PHYSICAL REVIEW D VOLUME 4, NUMBER 8 15 OCTOBER 1971

Bootstrap Conditions from the Peripheral Model Combined with Duality*

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(Received 6 July 1971)

By requiring that the $\pi\pi$ absorptive part generated by a peripheral model at moderately high energies be dual (in a semilocal sense) with the Regge behavior in a given process, it is possible to obtain information on the relevant trajectories. In practice we only had to consider singly-peripheral graphs, although at higher energies multiperipheral graphs would also come in. Our general presentation is therefore in terms of the multiperipheral model, although it does not require any of the usual formal apparatus of this approach. The input graphs are constructed in terms of the $\pi\pi$ amplitude, which is approximated by a Veneziano model. If we then require self-consistency between the input and output trajectories, we can obtain simple algebraic bootstrap conditions on the partial widths of the $\pi\pi$ resonances in that model. With the help of an additional duality argument it is also possible to obtain the *total* width of the g meson. Finally, the total cross section at high energies can be included in the bootstrap by making the Freund-Harari hypothesis, and bodily adding on a Pomeranchuk trajectory. The results are in reasonably good agreement with experiment.

I. INTRODUCTION

One of the limitations of a dual amplitude, as in the Veneziano model,[1] is that it contains free parameters. In particular, the over-all normalization, which is proportional to the resonance widths in the problem, is completely arbitrary. For this reason, it is necessary to complement it with a dynamical equation which incorporates the nonlinear constraint of unitarity. One promising candidate is the multiperipheral approach,[2-4] which is normally applied to high-energy phenomena but can also be used to calculate low-energy amplitudes. It has long been known to give Regge asymptotic behavior[2] and has recently been shown to be capable of generating an amplitude closely resembling the Veneziano amplitude in certain respects.[5] Thus, this amplitude can have zeros consistent with PCAC (partial conservation of axial-vector current) and Regge trajectories, which continue to rise up to fairly high energies in the crossed channel. Moreover, in contrast with most earlier dynamical schemes,[6] the output resonance widths lying on these trajectories are relatively narrow and in better agreement with experiment. Finally, it is possible to satisfy average duality down to fairly low energies, at least in the t-channel isospin $I_t = 1$ state, where the Pomeranchuk complication does not arise.

The input in a multiperipheral model is generally taken to be one or more Regge trajectories. If the model gives the same trajectories as output, we can then obtain bootstrap conditions by requiring self-consistency between them. In practice, this procedure has only had limited application so far, usually in the case of the Pomeranchuk trajectory.[3,7] This is partly because of the complicated nature of the equations involved and partly because of the inadequacy of some of the models used. We shall see, however, that the imposition of average or semilocal duality[8] on the absorptive part generated by the multiperipheral model leads to simple algebraic constraints on the Regge-residue parameters when self-consistency is required. In this respect, our method resembles somewhat the one proposed recently by Veneziano,[9] although our starting point and technique are quite different. Some of our results have been reported elsewhere.[10] Here we present a more detailed account of these and other calculations, and, in addition to the model assumed in Ref. 10, consider also one which is more elementary.

In practice, we considered $\pi\pi$ scattering and used models of the Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) type,[2] in which pairs of final particles are "bunched" together in the production amplitudes (see Fig. 1). It is then assumed that these pairs are dominated by resonances at low values of their subenergies s^i, although Regge behavior is assumed for higher values. In this way we avoid having to assume Regge behavior at very low values of the subenergies, as is done in the multi-Regge model. The mean value of a subenergy is known experimentally to be of the order of 1 GeV or less.

In Sec. II we discuss a very simple way of extracting Regge trajectories from our model. This approach, which is applicable to other dynamical equations besides the multiperipheral,[11] takes advantage of the fact that the $\pi\pi$ absorptive part can be calculated exactly up to any finite energy in terms of only a finite number of graphs. We can

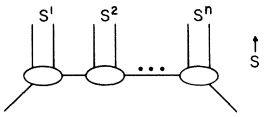

FIG. 1. ABFST model for a production amplitude. All our straight lines will represent pions.

then calculate the output trajectory by inserting this absorptive part into a finite-energy sum rule (FESR).[12] Alternatively, we could take the difference of two such sum rules (semilocal duality). If we apply this kind of duality in the region of the g resonance, our absorptive part will not involve anything more complicated than singly-peripheral graphs. Thus, we do not have any complications arising from the possibility of internal couplings being different from the external couplings in the multiperipheral chain of Fig. 1. It is also conceivable that this approximation would continue to be valid even if the multiperipheral model itself should prove to be incorrect.

In Sec. III, we consider a very specific model in which the parameters describing the blobs of Fig. 1 are calculated from the Veneziano model and pion exchange is assumed to dominate. By requiring self-consistency between the input and output Regge trajectories in the s-channel isospin state $I_s = 1$, we obtain a simple expression for the Veneziano normalization coefficient $\bar{\beta}$. If we then make the Freund-Harari hypothesis,[13] we can also include the effect of the Pomeranchuk trajectory. The residue for this trajectory can be calculated by going to the $I_s = 2$ state, where the Pomeranchuk trajectory alone contributes. This in turn gives us a value for σ_{tot}, the high-energy total cross section.

A knowledge of $\bar{\beta}$ gives us the partial widths of resonances for decay into $\pi\pi$. This is the same as the total width for the elastic resonances ρ and f^0, but is different for an inelastic resonance such as the g. In Sec. IV we therefore apply semilocal duality to the elastic absorptive part alone.[3,9] When combined with the result for $\bar{\beta}$, this immediately gives us a value for the elasticity of the g and enables us to calculate the total width of the g. We also discuss the connection of this procedure to previous suggestions about extending the concept of duality and reinterpret (in Sec. VI) some of the results obtained by Schmid in studying Argand-diagram loops in πN scattering.

Finally, in Sec. V we repeat some of the calculations of the preceding paragraphs for a model in which other effects besides pion exchange are roughly taken into account by the introduction of an "inelasticity" factor in the crossed channel. This factor can be approximately related to the high-energy $\sigma_{\text{tot}}/\sigma_{\text{el}}$ in that channel and leads to improved values for some of the parameters in our calculation.

II. THE ABFST MODEL AND ITS APPROXIMATE SOLUTION

The model for production which we shall use corresponds to taking pions for all the lines in Fig. 1.[2] This is based on the experimental fact that the large majority of produced particles at high energies are observed to be pions. Strictly speaking, the intermediate horizontal lines should be Reggeized and particles like the ω included. We shall, however, make the usual assumption that, since the physical pion has a very small mass, its pole will be close to the physical region and will dominate the amplitude. One possible way of improving on this assumption will be discussed in Sec. V.

If we now insert the graphs of Fig. 1 into a multiparticle unitarity relation, we will have an expression for the $\pi\pi$ absorptive part A_s which is given by the graphs of Fig. 2. We shall assume that the blobs of Figs. 1 and 2 are dominated by resonances for low values of the subenergies s^i of the vertical pion pairs emanating from them. The lowest such resonances would be the (degenerate) ρ and ϵ in our case. For $s^i > s_I$, where s_I is the value of s at which the higher-order graphs of Fig. 2 begin to become important (i.e., the 2ρ threshold), we shall use Regge exchange. It might be objected that there still are resonances, such as the g, for $s > s_I$, whereas our prescription will not give any. However, the inelastic contributions to A_s in Fig. 2 are given by one-pion (or Regge) exchange already, and so nothing is really gained by using a resonance form for the elastic part, which comes from the first term in Fig. 2. As usual, in a situation like this we fall back on duality to justify this use of Regge behavior; this version of average duality is actually somewhat different from that of Dolen, Horn, and Schmid[8] and will be discussed further in Secs. IV and VI. Exactly the same sort of argument has been used to justify the simple multi-Regge model.[3,14] However, whereas the multi-Regge model uses it at very low energies, we require it only for moderately high energies, above the 2ρ threshold.

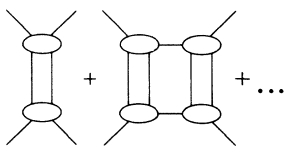

FIG. 2. Contributions to the $\pi\pi$ absorptive part.

The graphs of Fig. 2 have been shown to give Regge asymptotic behavior.[2] They also do not contribute to A_s below their corresponding production thresholds. With our resonance approximation for the blobs, this would correspond to the 2ρ, 3ρ, ... thresholds. If we therefore insert A_s into an FESR,

$$\int_{s_0}^{s_1} ds\, s^n [A_s(s,t) - \Sigma_i b_i(t) s^{\alpha_i(t)}] = 0\,, \tag{1}$$

where s, t, and u are the usual Mandelstam variables, we only have to calculate graphs with thresholds below s_1. By considering, say, $n=0$ and $n=1$, and assuming that a single trajectory dominates, we can then calculate the Regge residue b and trajectory α.[11] In all our calculations, however, we will only consider $n=0$, which simply places constraints on b and α. Instead of Eq. (1) we could also use continuous-moment sum rules and perhaps include cuts.[15,16] Such cuts are not likely to be important for the sort of values of s_1 we shall be concerned with, and so we shall defer any further discussion of them until we reach Sec. VII.

Equation (1) is automatically valid if we take a sufficiently large value of s_1. If we assume that our amplitude satisfies average duality, however, we should get reasonable results even for moderate values of s_1. Since duality was required to justify the multiperipheral model in the first place, as we have seen, this use of average duality should be no worse than the use of the multiperipheral model in the first place. Now normally s_1 would be taken at a resonance, or midway between two resonances. In our situation, however, a more appropriate value might be at a production threshold, since this is where discontinuities in A_s would occur. The lower limit s_0 would be the lowest threshold in the case of an ordinary FESR (average duality). But we can also take the difference between two such FESR's, in which case we again have an equation of the form of Eq. (1), but this time with s_0 chosen by the same sort of prescription as s_1 (semilocal duality). It is this latter form which we will use in our subsequent calculations.

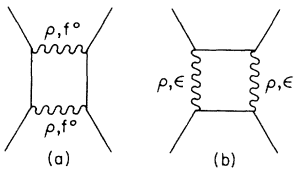

FIG. 3. (a) The first diagram of Fig. 2 with ρ, f^0 Regge exchange (wavy lines) for its blobs. (b) The second diagram of Fig. 2 with the ρ and ϵ resonances (wavy lines) for its blobs.

If we take s_1 to be at or below the 2ρ threshold, only the first term of Fig. 2 will contribute, and, since the blobs are dominated by resonances, Eq. (1) is simply a restatement of the usual Regge-resonance duality, at least for $I_s=0$ and $I_s=1$. The lowest value of s_1 which would yield any new results would be half-way between the g resonance and the next resonance above that. This also happens to be at about the $N\bar{N}$ threshold and means that our results would not be affected by the inclusion of baryonic effects. For s_0 we will take the 2ρ threshold, which also happens to be midway between the g and f^0 resonances. We thus have $(s_0,\ s_1)=(120,\ 180)$ in pion-mass units. The only diagrams in Fig. 2 which then contribute to Eq. (1) are those given in Fig. 3 where, in accord with our prescription, we have approximated the blobs in the first term in Fig. 2 by ρ and f^0 Regge exchange, and those in the second term by elastic resonances. The corresponding Fig. 1-type diagrams are shown in Fig. 4.

III. EXPRESSIONS FOR THE ABSORPTIVE PARTS

In evaluating the couplings in Figs. 3 and 4, or, equivalently, the blobs in Fig. 2, we will use the dual Veneziano model, which is then the basic input in our problem. For the $\pi\pi$ problem this gives amplitudes[17]

$$A^{I_s=0} = \tfrac{3}{2}[F(s,\ t)+F(s,u)] - \tfrac{1}{2}F(t,u)\,, \tag{2}$$

$$A^{I_s=1} = F(s,\ t) - F(s,\ u)\,, \tag{3}$$

$$A^{I_s=2} = F(t,\ u)\,, \tag{4}$$

where

$$F(x,\ y) = -\bar{\beta}\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} \tag{5}$$

and $\alpha(x)=\alpha_0+\alpha' x$. Equation (5) depends on three parameters. The α parameters can in principle be calculated in our method, but this is rather difficult in practice. We will therefore follow the usual practice of requiring the Veneziano amplitude to vanish at the Adler zero.[17] This gives $\alpha(1)=\frac{1}{2}$ or, since the pion mass is small, $\alpha_0\simeq 0.5$. The second parameter α' will be fixed by the requirement that the ρ mass have the correct experimental value, which gives $\alpha'=\frac{1}{60}$. It should

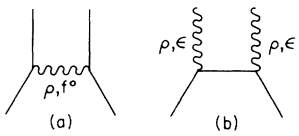

FIG. 4. (a) The $\pi\pi$ amplitude with Regge exchange. (b) The production amplitude corresponding to Fig. 3(b).

be emphasized, however, that because of the small value of the pion mass, this is not a true parameter but merely serves to fix the energy scale in our problem. The final parameter $\bar{\beta}$ will be determined self-consistently.

We will evaluate the diagrams of Fig. 3 at $t=0$, where they simplify considerably. Thus, Fig. 3(a) gives the absorptive part

$$A_s^{\pi\pi}(s,0)=\frac{1}{2q\sqrt{s}}\int_{-s+4}^{0}dt|W(s,t)|^2, \tag{6}$$

where W is given by Fig. 4(a) and has the form

$$W(s,t)=\beta(t)e^{-i\pi\alpha(t)}S^{\alpha(t)}-\beta(u)e^{-i\pi\alpha(u)}S^{\alpha(u)} \tag{7}$$

for $I_s=1$, with β given by Eqs. (3) and (5). Instead of using Eq. (7) directly, we have found it more convenient to take advantage of the fact that

$$F(t,u)\simeq\beta(t)S^{\alpha(t)}+\beta(u)S^{\alpha(u)}. \tag{8}$$

Now for the sort of energies we shall be concerned with, the first and second terms in Eqs. (7) and (8) dominate only in the forward and backward directions, respectively, and can be expected to have relatively little interference between them. Thus, as far as the integral in Eq. (6) is concerned, we can write

$$|W(s,t)|^2\approx|F(t,u)|^2, \tag{9}$$

where it must be remembered that $F(t,u)$ is a relatively smooth function in the region of interest; the integral itself was evaluated by using a simple Simpson's five-point rule. Since it is not possible to obtain a comparably simple and smooth expression for $I_s=0$, we did not consider this state.

Let us now turn to Fig. 4(b). This actually involves several processes, namely, $\pi\pi\to\rho\rho$, $\pi\pi\to\rho\epsilon$, and $\pi\pi\to\epsilon\epsilon$, which have the form

$$B_{\rho\rho}=\frac{C_{\rho\rho}}{1-t}p_1\cdot e_1\,p_2\cdot e_2, \tag{10}$$

$$B_{\rho\epsilon}=\frac{C_{\rho\epsilon}}{1-t}p_1\cdot e_1, \tag{11}$$

$$B_{\epsilon\epsilon}=\frac{C_{\epsilon\epsilon}}{1-t}. \tag{12}$$

The coefficients $C_{\rho\rho}$, $C_{\rho\epsilon}$, and $C_{\epsilon\epsilon}$ involve the couplings at the vertices of Fig. 2(b). The same couplings come into the ρ and ϵ poles in $\pi\pi$ scattering, so we can evaluate them in terms of the Veneziano parameters by comparing these poles with the ones predicted by Eqs. (2)–(5). The e_i are the usual polarization vectors for the ρ mesons and the p_i are the four-momenta of the adjoining pions. Figure 3(b) then gives an absorptive part

$$A_s^{\rho\epsilon}(s,0)=\frac{1}{4\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}\sum\int d\Omega|B|^2, \tag{13}$$

where m is the ρ, ϵ mass and the summation is over the intermediate states as well as over the polarizations of the ρ. For the latter, we use the standard formula

$$\sum e_\mu e_\nu=g_{\mu\nu}-k_\mu k_\nu/m^2 \tag{14}$$

but drop the $k_\mu\,k_\nu$ term. This is because this term becomes important only for large intermediate angles in the integration in Eq. (13) and would therefore be suppressed if we had used the more correct Reggeized forms for Eqs. (10)–(12). The resulting $A_s^{\rho\epsilon}$ is written down in Appendix B.

IV. BOOTSTRAP RESULTS

Using Eqs. (6) and (13), we can now calculate the total absorptive part,

$$A_s=A_s^{\pi\pi}+A_s^{\rho\epsilon}, \tag{15}$$

and insert it into Eq. (1). The $A_s^{\pi\pi}$ integration was done numerically, while the one for $A_s^{\rho\epsilon}$ could be performed analytically (see Appendix B). If, at the same time, we require the output trajectory itself to be given by Eqs. (3) and (5) for $I_s=1$, we would have $\alpha(0)=\frac{1}{2}$ and

$$b(0)=\pi\bar{\beta}(\alpha')^{\alpha_0}/\Gamma(\alpha_0). \tag{16}$$

Now, since $A_s\propto\bar{\beta}^2$, Eq. (1) gives $\bar{\beta}=0.527$. This corresponds to a ρ width of 110 MeV, which should be compared with the experimental value of 125 ± 20 MeV.

We have so far ignored the Pomeranchuk trajectory (P). It is not entirely clear how this should be put in in a multiperipheral model such as we are using. If we make the Freund-Harari hypothesis,[12] however, we would simply add its effect in Eq. (1) along with the Veneziano trajectory. Now $\alpha_P(0)\simeq1$ and

$$b_P(0)=3\sigma_{\rm tot}/32\pi^2. \tag{17}$$

If we therefore take $\sigma_{\rm tot}\simeq12$ mb (see below), we obtain $\bar{\beta}=0.75$, which corresponds to a ρ width of 156 MeV. This is not *too* different from the value obtained in the preceding paragraph by ignoring the P. We have, of course, ignored the effect of the P in the input blobs of Fig. 2. The justification for this is that our main contribution to A_s comes from the second diagram and for values of the pion-pair subenergies in the vicinity of the ρ mass. The P is likely to be much less important at these subenergies than in the energy range for which we are considering our output trajectories.[18]

Once we have $\bar{\beta}$, we can calculate the total widths of all elastic resonances, such as the ρ and f^0, from Eqs. (2)–(5). In the case of an inelastic resonance such as the g, however, we also need the elasticity of the resonance. Now, an average elas-

ticity r in the interval $(s_0, s_1) = (120, 180)$ can always be obtained from the ratio

$$r = \int_{s_0}^{s_1} A_s^{\pi\pi}(s, 0)\, ds \Big/ \int_{s_0}^{s_1} A_s(s, 0)\, ds\,, \qquad (18)$$

where A_s and $A_s^{\pi\pi}$ are the quantities given by Eqs. (15) and (6). If we use duality, we can also replace A_s and $A_s^{\pi\pi}$ by resonance forms. If, in addition, we assume that the g resonance dominates those resonances, then Eq. (18) gives the elasticity of the g resonance itself. Such a g-dominance assumption can be shown to be a good approximation for $I_s = 1$ if we use Eqs. (3) and (5). It is actually not even needed if, instead, we assume that all resonances in our energy range have approximately the same elasticity. If we now use Eqs. (6) and (15), Eq. (18) gives a g elasticity $r = 0.33$, a result which is independent of $\bar{\beta}$. If we combine this with the partial width obtained from Eqs. (3) and (5), we obtain a total g width $\Gamma_g = 112$ MeV with $\bar{\beta} = 0.527$ and $\Gamma_g = 159$ MeV with $\bar{\beta} = 0.75$. The corresponding experimental value is $\Gamma_g = 125$–170 MeV.

The above type of semilocal duality between $A_s^{\pi\pi}$ and the corresponding resonance form goes somewhat beyond the usual type of duality, which only involves A_s. It is equivalent to requiring that the first diagram of Fig. 2, with a resonance approximation for the blobs, is dual with Fig. 3(a). At $t = 0$ this is equivalent to requiring $\sigma_{\text{elastic}}^{\text{Regge}}(s) \approx \sigma_{\text{elastic}}^{\text{resonance}}(s)$ on the average. This version of it has recently been resurrected by Veneziano, who has emphasized its potential usefulness for extracting additional conditions from duality. The basic idea is, of course, much older. We have already seen in Sec. II how it can be used to justify our multiperipheral model and also how it had previously been used to justify the multi-Regge model.[3,14] A somewhat stronger version of it had also been used by Schmid when he originally proposed local duality.[19] This latter form is almost certainly not correct as it stands. We shall see in Sec. VI that *some* of his *results* can also be approximately obtained by using the weaker form, however.

We can get another relation between the Pomeranchuk trajectory and $\bar{\beta}$ by considering the $I_s = 2$ state. Here we do not have any resonances and so we might expect Eq. (1) to be valid with a smaller s_1. If we take the 2ρ threshold ($s_1 = 120$), only the first diagram of Fig. 2 will contribute. We will use Eq. (14) directly for its blobs, with $\bar{\beta} = 0.6$, corresponding to the experimental value of $\Gamma_\rho = 125$ MeV. Then, remembering that the Pomeranchuk is the only trajectory that comes into Eq. (1) and that its residue is given by Eq. (17), we obtain $\sigma_{\text{tot}} \simeq 12$ mb if we take $(s_0, s_1) = (60, 120)$ and $\sigma_{\text{tot}} \simeq 13$ mb with $(s_0, s_1) = (4, 120)$; the former s_0 is midway between the positions of the ρ and f^0 resonances while the latter is at the $\pi\pi$ threshold. The corresponding σ_{tot} obtained by factorization from the experimental πp and pp cross sections is $\sigma_{\text{tot}} = 15$ mb.

Finally, we may wish to combine the $I_s = 1$ and $I_s = 2$ conditions, requiring them to be consistent with each other. If we first ignore the P in the $I_s = 1$ state, we naturally get the same $\bar{\beta}$ as before, namely, $\bar{\beta} = 0.527$. The $I_s = 2$ state then gives $\sigma_{\text{tot}} = 9.3$ mb with $(s_0, s_1) = (60, 120)$. If we next include the P in both states, we obtain $\bar{\beta} = 0.99$ ($\Gamma_\rho = 210$ MeV) and $\sigma_{\text{tot}} = 32.7$ mb. These values are not as good as the values we obtained by looking at the two isospin states separately, but this is hardly surprising since the errors present for each state would tend to be compounded when we try, in addition, to make the two states consistent with each other. We shall see in the next section, however, that an improved treatment of $A_s^{\rho\epsilon}$, which takes some account of other exchanges besides an un-Reggeized pion, leads to a better self-consistent result.

V. AN EXTENDED ABFST MODEL

The best way of taking into account other exchanges besides the pion in our scheme would be to replace the horizontal lines in Fig. 2 by Regge poles, since this would bring in the exchange of entire families of particles lying on the assumed trajectories. Since this is rather involved, we shall, instead, follow a simpler but cruder procedure here. We first note that if we use the graphs of Fig. 2 to calculate the amplitude A rather than the absorptive part, then their sum satisfies the Bethe-Salpeter equation for a given isospin state I_t in the c.m. system in the t channel. This was first pointed out by ABFST in Ref. 2 and has recently been exploited by Ball and Marchesini.[20] We thus have

$$A(p', p, t) = V(p', p, t) + \frac{i}{\pi^3} \int d^4p''\, V(p', p'', t) \times G(p'', t) A(p'', p, t) R\,, \qquad (19)$$

where $P + p$ and $P - p$ are the four-momenta of the initial, and $P + p'$ and $P - p'$ are the four-momenta of the final particles in the t channel, while P is the four-vector $(\frac{1}{2}\sqrt{t}, 0, 0, 0)$, using a timelike metric. The Green's function G is given by

$$G^{-1}(p'', t) = [(P + p'')^2 - 1]\,[(P - p'')^2 - 1]\,, \qquad (20)$$

while the potential V is given by the first diagram of Fig. 2. An iterative expansion of Eq. (19) will give us back the graphs of Fig. 2.

The "inelasticity" factor R in Eq. (19) is simply unity if the horizontal lines in Fig. 2 are pions.

We will go beyond this approximation by taking a simple parametric form for it instead.[5] For simplicity we will assume that it depends only on p''^2, so $R = R(p''^2)$. To guarantee that Eq. (19) satisfies elastic unitarity below the inelastic threshold $t = t_{\text{in}}$, we must have $R = 1$ for $-p''^2 < (\frac{1}{4}t_{\text{in}} - 1)$. Suppose we now make a Logunov-Tavkhelidze-Blankenbecler-Sugar[21,22] approximation for Eq. (19) and make a partial-wave projection in the t channel. We then have exact elastic unitarity when $R = 1$ and

$$\sigma_l^{\text{tot}}(t)/\sigma_l^{\text{el}}(t) \approx R(1 - \tfrac{1}{4}t) \tag{21}$$

otherwise, where $\sigma_l^{\text{tot}}(t)$ and $\sigma_l^{\text{el}}(t)$ are the total and elastic partial-wave cross sections, respectively. Our parametrization therefore leads to an l-independent form for this ratio, and permits us to make the further identification

$$R(1 - \tfrac{1}{4}t) \approx \sigma^{\text{tot}}(t)/\sigma^{\text{el}}(t)\,, \tag{22}$$

where $\sigma^{\text{tot}}(t)$ and $\sigma^{\text{el}}(t)$ are now the full total and partial cross sections. As far as the important partial waves are concerned, Eq. (21) is just the sort of assumption one makes in a grey-disk diffraction model, for example. In such a model one also takes this ratio to be independent of t. This is also approximately true for Pomeranchuk Regge exchange at sufficiently high t (to within factors of $\ln t$), for which, moreover, the ratio is independent of I_t. All of these features seem to be borne out by the "experimental" cross sections for $t \gtrsim 3$ GeV^2 obtained by factorization from the pp and πp cross sections.

A simple expression which incorporates all the above properties is[5]

$$R(p''^2) = 1 + (R_0 - 1)\theta(-p''^2 - \tfrac{1}{4}t_I + 1)\,, \tag{23}$$

where θ is the step function and R_0 is a constant independent of I_t; we shall determine it from Eq. (22) in the limit of large t. Using the experimental values $\sigma_{\text{el}} \simeq 2.5$–$3.0$ mb and $\sigma_{\text{tot}} \simeq 15$ mb for $t \gtrsim 3$ GeV^2, we get $R_0 \simeq 5$–6. Equation (23) actually continues to be roughly valid even at the g-meson mass where $R_0 \approx \Gamma_g/\Gamma_g^{\pi\pi} \approx 4$, if we take the experimental value of 170 MeV for Γ_g and calculate the $\pi\pi$ partial width $\Gamma_g^{\pi\pi}$ from the Veneziano model with $\bar{\beta} = 0.6$. We can make it valid below the g mass by taking $t_I = 120$, i.e., the $\rho\rho$ threshold. We saw in Sec. II that this is the point at which inelastic effects can be expected to begin to become important.

From the first diagram of Fig. 2, it is evident that the potential V is given by the elastic absorptive part $A_s^{\pi\pi}$,

$$V(s,t) = \frac{1}{\pi}\int ds' \frac{A_s^{\pi\pi}(s',t)}{s'-s} + (u\text{-channel term})\,. \tag{24}$$

This is unambiguous when the pions are on their shells. In Eq. (19), however, we require an off-shell extrapolation for V. One such extrapolation is the one in effect used in Sec. III. However, this is not likely to make much sense except near $t = 0$, whereas our considerations of the inelasticity parameter R involved positive values of t well away from that point. We shall therefore, instead, make the original ABFST assumption that $A_s^{\pi\pi}(s,t)$ can be approximated by its on-shell value.[23] It has been argued[5] that this extrapolation is favored if we invoke a "criterion of maximal convergence." As in Secs. II and III, we shall make a resonance approximation for the on-shell $A_s^{\pi\pi}$ below the 2ρ threshold in the $I_s = 1$ state, taking directly the absorptive part of Eq. (3). Above the 2ρ threshold, and also in the $I_s = 2$ state, we again assume Regge behavior, which, as we have seen, amounts to using Eqs. (6) and (9), at least at $t = 0$.

The rest of our procedure is exactly the same as in Sec. IV. The only difference is that we evaluate $A_s^{\rho\epsilon}$ differently, using the integral term of Eq. (15) (with $A = V$) instead of Eq. (13) (see Appendix B). This means that, in addition to the masses in the problem, we must also know the value of R_0 in Eq. (23). We have already seen that experiment gives $R_0 \approx 5$–6. With $R_0 = 5$, we obtain $\bar{\beta} = 0.51$ ($\Gamma_\rho = 106$ MeV) and $\Gamma_g = 112$ MeV if we ignore the Pomeranchuk in the $I_s = 1$ state, and $\bar{\beta} = 0.73$ ($\Gamma_\rho = 150$ MeV) and $\Gamma_g = 160$ MeV if we include it (with $\sigma_{\text{tot}} = 12$ mb). These results, which were reported in Ref. 10, are very similar to those obtained in Sec. IV. The results for the $I_s = 2$ state are, of course, identical to those in Sec. IV since they do not involve $A_s^{\rho\epsilon}$.

Finally, we can require the $I_s = 1$ and $I_s = 2$ conditions to be consistent with each other. In addition, since σ_{tot} is part of the output in such a calculation, we can also require it to be consistent with the σ_{tot} which goes into the input R_0 through Eqs. (22) and (23). The only input is then the large-t value of σ_{el}. Experimentally, as we have seen, $\sigma_{\text{el}} = 2.5$–3.0 mb. With $\sigma_{\text{el}} = 3.0$ mb as input and with $(s_0, s_1) = (60, 120)$ in the $I_s = 2$ state, we obtain the output values $\bar{\beta} \simeq 0.74$ ($\Gamma_\rho = 154$ MeV), $\sigma_{\text{tot}} \simeq 18.8$ mb, and $\Gamma_g = 188$ MeV. If, instead, we take $\sigma_{\text{el}} = 2.5$ mb, we get $\bar{\beta} \simeq 0.69$ ($\Gamma_\rho = 144$ MeV), $\sigma_{\text{tot}} \simeq 16$ mb, and $\Gamma_g = 180$ MeV. Of course, we could also turn the problem around and take $\bar{\beta}$ or σ_{tot} as input.

Instead of considering states of definite I_s, it is sometimes more convenient to consider states of definite I_t. A calculation which does this is pre-

sented in Appendix A. The techniques are somewhat different from the ones discussed above, but most of the results are very similar.

VI. A REINTERPRETATION OF THE SCHMID ARGAND-DIAGRAM LOOP ANALYSIS

In Secs. IV and V the elasticity of the g meson was evaluated with the help of Eq. (18), which, in turn, assumed a semilocal Regge-resonance duality for the *elastic* absorptive part, in addition to the usual one involving the total absorptive part. We shall now argue that a partial-wave version of this is capable of approximately reproducing some of the results obtained by Schmid in analyzing Argand-diagram loops in πN scattering.[19] We can do this without having to identify the Regge-pole loops with actual resonances, an assumption which is almost certainly incorrect.

In his original analysis, Schmid found that a partial-wave projection of a smooth amplitude with Regge behavior leads to loops in the Argand diagram, and, hence, to bumps in the partial-wave amplitudes. He then identified these with resonances. Although we will avoid any such interpretation (and the accompanying debate on what is meant by a resonance), we shall nevertheless assume that Schmid's bumps can be approximately fitted by a Breit-Wigner formula in the region where they dominate. This fit should be thought of as a purely empirical one, without any connotations as to whether the amplitude has any poles on the second sheet. In other words, we are taking advantage of the fact that a function can usually be fitted quite easily in a limited region by another with quite different analyticity properties. We thus have a partial-wave amplitude

$$f_l^R \approx \frac{x' m' \Gamma'/\rho}{m'^2 - s - i m' \Gamma'}, \tag{25}$$

where ρ is the phase-space factor, and m', Γ', and x' are the fake mass, width, and elasticity that we would have had if Eq. (25) did in fact represent a resonance. They would be the values which would be read off from the Argand-diagram loops.

Suppose now that we did have a true resonance in the same angular momentum state. This would again be described by a Breit-Wigner formula

$$f_l^{\text{res}} \simeq \frac{x m \Gamma/\rho}{m^2 - s - i m \Gamma}, \tag{26}$$

but where m, Γ, and x are now the true mass, width, and elasticity. If we now require the elastic and total absorptive parts obtained from Eqs. (25) and (26) to be dual with each other, we have

$$\int_{s_0}^{s_1} ds' \rho (|f_l^{\text{res}}|^2 - |f_l^R|^2) = 0 \tag{27}$$

and

$$\int_{s_0}^{s_1} ds' (\text{Im} f_l^{\text{res}} - \text{Im} f_l^R) = 0 . \tag{28}$$

Combining this with Eqs. (25) and (26) and using a narrow-width approximation, we have $x'^2 m' \Gamma' = x^2 m \Gamma$ and $x' m' \Gamma' = x m \Gamma$. Using the values of $m' \Gamma'$ and $x' m' \Gamma'$ obtained by Schmid, we thus have $m\Gamma = (0.7, 0.8\ \text{GeV}^2)$ and $xm\Gamma = (0.12, 0.09\ \text{GeV}^2)$ for the 2190-MeV and 2420-MeV πN resonances, respectively. The corresponding experimental values[19] are $m\Gamma = (0.44, 0.67\ \text{GeV}^2)$ and $xm\Gamma = (0.13, 0.07\ \text{GeV}^2)$. The agreement is somewhat better for $xm\Gamma$, which followed from Eq. (28) alone. But it must be remembered that Eq. (25) is not a necessary assumption and that a more direct evaluation of the Regge terms in Eqs. (27) and (28) may lead to better results.

Note that the above analysis does not say anything precise about mass m, other than the fact that m^2 and m'^2 must both lie within the (s_0, s_1) bin, and therefore not be too far apart from each other. (Local duality would, of course, require $m = m'$.) In the Schmid analysis, m' and the experimental value of m for the above two resonances were in fact close to each other. This also means that an Argand-diagram loop prediction, such as that of Kugler,[24] which predicted resonances (and hence large imaginary parts) with $l \sim \sqrt{s}$, should continue to be approximately valid even without the assumption of local duality. This sort of resonance structure has recently been shown by Harari[25] to be capable of describing dips in inelastic hadron processes.

VII. CONCLUSION

We have seen that duality permits us to extract simple relations between resonance widths and other Regge-residue parameters from a multiperipheral model evaluated in the forward direction. By going away from the forward direction we can presumably obtain additional information such as the t dependence of the trajectories. Perhaps the main limitation of the method, at least when used in its simplest form, is its inability to *generate* some of the fine structure of the singularities in the angular momentum plane, although these can always be put in by hand. Thus, if we want to introduce cuts,[15] we would have to assume some model for them, such as the absorptive model.[16] A better procedure, however, would probably be to combine our method with some other simplifying technique which gives cuts, such as an effective factorizable-coupling method or the trace approximation used by Chew, Rogers, and Snider.[14] Such a method seems to complement, rather than be an alternative to the duality approach. For ex-

ample, we have already seen that, at least with s_1 below the 3ρ threshold, we do not even need to know the internal couplings in the multiperipheral chain of Fig. 1. Since something like the factorizable-coupling approximation requires both internal and external couplings (unless they are simply *assumed* to be equal), we could determine the effective strength of the latter by requiring that the output trajectory be consistent with the one calculated from duality. Or, more generally, if we simply want to introduce additional parameters to improve the approximation,[7] these could be determined in the same way.

In the calculations considered so far, the exchanges of other objects besides the pion were included only roughly, by introducing the inelasticity factor of Sec. V. A more natural way, as we have seen, would be to exchange Regge poles. In addition to the π trajectory, we can also have the ω, A_1, and A_2 trajectories. A general treatment of this problem might be quite difficult. If we take s_1 below the 3ρ threshold, however, then, as we saw in Sec. II, we only have to consider the graphs of Fig. 4, where the horizontal lines of Fig. 4(b) would now represent all of the aformentioned trajectories instead of just the pion. Since Fig. 4 only involves quasi-two-body processes, however, it should be relatively straightforward to construct these graphs. The Regge couplings, for example, could be related in the usual way to simple particle couplings through the use of ordinary Regge-resonance duality.[8]

Finally, we should be able to extend our techniques to processes involving baryons, particularly if we apply duality in a domain where a quasi-two-body approximation is still valid. Meson-baryon scattering processes should constitute a particularly fruitful domain, since there is considerable experimental data available on both the resonances and Regge poles which would be involved in such a calculation. Since there are usually many important partial waves coming in, only a few of which might be dominated by resonances in a given energy range, it may be more convenient to make a partial-wave projection of Eq. (1) instead of considering it at a fixed value of t. At the same time, the use of a partial-wave version of Eq. (18) should enable us to calculate the elasticities of resonances. This, as we saw, involves a different kind of average duality than that of Dolen, Horn, and Schmid and should lead to roughly the same elasticities as those obtained from the otherwise dubious Argand-diagram loop technique of Schmid.

APPENDIX A: AN $I_t = 1$ CALCULATION

We will consider here a calculation using the model of Sec. V, but in a state of definite I_t. In particular, it is convenient to look at $I_t = 1$, since there is no Pomeranchuk trajectory in this state. Since Fig. 3(a) is somewhat messy to deal with in this case, we followed the simpler but less productive procedure of using a resonance form for the blobs in the first diagram of Fig. 2. The resulting diagram, as we have argued already, is then dual to Fig. 3(a), and is expressible as a sum of resonances, each of which is multiplied by an elasticity factor r. Now in the $I_t = 1$ state the g meson does not dominate this sum to the same extent as it did in the $I_s = 1$ state. We will therefore simply take the elasticity $r(s) = R^{-1}(1 - \frac{1}{4}s)$, where R is the l-independent inelasticity factor given by Eq. (23), and calculate the parameters of our resonances from the Veneziano model. Then[5]

$$A_s^{\pi\pi}(s, t) = r(s) F_s(s, t), \tag{A1}$$

where F is given by Eq. (5).

We can now combine Eq. (A1) with Eqs. (15) and (1). If, in addition, we relate the Regge term in Eq. (1) through duality to the Veneziano absorptive part F_s, we obtain, for s_1 below the 3ρ threshold,

$$\int_{s_0}^{s_1} ds \{A_s^{\rho\epsilon}(s, 0) + F_s(s, 0)[r(s) - 1]\} = 0, \tag{A2}$$

where, as in Sec. V, $A_s^{\rho\epsilon}$ is evaluated from the integral term of Eq. (15) with $A = V$. If we now take, as before, $(s_0, s_1) = (120, 180)$ and the experimental value $R_0 \approx 5$, we obtain $\bar{\beta} = 0.79$ ($\Gamma_\rho = 164$ MeV). It is perhaps amusing to note that if we take $(s_0, s_1) = (120, 210)$ and require consistency with the $(s_0, s_1) = (120, 180)$ result, we must have $R_0 = 1.85$, which gives $\bar{\beta} = 1.05$ ($\Gamma_\rho = 220$ MeV). However, such a calculation entails rather large cancellations and is probably not very meaningful, particularly since we expect the $N\bar{N}$ intermediate state to begin contributing above $s \approx 180$.

Since the $I_t = 1$ and $I_s = 2$ states are probably the simplest to consider, it might be interesting to combine the two, using the above scheme for $I_t = 1$ and the conditions of Sec. IV for $I_s = 2$. Since σ_{tot} would again be part of the output in such a calculation, we can calculate R_0 through Eqs. (22) and (23) in terms of the large-t value of σ_{el}, which is then the input in our problem. If we take $\sigma_{el} = 2.5$ mb, which is consistent with experiment, we obtain $\bar{\beta} \simeq 0.67$ ($\Gamma_\rho = 140$ MeV) and $\sigma_{tot} \simeq 15$ mb.

APPENDIX B: FORMULAS FOR THE BOX DIAGRAM

We will write down here the formulas for Fig. 3(b), using Eqs. (10)–(12) for Fig. 4(b) and Eq. (14)

for the polarization sum, but with the $k_\mu k_\nu$ term dropped, as discussed at the end of Sec. III. From Eq. (13) we then have

$$A_s^{\rho\epsilon}(s,t)=\frac{1}{4\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}\int\frac{d\Omega}{D_iD_f}\{\tfrac{1}{4}C_{\rho\rho}{}^2[s-m^2-4q^2z-D_i-D_f]^2-\tfrac{1}{2}C_{\rho\epsilon}{}^2[s-m^2-4q^2z-D_i-D_f]+C_{\epsilon\epsilon}{}^2\}, \tag{B1}$$

where

$$D_{i,f}=(p^2+q^2+1)-2pqz_{i,f}\,, \tag{B2}$$

$s=4(q^2+1)=4(p^2+m^2)$, and z_i and z_f are the cosines of the scattering angles for the initial to intermediate and intermediate to final processes, respectively. To evaluate some of the resulting integrals we have used the expansions

$$\frac{1}{D_{i,f}}=\frac{1}{2pq}\sum_{l=0}(2l+1)Q_l\left(\frac{p^2+q^2+1}{2pq}\right)P_l(z_{i,f}) \tag{B3}$$

and

$$D_{i,f}=(p^2+q^2+1)P_0(z_{i,f})-2pzP_1(z_{i,f}) \tag{B4}$$

combined with the general property

$$\int d\Omega P_l(z_f)P_{l'}(z_i)=\frac{4\pi\delta_{ll'}}{2l+1}P_l(z)\,. \tag{B5}$$

We then have

$$\left(\frac{s}{s-4m^2}\right)^{1/2}A_s^{\rho\epsilon}(s,t)=X(s,t)[\tfrac{1}{4}C_{\rho\rho}{}^2(s-m^2-4q^2z)^2-\tfrac{1}{2}C_{\rho\epsilon}{}^2(s-m^2-4q^2z)+C_{\epsilon\epsilon}{}^2]$$
$$-\tfrac{1}{4}C_{\rho\rho}{}^2\{(s-m^2-4q^2z)(pq^{-1})Q_0(x)-[xQ_0(x)-zQ_1(x)]-1\}+C_{\rho\epsilon}{}^2(kq)^{-1}Q_0(x)\,, \tag{B6}$$

where

$$4\pi X(s,t)=\int d\Omega(D_iD_f)^{-1}\,, \tag{B7}$$

$$x=(p^2+q^2+1)/2pq\,, \tag{B8}$$

and the Q_l are the usual Legendre functions of the second kind.

To evaluate Eq. (B7), we used the Mandelstam relation[26]

$$\frac{1}{4\pi}\int\frac{d\Omega}{D_iD_f}=\frac{1}{2p}\int_{t_+}^{\infty}\{t'[(t'-4)p^2-m^4]\}^{-1/2}\frac{dt'}{t'-t}, \tag{B9}$$

where the lower limit t_+ is given by the upper zero of the term in the curly bracket. This integral is particularly easy to evaluate at $t=0$, so that we have

$$X(s,0)=(m^4+4p^2)^{-1}\,. \tag{B10}$$

In practice, for the range of s with which we are concerned, $m^4>>4p^2$, so that we can drop the latter term in Eq. (B10). It also means that we can make the asymptotic approximation $Q_0(x)\simeq x^{-2}$ and neglect the $Q_1(x)$ term in Eq. (B6). The resulting expression for $A_s^{\rho\epsilon}(s,0)$ can be integrated analytically when it is inserted through Eq. (15) into Eq. (1).

The evaluation of $A_s^{\rho\epsilon}$ within the model of Sec. V is very similar, except that we start from the integral term of Eq. (19) with $A=V$ and take its s discontinuity. We then have

$$\left(\frac{s}{s-4m^2}\right)^{1/2}A_s^{\rho\epsilon}(s,t)$$
$$=Y(s,t)[\tfrac{1}{4}C_{\rho\rho}{}^2(s-m^2-4q^2z)^2$$
$$-\tfrac{1}{2}C_{\rho\epsilon}{}^2(s-m^2-4q^2z)+C_{\epsilon\epsilon}{}^2]\,, \tag{B11}$$

where Y is a modified version of X and reduces to

$$Y(s,t)=\frac{1}{2p}\int_{t_+}^{\infty}\frac{dt'}{t'-t}R(1-\tfrac{1}{4}t')$$
$$\times\{t'[(t'-4)p^2-m^4]\}^{-1/2}\,. \tag{B12}$$

With R given by Eq. (23) this is again quite simple to evaluate at $t=0$ and leads to

$$Y(s,0)=\frac{1}{m^4+4p^2}\left[R_0-(R_0-1)\left(\frac{(t_I-4)p^2-m^4}{p^2t_I}\right)^{1/2}\right.$$
$$\left.\times\theta\left(s-4m^2\left(1+\frac{m^2}{t_I-4}\right)\right)\right]. \tag{B13}$$

Since $m^4\gg4p^2$ and $t_I=4m^2\gg4$, this can be further simplified to

$$Y(s,0)=\frac{1}{m^4}\left[R_0-(R_0-1)\left(1-\frac{m^2}{4p^2}\right)^{1/2}\theta(s-5m^2)\right], \tag{B14}$$

which again gives an expression for $A_s^{\rho\epsilon}(s,0)$ which can be integrated analytically when it is inserted through Eq. (15) into Eq. (1).

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*Supported in part by the U. S. Atomic Energy Commission and by an Alfred P. Sloan Foundation Fellowship.

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