like the Veneziano model; we have achieved a certain theoretical simplicity at the expense of rather complicated expressions for the final amplitudes.

We will devote most of this paper to a discussion of the boson Lagrangians. A complete discussion of the fermion field theories will be deferred to a later paper.

II. GAUGE INVARIANCE OF THE SECOND KIND AND INFINITE-COMPONENT FIELD THEORY

In this section we show how gauge invariance of the second kind forces the introduction of an infinite number of fields with bilinear interactions among them.

Let us start with a massless spin-zero theory. The Lagrangian for that theory is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi\,. \tag{2.1}$$

We notice that this theory is invariant under the constant translation $\varphi(x)\to\varphi(x)+\alpha$.

By the usual arguments of Gell-Mann and Lévy[11] this tells us that there is a conserved current

$$j^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\alpha(x)} = \partial^\mu\varphi \tag{2.2}$$

such that $\partial_\mu j^\mu=0$, i.e., $\Box^2\varphi=0$. Corresponding to this conservation law, there is also a charge which is the generator of the transformation:

$$Q=\int j^0 d^3x=\int\partial^0\varphi\, d^3x=\int \pi d^3x$$

such that

$$e^{iQ\alpha}\varphi(x)e^{-iQ\alpha}=\varphi(x)+\alpha\,. \tag{2.2a}$$

This invariance under a constant transformation is referred to as gauge invariance of the first kind.

Suppose we ask what field or fields must we introduce so that $\mathcal{L}$ is invariant under the nonconstant transformation

$$\varphi(x)\to\varphi(x)+\alpha(x)\,. \tag{2.3}$$

Under this transformation we find

$$\delta\mathcal{L}=\partial_\mu\varphi\,\partial^\mu\alpha(x)\,. \tag{2.3a}$$

Let us now introduce a vector field $\varphi_\mu(x)$ which interacts with φ such that $\mathcal{L}_{\rm int}=m\partial_\mu\varphi\varphi^\mu$; i.e., we consider

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\varphi+m\varphi_\mu)(\partial^\mu\varphi+m\varphi^\mu)\,. \tag{2.4}$$

If we make the simultaneous transformations

$$\begin{aligned}&\varphi(x)\to\varphi(x)+\alpha(x)\,,\\ &\varphi^\mu(x)\to\varphi^\mu(x)-\frac{1}{m}\partial^\mu\alpha(x)\,,\end{aligned} \tag{2.3$'$}$$

then we find $\mathcal{L}\to\mathcal{L}$. We see this Lagrangian [Eq. (2.4)] introduces the bilinear interaction $m\partial_\mu\varphi\varphi^\mu$ as a consequence of the gauge invariance of the quantity $G^\mu\equiv\partial^\mu\varphi+m\varphi^\mu$.

This discussion is very similar to the treatment of gauge invariance of the second kind in quantum electrodynamics (QED).[12] There one starts with the fermion Lagrangian,

$$\mathcal{L}=\bar\psi(i\gamma\cdot\partial-m)\psi\,, \tag{2.5}$$

which is invariant under $\psi\to e^{-i\alpha e}\psi$ and $\bar\psi\to\bar\psi e^{i\alpha e}$, with α constant. If we now let $\alpha=\alpha(x)$, we obtain

$$\delta\mathcal{L}=e\bar\psi\gamma^\mu\psi\partial_\mu\alpha(x)\,. \tag{2.6}$$

Thus to obtain gauge invariance of the second kind, one introduces the field $A^\mu(x)$, with interaction

$$e\bar\psi\gamma^\mu\psi A_\mu(x)$$

such that

$$A^\mu\to A^\mu-\partial^\mu\alpha(x) \tag{2.7}$$

when $\psi\to e^{-ie\alpha(x)}\psi$. If we now require that the free Lagrangian for the field A^μ or φ^μ be gauge-invariant of the second kind we are led to introduce the term

$$\mathcal{L}_{\rm free}=-\tfrac{1}{2}(\partial^\mu A^\nu-\partial^\nu A^\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)\,. \tag{2.8}$$

In the case of QED this requirement is the same as saying that A^μ is massless since $m^2A^\mu A_\mu$ is not gauge-invariant. Thus gauge invariance of the second kind for the above fermion Lagrangian "requires" the introduction of a zero-mass particle, the photon.

Suppose, however, we add the kinetic term

$$-\tfrac{1}{2}(\partial^\mu\varphi^\nu-\partial^\nu\varphi^\mu)(\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu) \tag{2.8a}$$

to our boson Lagrangian. Then we have

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}(\partial_\mu\varphi+m\varphi_\mu)(\partial^\mu\varphi+m\varphi^\mu)\\ &-\tfrac{1}{2}(\partial^\mu\varphi^\nu-\partial^\nu\varphi^\mu)(\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu)\,.\end{aligned} \tag{2.9}$$

But this is the Stueckelberg formalism for describing a particle of mass m and spin one, and nothing else.[10,13] To see this one lets $mA_\mu=m\varphi_\mu+\partial_\mu\varphi$. Then

$$\mathcal{L}'=-\tfrac{1}{2}(\partial^\mu A^\nu-\partial^\nu A^\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{2}m^2A_\mu A^\mu\,.$$

This is the ordinary Lagrangian for a field of spin one and mass m. We notice that the gauge invariance in this theory is "spurious". That is, we can introduce a new field A^μ which expresses the full content of the "gauge-invariant" theory, but whose Lagrangian admits no gauge group.

Let us instead choose for the kinetic term

$$-\tfrac{1}{2}(\partial^\mu\varphi^\nu+\partial^\nu\varphi^\mu)(\partial_\mu\varphi_\nu+\partial_\nu\varphi_\mu)\,, \tag{2.10}$$

or, so that we have an irreducible representation of the Lorentz group,

$$\begin{aligned}&-\tfrac{1}{2}(\partial^\mu\varphi^\nu+\partial^\nu\varphi^\mu-\tfrac{1}{2}g^{\mu\nu}\partial_\lambda\varphi^\lambda)\\ &\qquad\times(\partial_\mu\varphi_\nu+\partial_\nu\varphi_\mu-\tfrac{1}{2}g_{\mu\nu}\partial_\lambda\varphi^\lambda)\,.\end{aligned} \tag{2.11}$$

Then this new piece is not gauge-invariant under

$$\varphi^\mu(x)\to\varphi^\mu(x)-\frac{1}{m}\partial^\mu\alpha(x)$$

since

$$\delta\mathcal{L}=\frac{1}{m}\partial^\mu\varphi^\nu\delta^{\lambda\sigma}_{\mu\nu}\partial_\lambda\partial_\sigma\alpha(x)\,, \tag{2.12}$$

and we have to introduce a new field $\varphi_{\mu\nu}$ with bilinear coupling $\mathcal{L}_{\rm int}=\varphi_{\mu\nu}\delta^{\mu\nu}_{\lambda\sigma}\partial^\lambda\varphi^\sigma$ (where $\delta^{\mu\nu}_{\lambda\sigma}$ is the projection operator onto the space of traceless symmetric tensors). This makes the second term gauge-invariant of the second kind. That is, we have for the second term

$$-\tfrac{1}{2}(\partial_\mu\varphi_\nu+\partial_\nu\varphi_\mu-\tfrac{1}{2}g_{\mu\nu}\partial_\lambda\varphi^\lambda+m_2\varphi_{\mu\nu})^2\,, \tag{2.13}$$

which is invariant under

$$\varphi^{\mu}(x) \to \varphi^{\mu}(x) - \frac{1}{m_1}\partial^{\mu}\alpha(x),$$
$$\varphi^{\mu\nu}(x) \to \varphi^{\mu\nu}(x) + \delta^{\mu\nu}_{\lambda\sigma}\frac{1}{m_1 m_2}\partial^{\lambda}\partial^{\sigma}\alpha(x). \tag{2.14}$$

Again, if we introduce the kinetic term for $\varphi^{\mu\nu}$ in a gauge-invariant way, we can remove the gauge invariance by a redefinition of the fields. The only way to obtain a theory which is gauge-invariant of the second kind where the gauge fields have mass is to continually introduce the symmetric derivative kinetic term. Thus we are led to consider theories with an infinite number of gauge fields $\varphi^{\mu_1 \cdots \mu_k}$, which are symmetric and traceless and are the $(\frac{1}{2}k, \frac{1}{2}k)$ representations of the Lorentz group. The full action can be expressed as follows:

$$W = \int d^4x \sum_{k=1}^{\infty}[\eta_k G^{\nu_1 \cdots \nu_k}(\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}\partial_{\mu_k}\varphi_{\mu_1 \cdots \mu_{k-1}} + \alpha_k \varphi_{\nu_1 \cdots \nu_k}) - \tfrac{1}{2}\eta_k G^{\nu_1 \cdots \nu_k}G_{\nu_1 \cdots \nu_k}]. \tag{2.15}$$

Here η_k is $(-1)^{k+1}$ and is necessary to make the mass-term contribution to T^{00} have the correct sign, and $\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}$ is the projection operator for making tensors symmetric and traceless (the unit operator in the $(\frac{1}{2}k, \frac{1}{2}k)$ representation of the Lorentz group). A preliminary study of this field theory was given in an earlier paper.[10] Clearly this Lagrangian is invariant under the gauge transformations of the second kind,

$$\varphi_{\nu_1 \cdots \nu_k}(x) \to \varphi_{\nu_1 \cdots \nu_k}(x) + \gamma_k \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}\partial_{\mu_1} \cdots \partial_{\mu_k}\Lambda(x), \tag{2.16}$$

provided $\gamma_k + \alpha_{k+1}\gamma_{k+1} = 0$.

The presence of gauge invariance leads (as we shall see) to the absence of spin-zero states. This is fine if we are looking at the ρ Regge trajectory. In order to include the possibility of π mesons we must relax our stringent requirement that the Lagrangian be invariant under arbitrary gauge transformations. We look instead at Lagrangians invariant under restricted gauge transformations, or gauge transformations in which the gauge function $\Lambda(x)$ obeys the Klein-Gordon equation. This allows us to add a third term in defining $G^{\nu_1 \cdots \nu_k}$.

Specifically we consider Lagrangians of the form

$$\mathcal{L} = \sum_{k=0}^{\infty}[\eta_k G^{\nu_1 \cdots \nu_k}(\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}\partial_{\mu_k}\varphi_{\mu_1 \cdots \mu_{k-1}} + \alpha_k \varphi_{\nu_1 \cdots \nu_k} + \beta_k \partial^{\nu}\varphi_{\nu\nu_1 \cdots \nu_k}) - \tfrac{1}{2}\eta_k G^{\nu_1 \cdots \nu_k}G_{\nu_1 \cdots \nu_k}]. \tag{2.17}$$

This Lagrangian is invariant under the infinite set of gauge transformations

$$\varphi_{\mu_1 \cdots \mu_k}(x) \to \varphi_{\mu_1 \cdots \mu_k}(x) + \gamma^j_k \delta^{\nu_1 \cdots \nu_k}_{\mu_1 \cdots \mu_k}\partial_{\nu_{j+1}} \cdots \partial_{\nu_k}V_{\nu_1 \cdots \nu_j}(x), \tag{2.18}$$

provided

$$(\Box^2 + \overline{m}_j^{\,2})V_j(x) = 0, \quad \partial_{\nu_1}V^{\nu_1 \cdots \nu_j}(x) = 0$$

and

$$-\frac{\overline{m}_j^{\,2}}{2}\frac{(k-j+1)(k+j+2)}{(k+1)^2}\beta_k\gamma^j_{k+1} + \alpha_k\gamma^j_k + \gamma^j_{k-1} = 0.$$

III. BOSON FIELD THEORIES

In this section we will discuss in detail the physical attributes of the normal modes of the two boson Lagrangian field theories, i.e., the masses and electromagnetic form factors of the physical particles. The problem of finding the masses of the normal modes is equivalent to solving a second-order difference equation with the boundary condition that the physical states are normalizable. This boundary condition leads to an eigenvalue problem which we can solve explicitly for particular choices of the input parameters α_k and β_k. For arbitrary choices of α_k and β_k we can discuss the asymptotic behavior of the leading Regge trajectory for large j. The method of finding the eigenvalues is an extension of the techniques of Ref. 9. The canonical field-theory aspects of these boson Lagrangians have already been discussed in Ref. 10.

The Lagrangian, Eq. (2.17), leads to the following first-order field equations:

$$\eta_{k+1}\partial_{\lambda}G^{\lambda\mu_1 \cdots \mu_k} + \eta_{k-1}\beta_{k-1}\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}\partial^{\nu_1}G^{\nu_2 \cdots \nu_k} = \eta_k\alpha_k G^{\mu_1 \cdots \mu_k},$$
$$G_{\nu_1 \cdots \nu_k} = \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}\partial_{\mu_1}\varphi_{\mu_2 \cdots \mu_k} + \alpha_k\varphi_{\nu_1 \cdots \nu_k} + \beta_k\partial^{\lambda}\varphi_{\lambda\nu_1 \cdots \nu_k}, \tag{3.1}$$

which we write in shorthand as

$$\partial\cdot G^{k+1}+(\eta_{k-1}/\eta_{k+1})\beta_{k-1}\delta\partial G^{k-1}=(\eta_k/\eta_{k+1})\alpha_k G^k\;,\qquad G^k=\delta\partial\varphi^{k-1}+\alpha_k\varphi^k+\beta_k\partial\cdot\varphi^{k+1}\,. \tag{3.2}$$

This gives the following second-order equation for φ^k:

$$\frac{1}{k+1}\Box^2\varphi^k+\left(\frac{k}{k+1}\right)^2\delta\partial\partial\cdot\varphi^k+\left(\frac{\eta_{k-1}}{\eta_{k+1}}\right)\beta_{k-1}{}^2\delta\partial\partial\cdot\varphi^k-\frac{\eta_k}{\eta_{k+1}}\alpha_k{}^2\varphi^k$$
$$=-\beta_{k+1}\partial\cdot\partial\cdot\varphi^{k+2}+\left(\frac{\eta_k}{\eta_{k+1}}\alpha_k\beta_k-\alpha_{k+1}\right)\partial\cdot\varphi_{k+1}+\left(\frac{\eta_k}{\eta_{k+1}}\alpha_k-\frac{\eta_{k-1}}{\eta_{k+1}}\beta_{k-1}\alpha_{k-1}\right)\delta\partial\varphi_{k-1}-\frac{\eta_{k-1}}{\eta_{k+1}}\beta_{k-1}\delta\partial\partial\varphi_{k-2}\,. \tag{3.3}$$

These equations also describe the totally gauge-invariant Lagrangian [Eq. (2.15)] when we set $\beta_k=0$. Here we see that the fields G_k obey first-order wave equations with nearest-neighbor coupling in the Lorentz-index lattice, whereas when $\beta\neq 0$ φ has both nearest and next-nearest interactions. We also see that for the mass of the bare field (when the couplings are assumed to vanish) to be real, $\eta_k/\eta_{k+1}=-1$. If we look at how $\alpha_k{}^2$ enters T^{00}, we see in fact that we need $\eta_k=(-1)^{k+1}$.

We now assume that there exists an infinite number of free fields, labeled by an index $N=0,1,2,\ldots$ of definite spin j and mass m_{Nj} which correspond to the normal modes of the underlying set of coupled fields. These fields

$$\tilde\varphi^N_{\mu_1\cdots\mu_j}(x)$$

therefore satisfy the equations

$$(\Box^2+m_{Nj}{}^2)\tilde\varphi^N=0\quad\text{and}\quad\partial^{\mu_1}\tilde\varphi^N_{\mu_1\mu_2\cdots\mu_j}(x)=0 \tag{3.4}$$

as well as being symmetric and traceless in the indices $\{\mu_1\cdots\mu_j\}$. In terms of creation and destruction operators, we can expand $\tilde\varphi$ as

$$\tilde\varphi^N_{\mu_1\cdots\mu_j}(x)=\sum_s\int\frac{d^3p}{\sqrt{(2\pi)^3}\,2p_0(N)}\,[e^{-ip\cdot x}\epsilon_{\mu_1\cdots\mu_j}(p,s)a(p,j,s,N)+e^{ip\cdot x}\epsilon^*_{\mu_1\cdots\mu_j}(p,s)b^\dagger(p,j,s,N)]\;, \tag{3.5}$$

where

$$[a(p,j,s,N),a^\dagger(p',j',s',N')]=2p_0\delta^3(p-p')\delta_{jj'}\delta_{ss'}\delta_{NN'} \tag{3.6}$$

(and similarly for b and b^+) and where the $\epsilon_{\mu_1\cdots\mu_j}(p,s)$ are the polarization vectors for a spin-j particle, as given for example by Scadron.[14]

We now decompose our canonical gauge fields φ and G in terms of the normal modes $\tilde\varphi$:

$$\varphi^{\mu_1\cdots\mu_k}=\sum_{N=0}^{\infty}\sum_{j\le k}\alpha_k^{(Nj)}\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\partial^{\nu_{j+1}}\cdots\partial^{\nu_k}\tilde\varphi_N^{\nu_1\cdots\nu_j} \tag{3.7}$$

and

$$G^{\mu_1\cdots\mu_k}=\sum_{N=0}^{\infty}\sum_{j\le k}b_k^{(Nj)}\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\partial^{\nu_{j+1}}\cdots\partial^{\nu_k}\tilde\varphi_N^{\nu_1\cdots\nu_j}. \tag{3.8}$$

The requirement that the $\tilde\varphi^j_N$ diagonalize the field equations [i.e., that the Eqs. (3.1) hold separately for each component $\tilde\varphi^j_N$] leads to the following relations between the m_{Nj}, $a_k^{(Nj)}$, $b_k^{(Nj)}$, and the parameters of the underlying field theory α_k and β_k:

$$-\frac{m_{Nj}{}^2}{2}\frac{(k+j+2)(k-j+1)}{(k+1)^2}b_{k+1}^{(Nj)}\eta_{k+1}=-\eta_{k-1}\beta_{k-1}b_{k-1}^{(Nj)}+\eta_k\alpha_k b_k^{(Nj)} \tag{3.9a}$$

and

$$b_k^{(Nj)}=a_{k-1}^{(Nj)}+\alpha_k a_k^{(Nj)}-\frac{m_{Nj}{}^2}{2}\frac{(k-j+1)(k+j+2)}{(k+1)^2}\beta_k a_{k+1}^{(Nj)}\,. \tag{3.9b}$$

When $\beta\neq 0$, (3.8) is a second-order homogeneous difference equation for the $b_k^{(Nj)}$, and (3.9) is a second-order inhomogeneous difference equation for $a_k^{(Nj)}$. For $\beta=0$, we have two first-order difference equations which are easily solved.

The boundary conditions on the difference equation can be expressed as an eigenvalue problem for the m_{Nj}, the allowed masses of the physical particles. These boundary conditions are

$$a_k^{(Nj)} = b_k^{(Nj)} = 0, \quad \text{for} \quad k < j \tag{3.10a}$$

$$[\pi^k(x), \varphi^k(y)]_{x_0 = y_0} = -i\delta^3(x-y), \tag{3.10b}$$

and

$$[\tilde{\varphi}_N^{\dagger \nu_1 \cdots \nu_j}(x), \tilde{\varphi}_{N'\mu_1 \cdots \mu_j}(y)] = -i\delta_{NN'} P_{\mu_1 \cdots \mu_j}^{\nu_1 \cdots \nu_j}(x-y), \tag{3.10c}$$

where π^k is the canonical momentum conjugate to φ^k, as derived from the Lagrangian (2.19):

$$\pi_{\mu_1 \cdots \mu_{k-1}} = \eta_k G_{\mu_1 \cdots \mu_{k-1} 0} + \eta_{k-2}\beta_{k-2}\delta_{\mu_1 \cdots \mu_{k-1}}^{\lambda_1 \cdots \lambda_{k-1}} g_{\lambda_{k-1} 0} G_{\lambda_1 \cdots \lambda_{k-2}}, \tag{3.11}$$

and P is the projection operator

$$P_{\mu_1 \cdots \mu_j}^{\nu_1 \cdots \nu_j}(x-y) = \sum_s \int \frac{d^3p}{(2\pi)^3 2p_0} [e^{-ip\cdot(x-y)} \epsilon^*_{\mu_1 \cdots \mu_j}(p,s)\epsilon^{\nu_1 \cdots \nu_j}(p,s) - e^{ip\cdot(x-y)} \epsilon_{\mu_1 \cdots \mu_j}(p,s)\epsilon^{*\nu_1 \cdots \nu_j}(p,s)]\,. \tag{3.12}$$

Equations (3.10a) are the "initial" conditions for $k=j$. Equation (3.10b), when expanded using (3.7) and (3.8), gives us a sum over N and j for fixed k; thus it is not a boundary condition on a single one of the difference equations (3.9), but is rather a completeness sum over all the solutions. In Eq. (3.10c), if we knew the inverse of the expansions (3.7) and (3.8), we would indeed have a sum over k for fixed N and j, which would provide bounds of the growth of $a_k^{(Nj)}$ and $b_k^{(Nj)}$ for large k.

To avoid this rather cumbersome program, we make use of the fact that for charged bosons (3.10) leads to

$$[Q, \tilde{\varphi}(x)] = -e\tilde{\varphi}(x), \quad [Q, \tilde{\varphi}^\dagger(x)] = e\tilde{\varphi}^\dagger(x)\,. \tag{3.13}$$

Here Q is the electromagnetic charge operator $\int j^0(x)d^3x$ determined from the original Lagrangian, Eq. (2.17), by the principle of minimal coupling. This implies, for normalizable states $|Nj\rangle$, that

$$\langle Njp|Q|Njp'\rangle = 2p_0 e\delta^3(p-p')\,. \tag{3.14}$$

When we first expand Q in terms of the gauge fields φ and G, and then expand the φ and G in terms of physical-state operators, we shall obtain a normalization sum over the $a_k^{(Nj)}$'s and $b_k^{(Nj)}$'s. We get

$$1 = \sum_{k=j}^{\infty} \frac{(-1)^{k+j+1}}{2^{k-j+1}} \frac{(j!)^2}{(2j+1)!} \frac{(k+j+2)!(k-j+1)!}{[(k+1)!]^2} [b_{k+1}^{(Nj)}a_k^{(Nj)} + \beta_k b_k^{(Nj)} a_{k+1}^{(Nj)}] m_{Nj}^{2(k-j)} \tag{3.15}$$

as we shall show below. (Here we have assumed all the a's and b's real for simplicity.)

We shall also find that the expression $b_{k+1}^{(Nj)}a_k^{(Nj)} + \beta_k b_k^{(Nj)}a_{k+1}^{(Nj)}$ contains a factor of $(-1)^j$ which cancels the $(-1)^j$ that appears explicitly in (3.13). Thus we avoid one of the problems that plagued the first-order Lagrangian of Nambu, which had the structure

$$\mathfrak{L} = \bar{\psi}(\partial_\mu L^\mu + m)\psi$$

and which predicted that the matrix elements $\langle pj|J^0|pj\rangle$ were proportional to $(-1)^j$. We have instead a second-order Lagrangian, and the problem seems to disappear.

We now proceed explicitly to make our fields charged. In addition to allowing us to calculate (3.14), this will also enable us to show that the physical states have intrinsic form factors. We shall give expressions for the form factors and the transition matrix elements of $j_\mu(x)$ as a series in k involving the $a_k^{(Nj)}$, $b_k^{(Nj)}$, and the (3+1)-dimensional Legendre polynomials in $\gamma \equiv p\cdot p'/|p||p'|$, $|p| = m$ and $|p'| = m'$.

To introduce charge one must double the space by introducing two Hermitian fields φ_1^k and φ_2^k, or what is more commonly done, one introduces the non-Hermitian fields φ_k and $\varphi_k^\dagger$ which are eigenstates of the charge operator Q. In terms of the complex fields φ^k and $\varphi^{k\dagger}$ we can write the Lagrangian as

$$\mathfrak{L} = \tfrac{1}{2}\sum_{k=0}^{\infty} [\eta_k G^{k\dagger}(\delta\partial\varphi_{k-1} + \alpha_k\varphi_k + \beta_k\partial\cdot\varphi_{k+1}) + \eta_k(\delta\partial\varphi_{k-1}^\dagger + \alpha_k\varphi_k^\dagger + \beta_k\partial\cdot\varphi_{k+1}^\dagger)G^k] - \sum_{k=0}^{\infty}\eta_k G^{k\dagger}G_k\,. \tag{3.16}$$

We notice that this Lagrangian is invariant under the constant phase transformation

$$\begin{aligned} &\varphi_k \to e^{-i\alpha}\varphi_k, \quad G_k \to e^{-i\alpha}G_k, \\ &\varphi_k^\dagger \to e^{i\alpha}\varphi_k^\dagger, \quad G_k^\dagger \to e^{i\alpha}G_k^\dagger\,. \end{aligned} \tag{3.17}$$

Thus by the argument of Gell-Mann and Lévy, there is a conserved current

$$j_\mu = \frac{\delta \mathcal{L}}{\delta \partial^\mu \alpha} \text{ such that } \partial_\mu j^\mu = 0 \tag{3.18}$$

and $Q = \int j^0 d^3x$ is the generator of the above phase transformation. That is,

$$[Q, \varphi] = -\varphi \text{ and } [Q, \varphi^\dagger] = \varphi^\dagger . \tag{3.19}$$

This shows that φ destroys charge and $\varphi^\dagger$ creates charge. Letting $\alpha = \alpha(x)$ and using Eq. (3.16), we obtain

$$j^\mu(x) = \frac{i}{2} \sum_{k=0}^{\infty} \eta_k [G^{\dagger \nu_1 \cdots \nu_k \mu} \varphi_{\nu_1 \cdots \nu_k} + \beta_k \varphi^{\dagger \nu_1 \cdots \nu_k \mu} G_{\nu_1 \cdots \nu_k} - \varphi^\dagger_{\nu_1 \cdots \nu_k} G^{\nu_1 \cdots \nu_k \mu} - \beta_k G^\dagger_{\nu_1 \cdots \nu_k} \varphi^{\nu_1 \cdots \nu_k \mu}]. \tag{3.20}$$

One can easily verify that this current is conserved and that Q satisfies Eqs. (3.19) by using Eq. (3.11).

We should like to have an expression for all diagonal and transition matrix elements between physical states $a^\dagger(Njsp)|0\rangle$, which we designate by $|Njsp\rangle$. Using equations (3.20), (3.5), (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned} \langle N'j'p's' | j^\mu(x) | Njps \rangle = {} & \frac{1}{(2\pi)^3} e^{i(p'-p)\cdot x} (i)^{j-j'} \\ & \times \sum_{k=j}^{\infty} \eta_k \delta^{\mu_1 \cdots \mu_k \mu}_{\nu_1 \cdots \nu_k \nu_{k+1}} [(b^{*(N'j')}_{k+1} a^{(Nj)}_k + \beta_k a^{*(N'j')}_{k+1} b^{(Nj)}_k) \epsilon^{*\nu_1 \cdots \nu_{j'}}(p', s') \epsilon_{\mu_1 \cdots \mu_j}(p, s) \\ & \times p'^{\nu_{j'+1}} \cdots p'^{\nu_{k+1}} p_{\mu_{j+1}} \cdots p_{\mu_k} + (a^{*(N'j')}_k b^{(Nj)}_{k+1} + \beta_k b^{*(N'j')}_k a^{(Nj)}_{k+1}) \\ & \times \epsilon^{\nu_1 \cdots \nu_j}(p, s) \epsilon^*_{\mu_1 \cdots \mu_{j'}}(p', s') p^{\nu_{j+1}} \cdots p^{\nu_{k+1}} p'_{\mu_{j'+1}} \cdots p'_{\mu_k}]. \end{aligned} \tag{3.21}$$

We show in Appendix A that

$$\begin{aligned} \langle N'j'p's' | j^\mu(x) | Njps \rangle = {} & \frac{1}{(2\pi)^3} e^{i(p'-p)\cdot x} (-1)^j i^{j-j'} \frac{j!\,j'!}{[(2j+1)!\,(2j'+1)!]^{1/2}} \\ & \times \sum_{k=j}^{\infty} \eta_k \frac{m'^{k-j'} m^{k-j}}{2^{k-j/2-j'/2}} \{ m'(b'^*_{k+1} a_k + \beta_k a'^*_{k+1} b_k) [(k+1-j')!\,(k-j)!\,(k+j+1)!\,(k+j'+2)!]^{1/2} \\ & \times [D^{(\frac{1}{2}(k+1), \frac{1}{2}(k+1))}(L^{-1}(p')) L^\mu D^{(\frac{1}{2}k, \frac{1}{2}k)}(L(p))]_{j's', js} \\ & - m(a'^*_k b_{k+1} + \beta_k b'^*_k a_{k+1}) [(k-j')!\,(k+j'+1)!\,(k-j+1)!\,(k+j+2)!]^{1/2} \\ & \times [D^{(\frac{1}{2}k, \frac{1}{2}k)}(L^{-1}(p')) L^\mu D^{(\frac{1}{2}(k+1), \frac{1}{2}(k+1))}(L(p))]_{j's', js} \}. \end{aligned} \tag{3.22}$$

Here $b' \equiv b^{(N'j')}$, etc. The form factors are thus given by

$$\begin{aligned} \langle Njp's' | j^\mu(x) | Njps \rangle = {} & \frac{1}{(2\pi)^3} e^{i(p'-p)\cdot x} \frac{(-1)^j (j!)^2}{(2j+1)!} \\ & \times \sum_{k=j}^{\infty} \eta_k \frac{m^{2(k-j)}}{2^{k-j+1}} (k-j)!\,(k+j+1)!\,[(k+1-j)(k+j+2)]^{1/2} \,\mathrm{Re}(b^*_{k+1} a_k + \beta_k a^*_{k+1} b_k) \\ & \times [D^{(\frac{1}{2}(k+1), \frac{1}{2}(k+1))}(L^{-1}(p')) L^\mu D^{(\frac{1}{2}k, \frac{1}{2}k)}(L(p)) \\ & - D^{(\frac{1}{2}k, \frac{1}{2}k)}(L^{-1}(p')) L^\mu D^{(\frac{1}{2}(k+1), \frac{1}{2}(k+1))}(L(p))]_{j's', js} . \end{aligned} \tag{3.23}$$

For $j = 0$ we can express this very simply. Using the fact that

$$F_k \equiv \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} p^{\nu_1 \cdots \nu_k} p'_{\mu_1} \cdots p'_{\mu_k} = |p|^k |p'|^k \frac{k+1}{2^k} P_k(\gamma), \tag{3.24}$$

where

$$\gamma = \cosh\theta = \frac{p \cdot p'}{|p|\,|p'|}, \quad P_k(\gamma) = \frac{\sinh(k+1)\theta}{(k+1)\sinh\theta},$$

and that

$$\delta^{\mu_1 \cdots \mu_{k-1}\mu_k}_{\nu_1 \cdots \nu_{k-1}\mu} p^{\nu_1} \cdots p^{\nu_{k-1}} p'_{\mu_1} \cdots p'_{\mu_k} = \frac{1}{k}\frac{\partial}{\partial p^\mu} F_k$$
$$= \frac{k+1}{k}\frac{1}{2^k}\{p'^\mu |p|^{k-1}|p'|^{k-1} P'_k(\gamma) + p^\mu |p|^{k-2}|p'|^k [kP_k(\gamma) - \gamma P'_k(\gamma)]\}, \tag{3.25}$$

we find

$$\langle Nj=0p'|j^\mu(x)|Nj=0p\rangle = \frac{1}{(2\pi)^3}(p^\mu + p'^\mu)e^{i(p'-p)x}$$
$$\times \sum_{k=0}^\infty \eta_k \frac{m^{2k}}{2^k}\frac{k+2}{k+1}[(k+1)P_{k+1}(\gamma) + P_{k+1}'(\gamma)(1-\gamma)]\,\mathrm{Re}(b^{*(N)}_{k+1}a^{(N)}_k + \beta_k a^{*(N)}_{k+1} b^{(N)}_k). \tag{3.26}$$

Thus we obtain an expression for the form factor as a power series in the $(3+1)$-dimensional Legendre polynomials. To obtain the charge, we set $\mu = 0$ in Eq. (3.23) and use

$$L^0_{kj\sigma,k+1\,j'\sigma'} = -\tfrac{1}{2}[(k+1-j)(k+j+2)]^{1/2}\delta_{jj'}\delta_{\sigma\sigma'},$$
$$L^0_{k+1\,j\sigma,kj'\sigma'} = \tfrac{1}{2}[(k+1-j)(k+j+2)]^{1/2}\delta_{jj'}\delta_{\sigma\sigma'} \tag{3.27}$$

to obtain

$$\langle Njps|j^0(x)|Njps\rangle = \frac{1}{(2\pi)^3}e^{i(p'-p)x}(-1)^j\frac{(j!)^2}{(2j+1)!}$$
$$\times \sum_{k=j}^\infty \eta_k \frac{m^{2(k-j)}}{2^{k-j}}\frac{(k-j+1)!}{[(k+1)!]^2}(k+j+2)!\,\mathrm{Re}(b^{*(Nj)}_{k+1}a^{(Nj)}_k + \beta_k a^{*(Nj)}_{k+1}b^{(Nj)}_k). \tag{3.28}$$

Thus the normalization condition on the $a_k^{(Nj)}$ is

$$1 = (-1)^j \frac{(j!)}{(2j+1)!}\sum_{k=j}^\infty \eta_k \frac{m^{2(k-j)}}{2^{k-j}}\frac{(k-j+1)!}{[(k+1)!]^2}(k+j+2)!\,\mathrm{Re}(b^{*(Nj)}_{k+1}a^{(Nj)}_k + \beta_k a^{*(Nj)}_{k+1}b^{(Nj)}_k) \tag{3.28a}$$

as we recorded above in Eq. (3.15).

IV. FULLY GAUGE - INVARIANT THEORY

We turn now to a more detailed examination of theories governed by the Lagrangian defined in (2.15):

$$\mathcal{L} = \sum_{k=1}^\infty \eta_k [G^k(\delta\partial\varphi_{k-1} + \alpha_k\varphi_k) - \tfrac{1}{2}G^k G_k]. \tag{4.1}$$

As mentioned in Sec. II, this type of theory is invariant under the gauge transformation

$$\varphi_k \to \varphi_k + \gamma_k \delta\partial_{\nu_1}\cdots\partial_{\nu_k}\Lambda(x) \tag{4.2}$$

provided

$$\gamma_{k-1} + \alpha_k\gamma_k = 0. \tag{4.2a}$$

The equations of motion that follow from (4.1) are

$$G_k = \delta\partial\varphi_{k-1} + \alpha_k\varphi_k \tag{4.3}$$

and

$$\partial\cdot G_{k+1} = \alpha_k\zeta_k G_k, \tag{4.4}$$

with $\zeta_k = \eta_k/\eta_{k+1} = -1$.

Following the general philosophy of the preceding sections, we introduce the expansions

$$\varphi_k = \sum_{N,\, j\le k} a_k^{(Nj)}\delta\partial^{(k-j)}\tilde\varphi_j^{(N)} \tag{4.5}$$

and

$$G_k = \sum_{N,\, j\le k} b_k^{(Nj)}\delta\partial^{(k-j)}\tilde\varphi_j^{(N)}, \tag{4.6}$$

where the $\tilde\varphi_j^{(N)}$ are the normal modes of the field theory. By demanding that Eqs. (4.3) and (4.4) hold for each normal mode separately, we obtain the difference equations

$$b_k^{(Nj)} = a_{k-1}^{(Nj)} + \alpha_k a_k^{(Nj)} \tag{4.7}$$

and

$$-\left(\frac{m_{Nj}{}^2}{2}\right)\frac{(k-j+1)(k+j+2)}{(k+1)^2}b_{k+1}^{(Nj)} = -\alpha_k b_k^{(Nj)}. \tag{4.8}$$

In these equations, the boundary conditions

$$a_k^{(Nj)} = b_k^{(Nj)} = 0, \quad k<j \tag{4.9}$$

are to be understood. Solving (4.8) first, we have

$$b_k^{(Nj)} = \left(\frac{2}{m_{Nj}^2}\right)^{k-j} \left(\frac{k!}{j!}\right)^2 \frac{(2j+1)!}{(k-j)!(k+j+1)!} \left(\prod_{l=j}^{k-1} \alpha_l\right) b_j^{(Nj)} . \tag{4.10}$$

[Note that $\prod_{l=j}^{j-1} \alpha_l$ is defined to be $(1/\alpha_j)\prod_{l=j}^{j}\alpha_l = 1$.] The solution to (4.7) is then

$$a_k^{(Nj)} = (-1)^{k-j}\left(\prod_{l=0}^{k-j} \frac{1}{\alpha_{j+l}}\right) \times \left[b_j^{(Nj)} + \sum_{r=1}^{k-j} (-1)^r b_{j+r}^{(Nj)} \left(\prod_{p=0}^{r-1} \alpha_{j+p}\right)\right] . \tag{4.11}$$

We now define

$$f_k^{(j)}(m_{Nj}^2) = 1 + \sum_{r=1}^{k-j} \left(-\frac{2}{m_{Nj}^2}\right)^r \left(\frac{(r+j)!}{j!}\right)^2 \times \frac{(2j+1)!}{r!(r+2j+1)!} \prod_{p=0}^{r-1} (\alpha_{j+p}^2) \tag{4.12}$$

and rewrite (4.11) as

$$a_k^{(Nj)} = (-1)^{k-j}\left(\prod_{l=0}^{k-j} \frac{1}{\alpha_{j+l}}\right) b_j^{(Nj)} f_k^{(j)}(m_{Nj}^2) . \tag{4.13}$$

With the help of these formal solutions to the equations of motion, we shall now explore the properties of various systems determined by specific choices of the input parameters α_k. In particular, we shall discuss the following four cases:

A. An uncoupled theory (with alternate α_k's $=0$) which possesses a single trajectory and point form factors;

B. A coupled theory (i.e., no $\alpha_k = 0$) in which α_k behaves like the $1/k$ for large k, which possesses a leading trajectory and an infinite family of satellites, and which has structured form factors;

C. As we increase the α_k to behave like $1/\sqrt{k}$, we shall find that there are only a finite number of bound states;

D. Finally, if we take all the $\alpha_k = $ const, the bound-state spectrum disappears entirely.

A. Uncoupled Theory

We can simultaneously destroy the gauge invariance and the nearest-neighbor coupling of the theory by choosing some of the $\alpha_k = 0$. To be specific, let us consider the situation:

$$\alpha_k \neq 0, \quad k \text{ odd}, \qquad \alpha_k = 0, \quad k \text{ even}. \tag{4.14}$$

There are then three distinct possibilities:

Case 1. j even, $m_N^2 \neq 0$. Then from (4.7) we immediately have $b_j^{(Nj)} = \alpha_j a_j^{(Nj)} = 0$ (since $\alpha_j = 0$) so that

$$b_k^{(Nj)} = \alpha_k^{(Nj)} = 0 \quad \text{for all } k . \tag{4.15}$$

That is, there are no massive even-spin particles in the theory.

Case 2. j odd, $m_N^2 \neq 0$. In this case one can readily verify that the solutions are

$$b_j^{(j)} = \text{arbitrary constant}, \qquad b_{j+1}^{(j)} = a_j^{(j)} = b_j/\alpha_j , \tag{4.16}$$

and $b_k^{(j)} = a_k^{(j)} = 0$, otherwise. The mass eigenvalue condition is

$$m_j^2 = \alpha_j^2 (j+1) . \tag{4.17}$$

Thus there is one particle with each odd-integer value of spin, i.e., a single trajectory with odd signature. From (4.16), however, we learn that the expression for the form factor of a spin-j particle has only one term [only $k=j$ contributes in Eq. (3.23)]. In fact at zero momentum transfer we get

$$\langle p'j | \int J_0 d^3x | pj\rangle = 2p_0 \delta^3(p-p') = \frac{(j!)^2}{m_j} \frac{1}{2^{j-1}} |b_j|^2 \delta^3(p-p') \tag{4.18}$$

so that the choice $|b_j|^2 = 2^j m_j^2/(j!)^2$ properly normalizes the electromagnetic current. This is a noninteracting theory; there is no gauge invariance, there is no coupling between nearest neighbors, and the form factors have no structure.

Case 3. We mention before proceeding to the next choice of α_k's that in addition to the massive solutions we have found, there exist an infinity of massless solutions for each spin. Putting $m_N^2 = 0$ in (4.8), we see that $b_k^{(Nj)} = 0$ for k odd and $b_k^{(Nj)} = $ arbitrary for k even.

If j is even, we learn in addition from (4.7) that $b_j^{(Nj)} = 0$; but then all other $b_k^{(Nj)}$ are unconstrained for even k. The $a_k^{(Nj)}$ are given by

$$\begin{aligned} a_j &= -\alpha_{j+1} b_{j+2} , \\ a_{j+1} &= b_{j+2} , \\ a_{j+2} &= -\alpha_{j+3} b_{j+4} , \\ a_{j+3} &= b_{j+4} , \text{ etc.} \end{aligned} \tag{4.19}$$

The case j odd, $m_N^2 = 0$ is similar.

B. Coupled Theory with an Infinite Number of Bound States

We shall now examine a particular theory with all the α_k nonzero. This has the effect of restoring gauge invariance, and of coupling each field to both of its nearest neighbors. We see immediately that all massless solutions have disappeared, since from (4.8) and (4.7) the only solution for $m_N^2 = 0$ is $a_k^{(Nj)} = b_k^{(Nj)} = 0$ for all k. We shall also see below that the form factors have acquired structure.

To be specific, we choose

$$\alpha_k = \alpha/(k+1) . \tag{4.20}$$

We insert this choice into our general formulas (4.10), (4.12), and (4.13). We have, in (4.12),

$$f_k^{(j)}(m_N^{\,2})=\sum_{r=0}^{k-j} z^r\,\frac{(2j+1)!}{r!(r+2j+1)!}\,, \tag{4.21}$$

with

$$z=-2\alpha^2/m_N^{\,2}\,. \tag{4.22}$$

We also define, at this point, the limit function

$$f^{(j)}(z)=\lim_{k\to\infty} f_k^{(j)}(m_N^{\,2})\,. \tag{4.23}$$

In our case, we have

$$f^{(j)}(z)=\sum_{r=0}^{\infty}\frac{(2j+1)!}{(r+2j+1)!}\,\frac{z^r}{r!} \tag{4.24}$$

$$=\frac{(2j+1)!}{z^{j+\frac{1}{2}}}\,I_{2j+1}(2\sqrt{z})\,, \tag{4.24a}$$

where I is the modified Bessel function.

Unlike Case A, there is now an intimate connection between the allowed mass spectrum and the structure of the form factor. Specifically, if we calculate the form factor of a given state at zero momentum transfer, we find

$$\langle Njs|J_0|Njs\rangle =\frac{1}{(2\pi)^3}(j!)^2\,\frac{1}{2^{j-1}m_{Nj}}\,|b_j^{(Nj)}|^2\sum_{k=j}^{\infty} f_k^{(j)}(m_{Nj}^{\,2})\,. \tag{4.25}$$

If $|Njs\rangle$ is a true bound state, i.e., if its norm is finite, then the sum on the right-hand side of (4.25) must converge. But $\sum_{k=j}^{\infty} f_k^{(j)}$ can converge only if

$$\lim_{k\to\infty} f_k^{(j)}(m_N^{\,2})=f^{(j)}(z)=0\,, \tag{4.26}$$

i.e., we are looking for zeros of $f^{(j)}(z)$. Using the asymptotic form of the Bessel function, we conclude from (4.24a) that the criterion for bound states is

$$z=-[N+j+\tfrac{5}{4}]^2\,\tfrac{1}{4}\pi^2,\quad N \text{ an integer}\,, \tag{4.27}$$

for $(N+j)$ large. From (4.22) this is

$$m_{Nj}^{\,2}=\frac{8\alpha^2}{\pi^2}\,(N+j+\tfrac{5}{4}]^{-2}\,, \tag{4.27a}$$

i.e., the mass spectrum falls quadratically.

Notice that since we now have an infinite number of nonzero a_k's and b_k's, the form factor at arbitrary momentum transfer t, as calculated from (3.23), is an infinite series in t. Thus we have nontrivial vertex structure as a consequence of the infinite coupling of the theory.

C. Theory with a Finite Number of Zeros

Let us choose

$$\alpha_k=\alpha(k+\beta+1)^{1/2}/(k+1)\,. \tag{4.28}$$

Then

$$f^{(j)}(z)=\frac{(2j+1)!}{(j+\beta)!}\sum_{r=0}^{\infty}\frac{z^r}{r!}\,\frac{(r+j+\beta)!}{(r+2j+1)!}\,. \tag{4.29}$$

For $j+1<\beta$, the series (4.29) can be represented as a polynomial times an exponential, i.e.,

$$f^{(j)}(z)=\mathcal{P}^{(j,\beta)}(z)e^z \tag{4.30}$$

where $\mathcal{P}^{(j,\beta)}(z)$ is the polynomial

$$\mathcal{P}^{(j,\beta)}(z)=\frac{(2j+1)!}{(j+\beta)!}\left(\frac{1}{z^{2j+1}}e^{-z}\right)\left(\frac{d}{dz}\right)^{\beta-j-1}\left(z^{j+\beta}e^z\right). \tag{4.31}$$

Thus we have, for example,

$$\mathcal{P}^{(\beta-2,\beta)}(z)=\frac{1}{2\beta-2}\,(z+2\beta-2) \tag{4.32}$$

and

$$\mathcal{P}^{(\beta-3,\beta)}(z)=\frac{1}{(2\beta-3)(2\beta-4)}\times[z^2+2(2\beta-3)z+(2\beta-3)(2\beta-4)] \tag{4.33}$$

In general, $\mathcal{P}^{(j,\beta)}$ is a polynomial of degree $\beta-j-1$ and therefore we expect $\beta-j-1$ bound states for spin j. From (4.32) and (4.33) we learn that there is one bound state at

$$z=-2(\beta-1)$$

for spin $j_{\max}=\beta-2$, and two bound states at

$$z=-(2\beta-3)^{1/2}[(2\beta-3)^{1/2}\pm 1]$$

for spin $j_{\max}-1$. Using the definition of z, (4.22), we have

$$m^2(j_{\max})=\frac{\alpha^2}{\beta-1} \tag{4.34}$$

and

$$m_{\pm}^{\,2}(j_{\max}-1)=\frac{2\alpha^2}{(2\beta-3)^{1/2}}\,\frac{1}{(2\beta-3)^{1/2}\pm 1}\,. \tag{4.35}$$

Notice that

$$m_+^{\,2}(j_{\max}-1)<m^2(j_{\max})<m_-^{\,2}(j_{\max}-1)$$

so that, on the leading trajectory, the higher spin has the greater mass.

It is of course impossible in a theory with a finite number of bound states to ask for the asymptotic behavior of the trajectory. However, from (4.34), we see that

$$m^2(j_{\max})=\frac{\alpha^2}{j_{\max}+1}$$

so that, as we increase $j_{\max}$, the leading mass goes to zero as $1/j_{\max}$.

Finally, we remark that the existence of a structured form factor is independent of the existence of an infinite number of bound states. As we can see from (3.23), all that is required is that there be an infinite number of $a_k^{(Nj)}$ and $b_k^{(Nj)}$ for fixed (Nj), which is true in our case even though the number of bound states over which (Nj) can range is finite.

D. Theory with Constant Input Masses

Having treated cases where $\alpha_k \sim 1/k$ and $\alpha_k \sim 1/\sqrt{k}$, we now examine the situation

$$\alpha_k = \alpha \quad \text{for all} \quad k\,. \tag{4.36}$$

We have in (4.12)

$$f_{n+j}^{(j)}(m_N{}^2) = \sum_{r=0}^{n} (z)^r \frac{1}{r!} \frac{[(r+j)!]^2}{(r+2j+1)!} \frac{(2j+1)!}{(j!)^2} \tag{4.37}$$

and the limit function

$$f^{(j)}(z) = {}_2F_1(j+1, j+1; 2j+2; z)\,. \tag{4.38}$$

This is only true, however, inside the radius of convergence of the series (4.37), which is $|z|=1$. For $|z|>1$ the limit function does not exist. The criterion for a bound state is that $f^{(j)}(z)$ in (4.38) be zero. We now show, however, that there are no zeros of (4.38) for $-1 \leq z < 1$.

The demonstration is a simple consequence of an integral representation for the hypergeometric function, namely,

$$f^{(j)}(z) = \frac{(2j+1)!}{(j!)^2} \int_0^1 dx \frac{x^j(1-x)^j}{(1-zx)^{j+1}}\,. \tag{4.39}$$

Clearly for $-1 \leq z < 1$, the integrand in (4.39) is positive, and hence $f^{(j)}(z) > 0$; i.e., there are no zeros in that range.

E. General Comments

In this section we have gathered some data on the nature of particular bound-state spectra. What further conclusions can we draw?

First, let us remark that although we have unearthed several different kinds of spectra, the most popular kind of Regge trajectory, namely, an infinitely rising one, is not allowed. Recall that bound states for spin j are located at the zeros of a certain function $f^{(j)}(z)$ which is expressed as a power series about the origin in z. Since z is proportional to $1/m^2$, in order for us to have an infinitely rising sequence of masses, we must have a sequence of points $z_i \to 0$ with $f^{(j)}(z_i) = 0$, $i = 1, 2, \ldots$. But a function which is analytic in a neighborhood of the origin and which vanishes on a sequence of points that accumulate at the origin, must vanish identically. Hence such trajectories cannot occur.

However, we also discovered above that as we increased the strength of the input masses (and therefore the coupling between the fields) the number of bound states decreased. We went from an infinite number of bound states for $\alpha_k \sim 1/k$, to a finite number for $\alpha_k \sim 1/\sqrt{k}$, to none at all for $\alpha_k \sim$ const. The implication seems clear that as the α's increase, the bound states fall victim to an encroaching continuum. It will be of considerable interest to see in which cases such a continuum exists; the completeness relation, derived from the commutation relations (3.9), will be important in deciding this question. If the continuum is present, one will want to know whether it contains a resonance structure reminiscent of the bound states that have disappeared.

V. THEORY WITH $\beta \neq 0$

We turn now to the more general theory with

$$G_k = \delta\partial\varphi_{k-1} + \alpha_k \varphi_k + \beta_k \partial\cdot\varphi_{k+1}\,. \tag{5.1}$$

This implies the equations of motion

$$\eta_{k+1}\partial\cdot G_{k+1} + \eta_{k-1}\beta_{k-1}\delta\partial G_{k-1} = \alpha_k \eta_k G_k\,. \tag{5.2}$$

We see that, contrary to the case discussed in Sec. IV, the equations for G and φ are each second-order in the index k. From Eqs. (3.7) and (3.8) we obtain

$$\eta_{k+1}\left(\frac{-m_{Nj}{}^2}{2}\right)\frac{(k-j+1)(k+j+2)}{(k+1)^2}\, b_{k+1}^{(Nj)} = \eta_k \alpha_k b_k^{(Nj)} - \eta_{k-1}\beta_{k-1} b_{k-1}^{(Nj)} \tag{5.3}$$

and

$$\beta_k \left(\frac{-m_{Nj}{}^2}{2}\right)\frac{(k-j+1)(k+j+2)}{(k+1)^2}\, a_{k+1}^{(Nj)} = b_k^{(Nj)} - \alpha_k a_k^{(Nj)} - a_{k-1}^{(Nj)}\,. \tag{5.4}$$

As we showed in Ref. 10, (5.3) is equivalent to the difference equation derived from

$$(\partial_\mu L^\mu - M)\varphi = 0\,;$$

furthermore, from the results of Ref. 9, we expect the choice

$$\frac{\beta_{k-1}}{\alpha_k \alpha_{k-1}} = \frac{-\gamma k^2}{2(k+a)(k+b)(k+c)}$$

to lead to asymptotically linear trajectories. Finally, we can solve the equations exactly for a particular j if we let $a = j+1$, $b = \alpha - j$, and $c = \alpha - j - 1$ for some parameter α,

$$\frac{\beta_{k-1}}{\alpha_k \alpha_{k-1}} = \frac{-\gamma k^2}{2(k+j+1)(k-j+\alpha)(k-j+\alpha-1)}\,. \tag{5.5}$$

We shall concentrate from here on on this particular case. The solution to (5.3) with the choice (5.5) is given in detail in Appendix B. Here we shall use only the ground-state solution

$$b_k^{(0j)}=\left(\frac{\gamma}{2}\right)^k \frac{(k!)^2}{(k-j)!(k+j+1)!}\,\frac{1}{\Gamma(k-j+\alpha)}\left(\prod_{l=0}^{k-1}\alpha_l\right)\lambda(j), \tag{5.6}$$

with the ground-state mass given by

$$m^2=4\alpha/\gamma. \tag{5.7}$$

In (5.6), $\lambda(j)$ is a normalizing constant. As we show in the Appendix, the excited states are polynomials in k times the ground-state solution (5.6), and the higher masses for spin j are given by

$$m^2=4(\alpha+N)/\gamma, \quad N=1,2,\ldots\,. \tag{5.8}$$

Given the solution (5.6) of (5.3), we proceed to solve the inhomogeneous equation (5.4) for a_k by first solving the homogeneous problem

$$\beta_k\left(\frac{-m^2}{2}\right)\frac{(k-j+1)(k+j+2)}{(k+1)^2}\,\bar{a}_{k+1}+\alpha_k\bar{a}_k+\bar{a}_{k-1}=0, \tag{5.9}$$

and then using $\bar{a}_k$ to reduce the order of (5.4). Equation (5.9) is essentially the same as (5.3) [recall that η_k in (5.3) is given by (const)$\times(-1)^k$], with solution

$$\bar{a}_k^{(j)}=\left(\frac{\gamma}{2}\right)^k\frac{(k!)^2}{(k-j)!(k+j+1)!}\,\frac{1}{\Gamma(k-j+\alpha)}\prod_{l=0}^{k-1}\left(\frac{\alpha_l}{\beta_l}\right). \tag{5.10}$$

We now seek a solution to (5.4) in the form

$$a_k^{(0j)}=\bar{a}_k d_k^{(j)}. \tag{5.11}$$

We let

$$e_k\equiv d_k^{(j)}-d_{k-1}^{(j)}, \tag{5.12}$$

and rewrite (5.4) as

$$\beta_k\left(\frac{-m^2}{2}\right)\frac{(k-j+1)(k+j+2)}{(k+1)^2}\bar{a}_{k+1}(e_{k+1}+e_k+d_{k-1}^{(j)}) + \alpha_k\bar{a}_k(e_k+d_{k-1}^{(j)})+\bar{a}_{k-1}d_{k-1}^{(j)}=b_k^{(j)}. \tag{5.13}$$

By virtue of (5.9), the terms proportional to d_{k-1} vanish, so that (5.13) becomes

$$\beta_k\frac{m^2}{2}\frac{(k-j+1)(k+j+2)}{(k+1)^2}\,e_{k+1}+\bar{a}_{k-1}e_k=-b_k^{(j)}. \tag{5.14}$$

Inserting the known values of b_k and $\bar{a}_k$, Eqs. (5.6) and (5.10), respectively, we find after some algebra that

$$e_k=\frac{(k-j-1)!}{\alpha^k}\sum_{n=j}^{k-1}\frac{\alpha^{n+1}g_n}{(n-j)!}+c_1\frac{(k-j-1)!}{\alpha^k}, \quad k=j+1, j+2, \ldots, \tag{5.15}$$

with

$$g_n=\frac{-(n-j+\alpha)}{\alpha}\,\frac{\lambda(j)}{\alpha_n}\prod_{l=0}^{n-1}\beta_l \tag{5.16}$$

and c_1 a constant. The boundary condition

$$e_{j+1}=-\frac{\lambda(j)}{\alpha_j}\prod_{l=0}^{j-1}\beta_l,$$

derived directly from (5.13), determines $c_1=0$. [We note that we have not determined e_j, but that e_j is not needed, since its coefficient vanishes in (5.13) with $k=j$.]

Now that we have e_k, we put

$$d_k^{(j)}=\left(\sum_{n=j+1}^{k}e_n\right)+d_j^{(j)}. \tag{5.17}$$

Except for the value of $d_j^{(j)}$ (which does not seem to be determined at this level by the theory), we now have all the information we need to calculate the form factor of the $N=0$, spin-j state; in fact, we merely have to insert Eqs. (5.6), (5.10), (5.11), and (5.15)–(5.17) into (3.23). Of course a sum such as (3.23) now represents can at best be done numerically on a computer. However, we make the following remarks:

(i) In principle, we can evaluate the form factor at zero momentum transfer [Eq. (3.28)] in order to determine the constant $\lambda(j)$ in (5.6).

(ii) We notice that the mass spectrum is determined by the combination of parameters (5.5). However, the expression for the form factor, through the g_n in (5.16), also depends independently on the combination $(\prod_{l=0}^{k-1}\beta_l)$. Thus in the theory with $\beta\neq 0$ we have, to some extent, the opportunity of choosing the form factor independently of the mass spectrum.

VI. CURRENT ALGEBRA AND INTERACTIONS

In the previous sections we have tried to suggest that the physical particles are well approximated by the normal modes of an underlying field theory which has bilinear interactions given by the principle of gauge invariance of the second kind. We have seen that if we assume the electromagnetic field interacts locally with the underlying field, the matrix elements of the electromagnetic current of the field theory between physical states give us nontrivial form factors for the physical particles.

Since we are considering the underlying fields as fundamental, we picture scattering in an identical way to Susskind's rubber-band interpretation[6] of the Veneziano model – it is merely the absorption and emission of quanta by the underlying field. Compton scattering is thought of as a photon being absorbed, exciting the normal modes, and then being reemitted, with subsequent deexcitation. The vertex is well known, since the photon couples to the electromagnetic current which is derived from minimal interaction.

For strong interactions we will use the Yang-Mills idea that the external quanta which are vector mesons and axial-vector mesons couple to the chiral SU(2)⊗ SU(2) currents, and that the external quanta which are pseudoscalar mesons couple to the divergence of the axial-vector current. The vector and axial-vector currents are generated in the usual way from the underlying Lagrangian field theory. These ideas allow us to calculate all 4-point processes of the form

$$j^{\mu}+A\rightarrow j^{\nu}+B,$$
$$X+A\rightarrow Y+B,$$

where A and B are arbitrary particles on the SU(2)⊗ SU(2) trajectories (π-A_1 or ρ); X, Y stand for A_1, π or ρ; and j_μ is any component of the vector or axial-vector current.

A. Electromagnetic Interactions

The principle of minimal coupling tells us that we find the coupling of the photons to the underlying fields by replacing ∂_μ by $\partial_\mu - ieA_\mu^{\text{ext}}$; the current is then given by

$$j^{\mu}_{\text{e.m.}}=\frac{\delta\mathcal{L}}{e\,\delta A_\mu^{\text{ext}}}\,.$$

Aside from seagull terms, this is the same current we looked at in Sec. III. The effective Hamiltonian is

$$H_I=e\int j_\mu^{\text{e.m.}}(x)A^\mu_{\text{ext}}(x)d^3x \tag{6.1}$$

and the amplitude for Compton scattering (elastic or quasielastic) is given by

$$_{\text{out}}\langle\gamma(k'\lambda')N'j's'p'|\gamma(k\lambda)Njsp\rangle_{\text{in}}=\delta_{fi}+\frac{i^2}{2}\,\frac{\epsilon^{\mu}(k,\lambda)\epsilon^{\nu}{}^*(k'\lambda')}{(2\pi)^3}\int d^4x\,d^4y\,e^{-i(kx-k'y)}\langle N'j's'p'|T(j_\mu^{\text{e.m.}}(x),j_\nu^{\text{e.m.}}(y))\,|Njsp\rangle\,, \tag{6.2}$$

with j_μ given essentially by (3.20) and (3.22). Equation (6.2) leads to the two diagrams of Fig. 1. Since, in principle, we know $j^\mu(x)$ for arbitrary k^2 and k'^2, we can determine the structure functions for inelastic electron scattering by calculating

$$W_{\mu\nu}=\tfrac{1}{2}\sum_{s}\sum_{N'}\langle Njsp|j_\mu(0)|N'j's'p_{N'}\rangle\langle N'j's'p_{N'}|j_\nu(0)|Njsp\rangle(2\pi)^3\delta^4(p+q-p_{N'}). \tag{6.3}$$

The diagonal and transition matrix elements of j_μ are given by Eq. (3.21). However, this calculation involves knowledge of the $a_k^{(Nj)}$ and $b_k^{(Nj)}$, for which we have explicit expressions only in particular cases. We hope to calculate these matrix elements by numerical methods in the near future. We point out that Domokos *et al.*[5] claim that one can get a reasonable fit to W_1 and νW_2 by using an infinite number of poles in the s channel lying on rising trajectories and putting in appropriate widths and form factors. We feel that if we put in widths by hand and use an appropriate fermion Lagrangian, we should be able to reproduce their successes. We will defer discussion of fermion Compton scattering to a subsequent paper on fermions.

B. The Lagrangian for the π-A_1 and ρ Trajectories and Chiral SU(2)⊗ SU(2)

Before we can consider interactions we have to generalize our Lagrangian to describe the physical $I=1$ trajectories for the π-A_1 and ρ systems. We add two indices to the underlying fields, an isospin a ($a=1,2,3$) and a normality index η ($\eta=0,1$). The parity of a particle on a trajectory is $(-1)^{j+\eta}$. Thus $\eta=0,1$ for the ρ and π-A_1 trajectories, respectively. The quantum number is additive, and is conserved (mod 2) at any vertex. The vector and axial-vector currents have $\eta=0,1$, respectively.

With these extra degrees of freedom we can construct a Lagrangian that describes the entire (π-A_1, ρ) system of trajectories:

$$\mathcal{L}=\sum_{k,a,\eta}\eta_k\,G^k_{a\eta}(\delta\partial\varphi^{a\eta}_{k-1}+\alpha^\eta_k\varphi^{a\eta}_k+\beta^\eta_k\partial\cdot\varphi^{a\eta}_{k+1})-\tfrac{1}{2}\eta_k G^k_{a\eta}G^{a\eta}_k. \tag{6.4}$$

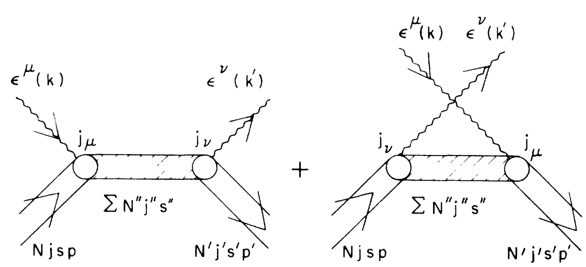

FIG. 1. Compton scattering with the exchange of a Regge trajectory.

If we want the ρ to be described by the gauge-invariant theory, we set $\beta_k^{\eta=0}=0$.

Using the methods of Gell-Mann and Lévy,[11] we consider the following transformations on the fields $\varphi_k^{a\eta}$:

$$\varphi_a \to \varphi_a + \epsilon_{abc}\Lambda_b\varphi_c\,, \tag{6.5a}$$

$$\varphi_a \to \varphi_a + \epsilon_{abc}\Lambda_b\epsilon\varphi_c\,. \tag{6.5b}$$

Here ϵ is the exchange operator in η space,

$$\epsilon_{\eta\eta'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and it plays a role similar to γ_5 in a fermion theory.

The transformation (6.5a) is an invariance of the Lagrangian and leads to a conserved vector current:

$$\begin{aligned} V_a^\mu &= \frac{\delta\mathfrak{L}}{\delta\partial_\mu\Lambda_a} \\ &= -\sum_{\eta k}\eta_k\epsilon_{abc}(G_{b\eta}^{k-1\,\mu}\;\varphi_{k-1}^{c\eta} + \beta_k^\eta G_k^{b\eta}\varphi_{c\eta}^{k\mu})\,, \\ \partial_\mu V^\mu &= 0\,. \end{aligned} \tag{6.6}$$

Using Eq. (3.11) we see that

$$V_0^a(x) = \sum_{k\eta}\epsilon^{abc}\pi_{b\eta}^k\varphi_k^{c\eta}. \tag{6.7}$$

From the commutation relations

$$[\pi_{a\eta}^{\mu_1\cdots\mu_k}(x), \varphi_{\nu_1\cdots\nu_k}^{a'\eta'}(y)]_{x_0=y_0} = -i\delta_{\nu_1\cdots\nu_k}^{\mu_1\cdots\mu_k}\delta_{aa'}\delta_{\eta\eta'} \tag{6.8}$$

we easily verify

$$[V_0^a(x), V_0^b(y)] = i\epsilon^{abc}V_0^c(x)\delta^3(x-y). \tag{6.9}$$

As in Ref. 10, we expect Schwinger terms in the $[V_0, V_j]$ commutation relations.

The transformation (6.5b) leads to an axial-vector current

$$\begin{aligned} A^{a\mu} &= \frac{\delta\mathfrak{L}}{\delta\partial_\mu\Lambda_a} \\ &= -\sum_{k,\eta\eta'}\eta_k\epsilon^{abc}(G_{b\eta}^{k-1\,\mu}\epsilon_{\eta\eta'}\varphi_{k-1}^{c\eta'} + \beta_k^\eta G_k^{\eta b}\epsilon_{\eta\eta'}\varphi_{c\eta'}^{k\mu}). \end{aligned} \tag{6.10}$$

We find

$$A_0^a = \sum_{k\eta\eta'}\epsilon^{abc}\pi_{b\eta}^k\epsilon_{\eta\eta'}\varphi_{k\eta'}^c \tag{6.11}$$

Using (6.8), we verify that these currents obey the chiral SU(2)⊗SU(2) algebra. That is, we have

$$\begin{aligned} [A_0^a(x), A_0^b(y)]_{x_0=y_0} &= i\epsilon^{abc}V_0^c(x)\delta^3(x-y), \\ [V_0^a(x), A_0^b(y)]_{x_0=y_0} &= i\epsilon^{abc}A_0^c(x)\delta^3(x-y). \end{aligned} \tag{6.12}$$

The axial-vector current is not *a priori* conserved. In fact

$$\partial_\mu A^{\mu a} = \frac{\delta\mathfrak{L}}{\delta\Lambda_a} = \sum_k \epsilon^{abc}G_{kb}\epsilon\tilde{G}_c^k\,. \tag{6.13}$$

The notation $\tilde{G}$ means that the α_k and β_k in $\tilde{G}_k$ are for the trajectory of opposite η; for example, for $\eta=0$, $\tilde{G}$ contains the fields corresponding to the ρ trajectory, but the α's and β's corresponding to the π trajectory. Clearly if the α's and β's are the same for both trajectories the current will be conserved. But the ρ and π trajectories are not degenerate in nature, so this possibility is not attractive. It might be possible to make (6.13) vanish by assuming a ρ trajectory with $\beta_k=0$, and then suitably relating the rest of the parameters $\alpha_k(\rho,\pi)$ and $\beta_k(\pi)$. This possibility has not yet been checked; it will be interesting to see if it leads to a reasonable relationship between the π-A_1 and ρ trajectories.

C. Rules for Constructing Scattering Amplitudes

Using the ideas of Yang and Mills,[15] we assume that an external ρ meson couples to the underlying field theory in the same manner as the electromagnetic field; however, instead of the gauge-invariant derivative being

$$\partial_\mu - ieA_\mu^{\text{ext}},$$

we now use

$$\partial_\mu - ieg\vec{t}\cdot\vec{\rho}_\mu^{\,\text{ext}}, \tag{6.14}$$

where $\vec{t}$ is a vector in isospin space. This is just the statement that the ρ couples universally to the isospin current, an idea espoused especially by Schwinger and Sakurai.[16] Thus when an external ρ meson is absorbed or emitted by the underlying fields, the effective vertex is

$$\rho_a^{\mu\,\text{ext}}(x)V_\mu^a(x), \tag{6.15}$$

where V_μ^a is given by Eq. (6.6).

We next examine the coupling of an external pion. If we look at the s-channel ρ pole in π-ρ scattering, we have that the ordinary field-theory vertex is given by

$$g_{\rho\pi\pi}\epsilon^{abc}\rho_a^\mu\varphi_b\partial_\mu\varphi_c\,. \tag{6.16}$$

In our picture of scattering this s-channel pole can be produced in several ways. We look at Figs. 2(a1) and 2(b5). In Fig. 2(a1), the underlying field is in a normal mode with $\eta=1$, it absorbs a ρ meson ($\eta=0$), then all the normal modes of $\eta=1$ propagate, and then a ρ is emitted with a return to the original state. In Fig. 2(b5), the underlying field is in the ρ normal mode, absorbs a π with excitation of the $\eta=1$ modes, and then emits a π and returns to its initial state.

In Eq. (6.16), we consider the s-channel pole term in two ways: Either as

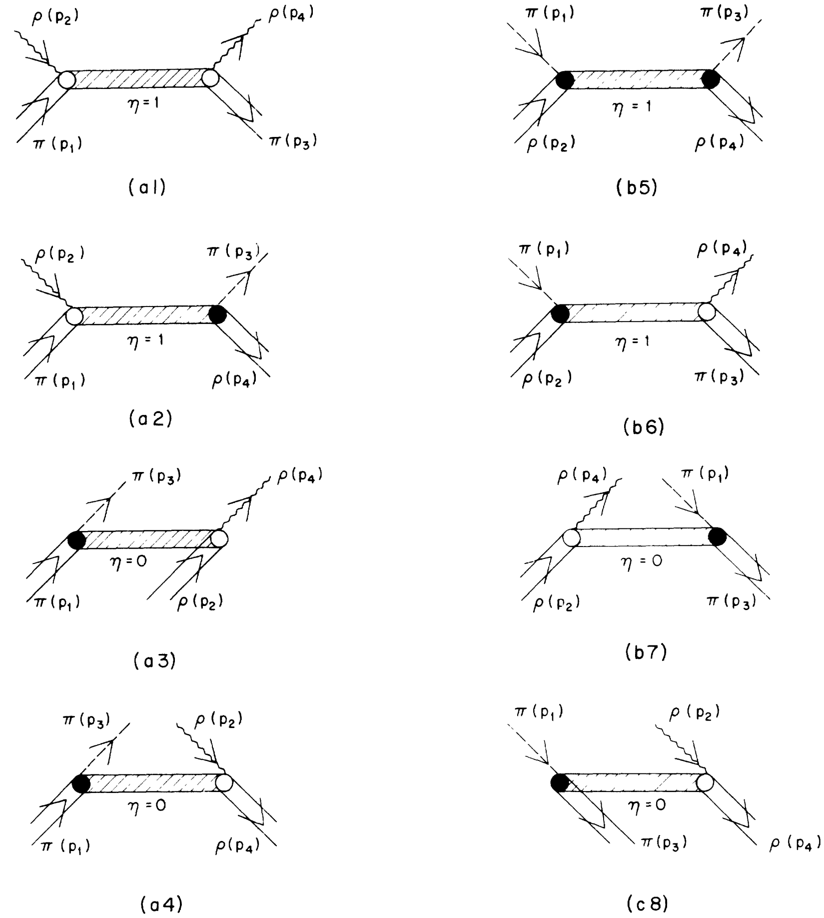

FIG. 2. Diagrams contributing to $\pi^+\rho^-$ scattering. Two particles (dashed line for the pion, wavy line for the ρ) are considered to be external quanta, while the other two are projected out from the appropriate currents. The external ρ couples to the vector current at vertices denoted by blank circles, while dark circles indicate vertices at which the pion couples to the divergence of the axial-vector current. The shaded line denotes a propagating sum of normal modes; the value of η is explicitly given for each diagram.

$$g\rho^{\mu}_{\text{ext}}V_\mu, \quad V^a_\mu=\epsilon^{abc}\varphi^b\partial_\mu\varphi^c \tag{6.16a}$$

or as

$$f\partial_\mu\varphi_{\text{ext}}A^\mu, \quad A^a_\mu=\epsilon^{abc}\rho_{\mu b}\varphi_c\,. \tag{6.16b}$$

Clearly (6.6) is the generalization of (6.16a), and (6.10) is the generalization of (6.16b).

Therefore, in what follows, we assume that an external pion couples to the axial-vector current with the effective vertex

$$f_1\partial_\mu\varphi_a^{\text{ext}}A^\mu_a\,. \tag{6.17}$$

Similarly we expect an external A_1 (if it really lies on the π trajectory) to couple directly to the axial-vector current with the effective vertex

$$f_2A^\mu_{1\,\text{ext}}A_\mu\,. \tag{6.18}$$

Now that we have the vertices for external π, ρ, and A_1 mesons we can write down rules for arbitrary n-point functions with π, A_1, ρ plus any two excited-state particles as the external particles. The "Feynman rules" are that in a given channel for the scattering process, we let any external particle be the underlying field, let it absorb and emit quanta consistent with the channel looked at and with conservation of η at the vertices. Then, we sum over the possible ways of having each external particle as the underlying field, careful not to count any diagram twice if the two diagrams are alike according to the Feynman prescription. For example, consider the process

$$\pi^+(p_1)\rho^-(p_2)\rightarrow\pi^+(p_3)\rho^-(p_4).$$

This is a simple process since there are no resonances in the u channel. We start off with the $\pi^+(p_1)$ as the underlying field. We get the four diagrams of Fig. 2(a). Next we let $\rho^-(p_2)$ be the underlying field, to obtain the diagrams of Fig. 2(b). The other possible choices for the underlying field lead to one more diagram shown in Fig. 2(c). We see that we have four diagrams corresponding to the π^0 trajectory in the s channel [Figs. 2(a1), 2(a2), 2(b5), and 2(b6)]. The four other diagrams correspond to the ρ^0 trajectory in the t channel. Clearly the t-channel process $\pi^+ + \pi^- \to \rho^+ + \rho^-$ has exactly the same generalized Feynman diagrams, so that crossing is ensured. In order to calculate these "Feynman diagrams" we use the effective vertices Eqs. (6.15) and (6.17). The result is similar to that of Compton scattering (where, however, there is no t-channel exchange allowed). For example, the amplitude T (where $S = 1 + iT$) for Fig. 2(a1) is given by

$$\frac{ig^2}{(2\pi)^3}\, \epsilon^{\mu}(x)\epsilon^{\nu *}(y)\int d^4x\, d^4y\, e^{-i(p_2 x - p_4 y)} \langle N=0\; j=0\; \eta=1\; p_3(1+i2)|T(V_{\mu}^{(+)}(x), V_{\nu}^{(-)}(y))\,|\,0\,0\,1\; p_1(1+i2)\rangle . \qquad (6.19)$$

This calculation can be done using Eqs. (5.6), (3.21), and (3.5), knowing that the $\tilde{\varphi}_N$ propagate as free fields. The only thing that prevents this calculation at this time is the lack of explicit expressions for the $a_k^{(Nj)}$ and the need to put in the widths of the resonances by hand. To calculate the other diagrams, we need to know the matrix elements of the axial-vector current. From Eq. (6.10) it is clear that one can obtain the matrix element of the axial-vector current from that of the vector current by a simple substitution rule in Eq. (3.22). That is,

$$\begin{aligned}
\langle N'j'p's'\eta'\,|A^{\mu}_{(3)}(x)|\,Njps\eta\rangle = {} & \frac{1}{(2\pi)^3}\, e^{i(p'-p)x}\, \frac{j'!\, j!\, i^{\,j-j'}}{(2j'+1)^{1/2}(2j+1)^{1/2}} \\
& \times \sum_{k=j}^{\infty} \eta_k (m_{N'j'})^{k-j'} (m_{Nj})^{k-j}\, 2^{[(j+j')/2]-k} \{ (b^{*\prime}_{k+1} a_k + \beta'_k a^{*\prime}_{k+1} b_k)\, m_{N'j'\eta'} \\
& \quad \times [(k+1-j')!\,(k+j'+2)!]^{1/2} [D^{(\frac{1}{2}(k+1),\, \frac{1}{2}(k+1))}(L^{-1}(p'))\, L^{\mu} D^{(\frac{1}{2}k,\, \frac{1}{2}k)}(L(p))]_{j's',js} \\
& \quad \times [(k-j)!\,(k+j+1)!]^{1/2} - (a^{*\prime}_k b_{k+1} + \beta_k b^{*\prime}_k a_{k+1})\, m_{Nj\eta} [(k-j')!\,(k+j'+1)!]^{1/2} \\
& \quad \times [D^{(\frac{1}{2}k,\, \frac{1}{2}k)}(L^{-1}(p)) L^{\mu} D^{(\frac{1}{2}(k+1),\, \frac{1}{2}(k+1))}(L(p))]_{j's',\,js} [(k-j+1)!\,(k+j+2)!]^{1/2} \} \,, \qquad (6.20)
\end{aligned}$$

where $a' = a^{N'j'\eta'}$ and $b' = b^{N'j'\eta'}$. By normalizing the amplitude for $\pi\rho \to \pi\rho$ at the s-channel π pole, and t-channel ρ pole, we can determine f_1 and g in terms of $g_{\rho\pi\pi}$ and $g_{\rho\rho\rho}$. Once this is done, all processes of the form $(\pi, \rho) + A \to (\pi, \rho) + B$ are completely specified. It is easy to see that our prescription for calculating the 4-point function leads to all the tree graphs for the 5-point function. However, when we get to the 6-point function, we seem not to have the three-Regge vertex, because of our ansatz of the continuity of propagation of the underlying field. Knowing what this vertex is is equivalent to knowing how to excite the fields with an external quantum of any spin. At present we can only guess, for example, that the spin-2 external field couples to the stress tensor, etc. If we can better understand what the triple Regge vertex means in this language, then we will in principle be able to calculate all the tree graphs for n-point functions in the narrow-resonance approximation.

VII. CONCLUSIONS

In this paper we have shown how one uses gauge invariance of the second kind to generate Lagrangians that describe particles which lie on Regge trajectories and have internal structure. Assuming that the underlying fields are basic, we have given a prescription for calculating the n-point functions in the tree-diagram approximation. We also show that the vector and axial-vector currents satisfy the chiral $SU(2) \otimes SU(2)$ algebra. The difficulty in this approach is that although the theory is relatively simple in terms of the fundamental fields, the interesting quantities, such as the currents, have very complicated matrix elements between physical states. Thus we are forced to rely on a computer to obtain solutions. We hope that in the near future these calculations can be done so that we can directly compare this theory with experiment.

ACKNOWLEDGMENTS

We would like to thank Michael Whippman for many useful conversations and suggestions. One of us (F.C.) would also like to thank Ralph Behrends and Graham Frye for some discussions.

APPENDIX A

We wish to show how (3.21) can be simplified to be given just in terms of a sum over two neighboring representations of the Lorentz group. For the case $j=0$, this reduces to the usual addition theorem for (3+1) spherical harmonics. In Ref. 10, we showed how to go from the "$kj\sigma$" basis for the $(\frac{1}{2}k, \frac{1}{2}k)$ fields to the more standard representation in terms of symmetric traceless tensors of rank k:

$$\varphi_{\mu_1 \cdots \mu_k}(x) = \sum_{j\sigma} T_{\mu_1 \cdots \mu_k, j\sigma} \varphi_{kj\sigma}(x), \tag{A1}$$

where

$$T_{\mu_1 \cdots \mu_k, j\sigma} = [L_{\mu_1} \cdots L_{\mu_k}]_{0,kj\sigma} . \tag{A2}$$

The L_μ are a 4-vector set of matrices in $kj\sigma$ space, chosen so that

$$[L_\mu, L_\nu] = iM_{\mu\nu}, \tag{A3}$$

where $M_{\mu\nu}$ are the generators of the Lorentz group. Explicitly,

$$(L^0)_{kj_1s_1,k-1\, j_2s_2} = a_k^{(j_1)} \delta_{j_1j_2} \delta_{s_1s_2}, \tag{A4a}$$

$$(L^0)_{k-1j_1s_1,kj_2s_2} = -a_k^{(j_2)} \delta_{j_1j_2} \delta_{s_1s_2}, \tag{A4b}$$

where

$$a_k^{(j)} = \tfrac{1}{2}[(k-j)(k+j+1)]^{1/2} . \tag{A4c}$$

The other L's can be calculated from L^0 since

$$[L^0, M^{k0}] = iL^k . \tag{A5}$$

In analogy with (A1) we have

$$\epsilon_{\mu_1 \cdots \mu_j}(p, s) = T_{\mu_1 \cdots \mu_j, j_1s_1} D^{(\frac{1}{2}j, \frac{1}{2}j)}_{j_1s_1, js}(L(p)) 2^{j/2}/j!, \tag{A6}$$

where $D^{(\frac{1}{2}j, \frac{1}{2}j)}$ is the $(\frac{1}{2}j, \frac{1}{2}j)$ representation of the homogeneous Lorentz group, and $L(p)$ is the standard boost:

$$L(p)(m, \vec{0}) = (p_0, \vec{p}) . \tag{A7}$$

We can use the relations

$$QL_\mu^\dagger Q^{-1} = -L_\mu \tag{A8a}$$

and

$$QD(L(p))Q^{-1} = D(L^{-1}(p)), \tag{A8b}$$

where

$$Q = (-1)^j \delta_{kk'} \delta_{jj'} \delta_{\sigma\sigma'}, \tag{A8c}$$

to derive

$$\epsilon^*_{\mu_1 \cdots \mu_j}(p, s) = \frac{2^{j/2}}{j!} D^{(\frac{1}{2}j, \frac{1}{2}j)}_{js, j_1s_1}(L^{-1}(p)) [L_{\mu_1} \cdots L_{\mu_j}]_{jj_1s_1,0} \tag{A9}$$

from (A6). This leads to the normalization

$$\epsilon^*_{\mu_1 \cdots \mu_j}(p, s) \epsilon^{\mu_1 \cdots \mu_j}(p, s) = (-1)^j . \tag{A10}$$

Using the fact that

$$\delta^{\nu_1 \cdots \nu_k}_{\mu_1 \cdots \mu_k} = \frac{2^k(-1)^k}{(k!)^2} [L_{\mu_1} \cdots L_{\mu_k}]_{0,kj\sigma} [L^{\nu_1} \cdots L^{\nu_k}]_{kj\sigma,0}, \tag{A11}$$

we find, for $j, m \le k$,

$$\delta^{\mu_1 \cdots \mu_{k-1}\mu}_{\nu_1 \cdots \nu_k} \epsilon^{*\nu_1 \cdots \nu_m}(p', s') \epsilon_{\mu_1 \cdots \mu_j}(p, s) = \frac{2^k(-1)^{k+j+m}}{2^{(m+j)/2}} \frac{m!\,j!}{(k!)^2} D^{(\frac{1}{2}m, \frac{1}{2}m)}_{ms', j_2\sigma_2}(L^{-1}(p'))$$
$$\times [L_{\nu_{m+1}} \cdots L_{\nu_k}]_{mj_2\sigma_2, kj_3\sigma_3} [L^\mu L^{\mu_{k-1}} \cdots L^{\mu_{j+1}}]_{kj_3\sigma_3, jj_4\sigma_4} D^{(\frac{1}{2}j, \frac{1}{2}j)}_{j_4s_4, js}(L(p)) \tag{A12a}$$

and

$$\delta^{\mu_1 \cdots \mu_{k-1}\mu}_{\nu_1 \cdots \nu_k} \epsilon^*_{\mu_1 \cdots \mu_m}(p', s') \epsilon^{\nu_1 \cdots \nu_j}(p, s) = \frac{2^k(-1)^{k+j+m}}{2^{(m+j)/2}} \frac{m!\,j!}{(k!)^2} D^{(\frac{1}{2}m, \frac{1}{2}m)}_{ms', j_2\sigma_2}(L^{-1}(p'))$$
$$\times [L^{\mu_{m+1}} \cdots L^{\mu_{k-1}} L^\mu]_{mj_2\sigma_2, kj_3\sigma_3} [L_{\nu_{j+1}} \cdots L_{\nu_k}]_{kj_3\sigma_3, jj_4\sigma_4} D^{(\frac{1}{2}j, \frac{1}{2}j)}_{j_4\sigma_4, js}(L(p)). \tag{A12b}$$

All but one L^μ in (3.21) are multiplied by the appropriate p or p'. We can move the D's through the $p \cdot L$'s using

$$D(\Lambda^{-1}) L^\nu D(\Lambda) = \Lambda^\nu{}_\mu L^\mu \tag{A13}$$

and

$$p_\nu \Lambda^\nu{}_\mu = m\delta^0_\mu \quad \text{if} \quad \Lambda = L(p) . \tag{A14}$$

Therefore,

$$p_\mu L^\mu D(L(p)) = D(L(p))(mL^0) \tag{A15a}$$

and

$$D(L^{-1}(p'))\, p'_\mu L^\mu = (m' L^0) D(L^{-1}(p')) \ . \qquad \text{(A15b)}$$

Using (A4), (A12), and (A15) we can now convert (3.21) into (3.22), the desired result.

APPENDIX B

We use techniques developed in Ref. 9 to solve (5.3). Our first step is to let

$$b_k^{(Nj)} = \left(\prod_{l=0}^{k-1} \alpha_l\right) \hat{b}_k \qquad \text{(B1)}$$

and consider (5.3) for large k. We get

$$\tfrac{1}{2} m^2 \hat{b}_{k+1} = \hat{b}_k - \frac{\gamma}{2k} \hat{b}_{k-1} \ . \qquad \text{(B2)}$$

The boundary condition that we impose is that b_k shall have the best possible large-k behavior. From (B2) we see that the solutions for large k are

$$\hat{b}_k = \left(\frac{2}{m^2}\right)^k \qquad \text{(B3a)}$$

and

$$\hat{b}_k = \left(\frac{\gamma}{2}\right)^k \frac{1}{k!} \ . \qquad \text{(B3b)}$$

Thus our boundary condition is that $\hat{b}_k$ shall have the asymptotic behavior (B3b).

Next we define a variable f_n, with $n \equiv k - j$, as

$$\hat{b}_k = \frac{[(n+j)!]^2}{n!(n+2j+1)!} \left(\frac{\gamma}{2}\right)^n \frac{1}{\Gamma(n+\alpha)} f_{n+1} \ . \qquad \text{(B4)}$$

Notice that the coefficient of f_n in (B4) already has the asymptotic behavior (B3b), so that f_n can at most be a polynomial in n. Putting (B4) in (5.3), we find that f_n obeys

$$\left(\frac{\gamma m^2}{4}\right) f_{n+2} = (n+\alpha) f_{n+1} - n f_n \qquad \text{(B5)}$$

or, introducing $\Delta f = f_{n+1} - f_n$,

$$(\tfrac{1}{4}\gamma m^2)\Delta^2 f + [\tfrac{1}{2}\gamma m^2 - n - \alpha]\Delta f + [\tfrac{1}{4}\gamma m^2 - \alpha] f = 0 \ . \qquad \text{(B6)}$$

We assume that f has the expansion

$$f_n = \sum_l c_l [n]_l \ , \qquad \text{(B7)}$$

where

$$[n]_l = \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda-l)} \ . \qquad \text{(B8)}$$

Here λ is an arbitrary parameter. Since $[n]_l$ obeys

$$\Delta[n]_l = l[n]_{l-1} \qquad \text{(B9a)}$$

and

$$n[n]_l = [n]_{l+1} - (\lambda - l - 1)[n]_l \ , \qquad \text{(B9b)}$$

we can replace the difference equation (B6) by an equivalent differential equation in a variable r, whose solution will be

$$f(r) = \sum_l c_l r^l \ . \qquad \text{(B10)}$$

The coefficients c_l in (B10) will be the same as those in (B7), provided we set

$$\Delta \to \frac{d}{dr} \ , \qquad \text{(B11a)}$$

$$n \to r - \lambda + 1 + r\frac{d}{dr} \qquad \text{(B11b)}$$

in (B6), as dictated by (B9). Making these replacements, we find

$$z f'' + (1 + \alpha - \lambda - \tfrac{1}{4}\gamma m^2 - z) f' + (\tfrac{1}{4}\gamma m^2 - \alpha) f = 0 \ . \qquad \text{(B12)}$$

Here $z = \frac{1}{4}\gamma m^2 - r$ and $f' = df/dz$.

The condition that f be a polynomial of degree N is

$$\tfrac{1}{4}\gamma m^2 - \alpha = N, \quad N = 0, 1, 2, \ldots \qquad \text{(B13)}$$

which gives us the mass spectrum (5.8). Equation (B12) is then the associated Laguerre equation, with solution

$$f(r) = L_N^{(-N-\lambda)}(z) \ . \qquad \text{(B14)}$$

By expanding (B14), we can obtain the coefficients c_l in (B7), and hence, via (B4) and (B1) we can obtain the values of $b_k^{(Nj)}$ for the excited states.

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PHYSICAL REVIEW D VOLUME 4, NUMBER 8 15 OCTOBER 1971

Inelastic Lepton Scattering in Gluon Models*

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A formal study of quark models with interactions due to scalar, pseudoscalar, or vector fields is presented. It is shown that all the results which have been derived in quark-parton models in which the details of the nucleon's constitution are not specified can be obtained formally using naive canonical manipulations of operators. In the case that there is no vector field some new results are obtained which would provide an experimental measurement of the proportion of scalar or pseudoscalar gluons in the nucleon.

I. INTRODUCTION

Some time ago we studied generalized quark-parton models and abstracted those results which might be true more generally.[1] In fact, we showed[1] that the most easily tested consequences of the model could all be derived formally in the gluon model using the Bjorken limit with naive canonical values for the equal-time commutators. In this paper we show that *all* the old results of generalized parton models can be formally derived in renormalizable quark models. We also present some new results which depend essentially on the assumption that none of the partons travel backwards in the infinite-momentum frame; it turns out that these results can be rederived formally if the interaction between the quarks is due to a scalar or pseudoscalar field but not if it is due to a vector field (the conventional gluon model).[2-4]

In perturbation theory the formal arguments used in this paper are invalid[5,6] and scale invariance is broken by logarithmic terms. Although they are not excluded by the data we shall assume that such terms are absent and that, therefore, arguments based on perturbation theory may be irrelevant. In this sense perhaps "Nature reads books on free field theory."[7]

We will not dwell on the experimental implications of the results, which have been reviewed elsewhere.[8] After completing this work we received an elegant preprint from Gross and Treiman,[9] who have independently rederived the "old" parton results in the gluon model.[10] They have actually gone further and derived the explicit form of the light-cone commutators in the presence of a vector interaction. This has also been done independently by Cornwall and Jackiw in a recent paper.[11]

II. FORMAL DERIVATION OF ALL "OLD" PARTON-MODEL RESULTS

Inelastic electron and neutrino scattering processes in which only the final lepton is observed are described by the tensors

$$W^{\gamma}_{\mu\nu} = \overline{\sum}\int \frac{d^4x}{4\pi} e^{iq\cdot x}\langle P|[J^{\gamma}_{\mu}(x), J^{\gamma}_{\nu}(0)]|P\rangle = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)W^{\gamma}_1 + \left(P_\mu - \frac{q_\mu \nu}{q^2}\right)\left(P_\nu - \frac{q_\nu \nu}{q^2}\right)\frac{W^{\gamma}_2}{M^2},$$

$$W^{\nu,\bar\nu}_{\mu\nu} = \overline{\sum}\int \frac{d^4x}{4\pi} e^{iq\cdot x}\langle P|[J^{\mp}_{\mu}(x), J^{\pm}_{\nu}(0)]|P\rangle = -g_{\mu\nu}W^{\nu,\bar\nu}_1 + \frac{P_\mu P_\nu W^{\nu,\bar\nu}_2}{M^2} - \frac{i\epsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta W^{\nu,\bar\nu}_3}{2M^2} + \frac{q_\mu q_\nu W^{\nu,\bar\nu}_4}{M^2} + \frac{(q_\mu P_\nu + q_\nu P_\mu)W^{\nu,\bar\nu}_5}{2M^2} + \frac{i(q_\mu P_\nu - q_\nu P_\mu)W^{\nu,\bar\nu}_6}{2M^2}, \tag{1}$$

where $\nu = q\cdot P$, J^{γ}_{μ} is the electromagnetic current, J^{+}_{μ} $[J^{-}_{\mu} = (J^{+}_{\mu})^{\dagger}]$ is the current which couples to the neutrino (antineutrino) current, $\overline{\sum}$ indicates an average over the spin states of the target, and the states are normalized to $2E$ per unit volume. We assume the conventional Cabibbo current and