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# Analytic Continuation in Helicity and O(2,1) Expansions*

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We discuss the analytic continuation of an azimuthal angular variable in a scattering process to a channel where it may take on large imaginary values. The asymptotic behavior of the many-particle amplitude for large values of that variable is shown to be governed by singularities in an analytically continued helicity amplitude. The procedure is analogous to the usual Regge-Sommerfeld-Watson transformation relating large-$\cos\theta$ behavior to singularities in analytically continued angular momentum amplitudes.

## I. INTRODUCTION

We present here a discussion of analytic continuation in the helicity variable $m$ for scattering amplitudes. This continuation is in many ways analogous to the Froissart-Gribov continuation in angular momentum $l$. In both cases the original variable ($l$ or $m$) as physically defined for a scattering problem is an integer. To define the Froissart-Gribov partial-wave amplitude for complex $l$, it is necessary to know how to continue analytically in the projection variable $z$ (cosine of scattering angle) from the physical region $-1 \leq z \leq 1$ to very large values. Similarly, to define a helicity amplitude for complex $m$ it is necessary to make a continuation in the relevant projection angle $\phi$ from the physical region $0 \leq \phi \leq 2\pi$ to large values in the complex plane. It is, of course, just because the definition of analytically continued helicity or partial-wave amplitudes involves infinite integrations in the projection variables that singularities in $m$ or $l$ control the asymptotic behavior in these same projection variables. Often these asymptotic limits are physical regions in some channel, giving a potentially simple description of very high-energy processes.

It is no great surprise that making $m$ continuous and $\phi$ infinite in range results in Fourier integrals rather than Fourier series. What is not quite so clear is how to carry out a simultaneous continuation in total angular momentum *and* helicity. The whole question is discussed here by using space-time O(2, 1) expansions which are defined by conditions (a) and (b) in Sec. II. The boon of the O(2, 1) treatment is that the expansion already takes place in the relevant "crossed" channel where $z$ and $\phi$ can physically be at large values in their complex plane. Thus the simultaneous continuation in $l$ and $m$ is essentially contained in the O(2, 1) analysis and there remains only to carry out a careful identification of crossed-channel quantities and their analytic continuations.

Simultaneous evaluation of helicity and angular momentum pole contributions permits a discussion of multiple asymptotic limits that cannot be deduced from simple multi-Regge theory[1] alone. The general technique of complex helicity may also conceivably be useful in studying infinite helicity sums, such as occur in many-body Regge dynamics.[2]

*Note added.* After the completion of this work we learned of a similar treatment of complex helicity by Goddard and White.[3] They also discuss the Sommerfeld-Watson transformation of the Fourier series as well as the simultaneous continuation in angular momentum and helicity, overlapping our Sec. II.

In Sec. II we briefly review O(2, 1) expansions and identify the helicity variable and its analytic continuation. In Sec. III we locate those singularities in helicity which are relevant to the calculation of inclusive processes.

## II. ANALYTIC CONTINUATION IN HELICITY

In this section we connect a continuous expansion parameter in space-time O(2, 1) expansions[4,5] for scattering amplitudes with the analytic continuation from integers of a helicity index for a crossed-channel amplitude. The parameter under discussion is the eigenvalue $\mu$ of the generator of $y$ boosts, $K_y$.

We start by reviewing O(2, 1) expansions. We recall that an arbitrary O(2, 1) transformation $g$ can be parametrized as follows:

$$g(\alpha, \xi, \psi) = e^{-iK_y\alpha}e^{-iK_x\xi}e^{-iJ_z\psi}, \tag{2.1}$$

where $K_y$ and $K_x$ are the generators of $y$ and $x$ boosts and $J_z$ is the generator of $z$ rotations. The parametrization (2.1) differs from the Euler-type expression which would have $z$ rotations on each end. One virtue of (2.1) is that both $\alpha$ and $\xi$ have an infinite range. When writing scattering amplitudes as functions of $\alpha$, $\xi$, and $\psi$, the asymptotic $\xi$ limit is governed by ordinary Regge poles, whereas we shall show that the asymptotic $\alpha$ limit is governed by helicity poles. This gives a double asymptotic limit. If the $z$ rotation in (2.1) is replaced by a second $y$ boost, a triple asymptotic limit may also be found,[6] but it is a straightforward generalization of the double limit and we shall not discuss it here.

We consider scattering processes of the type (see Fig. 1)

$$P_a + P_b \to P_a' + P_b', \tag{2.2}$$

where these four-momenta generally represent clusters of spinless particles. The momentum transfer $Q = P_b' - P_b = P_a - P_a'$ is assumed to be spacelike, which makes O(2, 1) the relevant symmetry group, since O(2, 1) is the little group of a spacelike vector. The parameters $\alpha$, $\xi$, and $\psi$ of (2.1) are useful variables for describing the relation (2.2) in a frame where $Q$ has only a positive $z$ component if the following conditions are satisfied:

(a) The two equal O(2, 1) three-vectors $\vec{P}_a$ and $\vec{P}_a'$ defined by omitting the $z$ component of $P_a$ and $P_a'$ are required to be spacelike: $\vec{P}_a'^2 = \vec{P}_a^2 < 0$. It then follows that $P_a^2 < 0$ and $P_a'^2 < 0$.

(b) $\vec{P}_b^2 = \vec{P}_b'^2 > 0$.

The physical meaning of the variables $(\alpha, \xi, \psi)$, in the case of a reaction of the type (2.2) where the conditions (a) and (b) are fulfilled, can be briefly described as follows:

(i) In a frame where $\vec{P}_a$ and $\vec{P}_a'$ have only an $x$ component, $\alpha$ describes the variation of the scattering amplitude when the clusters $P_a$ and $P_a'$ are subjected to the $y$ boost $e^{iK_y\alpha}$.

(ii) In a frame where $\vec{P}_b$ and $\vec{P}_b'$ have only a time component, $\psi$ describes the variation of the scattering amplitude when the clusters $P_b$ and $P_b'$ are subjected to a $z$ rotation $e^{-iJ_z\psi}$.

(iii) The variable $\xi$ relates the frames discussed in (i) and (ii). In the frame defined in (ii) we let $\vec{P}_a$ and $\vec{P}_b$ define the $x$-$t$ plane; then $e^{-i\xi K_x}$ is the $x$ boost which eliminates the time component of $\vec{P}_a$.

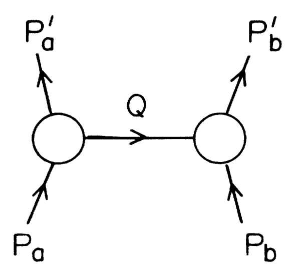


FIG. 1. Scattering process.

We note that if $P_a$ and $P_a'$ are single-particle clusters, then there is no $\alpha$ dependence. Similarly if $P_b$ and $P_b'$ are single-particle clusters, there is no $\psi$ dependence. This shows the usefulness of the parametrization $(\alpha, \xi, \psi)$ for the process (2.2) when conditions (a) and (b) are obeyed. In what follows we are uninterested in $\psi$ dependence so we shall assume that $P_b$ and $P_b'$ are single-particle clusters.

We now expand the scattering amplitude $A(\alpha, \xi)$ for the process (2.2) in terms of the irreducible representations of O(2, 1) where we suppress all but the group variables:

$$A(\alpha, \xi) = \int_{-\infty}^{\infty} d\mu\, e^{-i\mu\alpha} \times \sum_\rho \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dl}{2i} \frac{2l+1}{\tan\pi l} a_{\mu,\rho}(l) d^{l\,*}_{\mu,\rho}(\xi), \tag{2.3}$$

with

$$a_{\mu,\rho}(l) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\alpha\, e^{i\mu\alpha} \int_{-\infty}^{\infty} d(\sinh\xi) d^l_{\mu,\rho}(\xi) A(\alpha, \xi). \tag{2.4}$$

The above expansion exists provided that $A$ is square-integrable in the variables $\alpha$ and $\sinh\xi$. This expansion is derived, fully discussed, and generalized to nonintegrable functions in another paper.[5] We only remark here that the O(2, 1) representations used in the expansion are the matrix elements of $g(\alpha, \xi, \psi)$ in a mixed basis, consisting of eigenstates of $K_y$ on the left (defined by an eigenvalue $\mu$ and a two-valued degeneracy index $\rho$) and an eigenstate of $J_z$ on the right (which, for the case of no $\psi$ dependence, consists only of the zero eigenvalue of $J_z$). The mixed basis is important because it produces the simple $\alpha$ dependence given by the exponential in (2.3) and this in turn permits the identification of $\mu$ with an analytically continued helicity index.

It is shown in JLY that if a continuation is made of the amplitude $A(\alpha, \xi)$ to the crossed channel, $P_a + \bar{P}_a' \to P_b' + \bar{P}_b$, then the physically allowed values of $\alpha$ are

$$0 \le i\alpha \le 2\pi. \tag{2.5}$$

In this channel $\phi = i\alpha$ describes an azimuthal variation of the clusters $P_a$ and $\bar{P}_a'$ with respect to an axis along the ordinary three-vector part of $P_a$ or $\bar{P}_a'$ in the center-of-mass frame. The projection $\int_0^{2\pi} d\phi\, e^{im\phi}$ of such a $\phi$ dependence clearly gives a helicity amplitude. Our task is to show that the $a_{\mu,\rho}(l)$ of (2.4) is related to the analytic continuation of such a helicity amplitude with $m = i\mu$.

We begin by noting that the $\alpha$ dependence of

$A(\alpha, \xi)$ is simply Fourier-decomposed by (2.3) and (2.4). That is, (2.3) can obviously be rewritten

$$A(\alpha, \xi) = \int_{-\infty}^{\infty} d\mu \, e^{-i\mu\alpha} A_\mu(\xi) , \tag{2.6}$$

where

$$A_\mu(\xi) = \sum_\rho \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dl}{2i} \frac{2l+1}{\tan\pi l} d^{l*}_{\mu,\rho}(\xi) a_{\mu,\rho}(l) \tag{2.7}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \, e^{i\mu\alpha} A(\alpha, \xi) . \tag{2.8}$$

A comparison of (2.3) and (2.4) with (2.6)–(2.8) shows that

$$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dl}{2i} \frac{2l+1}{\tan\pi l} \sum_\rho d^{l*}_{\mu,\rho}(\xi) d^l_{\mu,\rho}(\xi') = \delta(\sinh\xi - \sinh\xi') ,$$

which is *independent* of $\mu$. These equations show that the functional dependence of $A_\mu(\xi)$ on the variable $\mu$ has nothing to do with the dependence of $d^l_{\mu,\rho}$ on $\mu$, a fact which is perfectly clear from (2.8) but not so obvious from (2.7).

We now wish to verify that $A_\mu(\xi)$ is the analytic continuation of a helicity amplitude for the crossed reaction $P_a + \overline{P}'_a \to P'_b + \overline{P}_b$. As mentioned earlier $\phi = i\alpha$ becomes, for this crossed process, an azimuthal angle for clusters $P_a$ and $\overline{P}'_a$. A helicity amplitude can then be defined in this crossed channel by

$$A_m = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-im\phi} A(\phi) , \tag{2.9}$$

with

$$A(\phi) = \sum_{m=-\infty}^{\infty} A_m e^{im\phi} . \tag{2.10}$$

It is useful to define the complex variable $z = e^{i\phi}$ and then to rewrite (2.9) and (2.10) as

$$A_m = \frac{1}{2\pi i} \int_{C_1} \frac{dz}{z^{m+1}} A(z) \tag{2.11}$$

and

$$A(z) = \sum_{m=0}^{\infty} (A_m z^m + A_{-m-1} z^{-m-1})$$
$$= A_1(z) + A_2(z) . \tag{2.12}$$

The contour $C_1$ is the unit circle and (2.12) is the Laurent expansion. We assume that $A(z)$ is an analytic function of $z$ free of singularities and single valued in some annular region that includes the unit circle, i.e., in a neighborhood of the physical region (see Fig. 2). The expansion (2.12) converges in such an annular region up to the point where singularities are reached.

To proceed further we wish to continue $z$ out of the annulus to the region where $\phi = i\alpha$ and $-\infty < \alpha < \infty$, where identification with the $O(2,1)$ expansion can be made. This is just the region $0 \leqslant z \leqslant +\infty$. In order to effect this continuation in $z$, we must also continue the amplitude $A_m$ away from the integers to complex $m$. To see how this is done we treat in (2.12) only the first sum, $A_1(z)$, which is analytic at $z = 0$. The second sum can be treated by transforming a new variable $z' = 1/z$. In general, the singularity structures of the two functions $A_1(z)$ and $A_2(z')$ are similar, although not identical. In the forward direction, with final cluster = initial cluster, inversion invariance makes the transformation $z \to 1/z$ a symmetry of the amplitude, and the two functions are identical to within a constant.

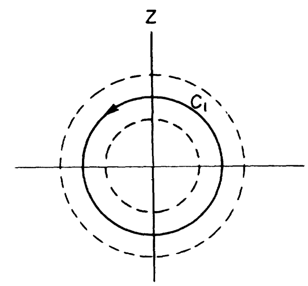


FIG. 2. Projection contours.

We now distort the contour $C_1$ in (2.11) to $C_2$, extending it to infinity (as shown in Fig. 3) in such a way that all points of the contour have $|z| > 1$. If there are singularities exactly on the real axis, they may be displaced slightly above or below the real axis and then allowed to approach the real axis after the expansion is performed. The direction of displacement is determined by whether we wish to calculate the continued function above or below the real axis.

The contour $C_2$ can be used to define $A_m$ when $m$ is not an integer provided Re$m$ is sufficiently large and $A_1(z)$ is power bounded as $z \to \infty$. Determination of $A_m$ for low values of Re$m$ can be made by means of analytic continuation. In particular, if $|A_1(z)| < |z|^x$ as $|z| \to \infty$, then $A_m$ is analytic for Re$m >$ Re$x$. A contour integral can now be written for $A_1$:

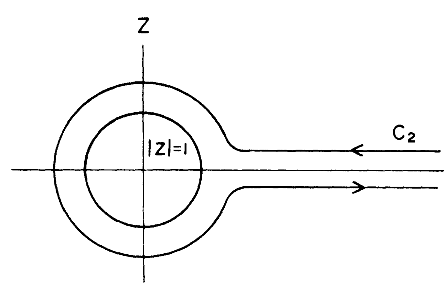


FIG. 3. Distortion of original contour in Fig. 2.

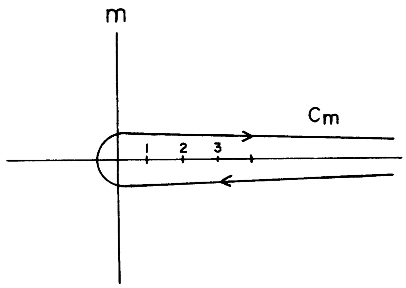


FIG. 4. Contour for helicity sum.

$$A_1(z) = -\frac{1}{2i}\int_{C_m} \frac{dm\, e^{i\pi m}}{\sin\pi m} z^m A_m, \tag{2.13}$$

where $C_m$ encloses the poles of $1/\sin\pi m$ along the real axis as shown in Fig. 4, detouring, if necessary, any singularities in $A_m$ along the real axis.

To make the identification of $A_m$ with $A_\mu$, we shall first assume that $\mathrm{Re}x < 0$. Now $A_m$ is analytic for $\mathrm{Re}m \geqslant 0$ and the contour $C_m$ in Fig. 4 can be distorted to the imaginary axis as shown in Fig. 5. The right-hand semicircular contour in Fig. 5 gives a vanishing contribution and can thus be dropped, leaving only the integral along the imaginary axis. This step is justified by examining $A_m$ in the limit $|m| \to \infty$ in the right-half plane. This behavior for a point $z$ on the contour $C_2$ (Fig. 3) is seen to be roughly

$$A_m \sim z^{-m-1} = e^{-i\theta(m+1)} e^{(\ln|z|)(m+1)}, \tag{2.14}$$

where $0 < \theta < 2\pi$ and $\ln|z| > 0$. This assumes a zero phase for $z$ on the upper lip of the infinite contour in Fig. 3. The behavior (2.14) establishes the convergence of (2.13) and validates the dropping of the semicircular contour in Fig. 5.

The new $m$ contour in Fig. 5 thus leads to the formula

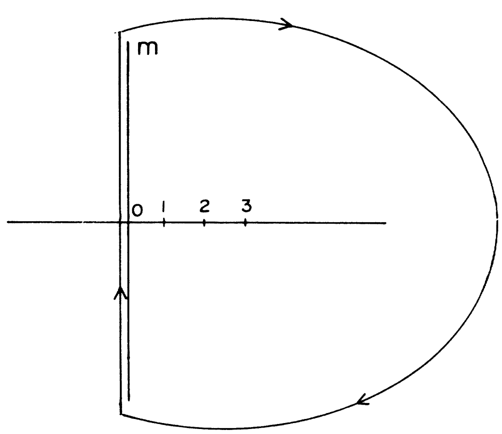


FIG. 5. Distortion of contour $C_m$ in Fig. 4.

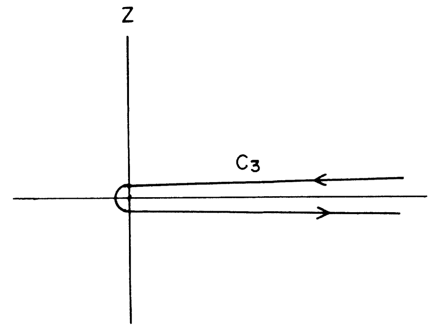


FIG. 6. Flattening of contour $C_2$ in Fig. 3 to give Fourier result.

$$A_1(z) = -\frac{1}{2i}\int_{-i\infty}^{i\infty} \frac{dm\, e^{i\pi m}}{\sin\pi m} z^m A_m. \tag{2.15}$$

Since $A_1(z)$ is analytic at $z=0$, the contour $C_2$ can be flattened to the real axis as shown in Fig. 6 to give contour $C_3$. The following formula for $A_m$ then results:

$$A_m = \frac{1}{2\pi i}\int_{C_3} \frac{dz}{z^{m+1}} A_1(z)$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{dz}{z^{m+1}} A_1(z)(e^{-2\pi i(m+1)} - 1) \tag{2.16}$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{dz}{z^{m+1}} A_1(z) e^{-i\pi m} \sin\pi m\,. \tag{2.17}$$

Using (2.15) and (2.17) and the substitution $m = i\mu$ and $\phi = i\alpha$, we obtain (with some reshuffling of constant factors) exactly (2.6) and (2.8) with, of course, the $\xi$ variable suppressed.

We have thus identified $A$ in (2.8) with the analytic continuation of the helicity amplitude $A_m$ from the crossed channel. If $\mathrm{Re}x > 0$, the $\alpha$ expansion (2.6)–(2.8) is no longer valid and the formalism involving the helicity shows readily how it is to be generalized, still using the analytic continuation of $A_m$. The first step is to distort the $C_m$ to $C'_m$ as shown in Fig. 7 with a vertical piece at $\mathrm{Re}x + \epsilon$. The right-hand semicircular contour vanishes just as before. We can no longer

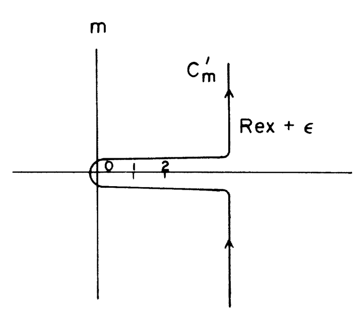


FIG. 7. Distortion of $C_m$ contour when $\mathrm{Re}\,x > 0$.

collapse the $z$ contour in Fig. 3 to the real axis since there would be convergence difficulties at $z=0$ for the first few terms in the $z$ expansion for $A_1$. This gives simply the expansion valid for $\mathrm{Re}\,x>0$:

$$A_1(z) = -\frac{1}{2i}\int_{C_m'} dm \frac{e^{i\pi m}}{\sin \pi m} z^m A_m, \tag{2.18}$$

$$A_m = \frac{1}{2\pi i}\int_{C_2} \frac{dz}{z^{m+1}} A(z). \tag{2.19}$$

If desired, one can substitute $m=i\mu$, $\phi=i\alpha$, to get the generalized O(2, 1) expansion in terms of the O(2, 1) variables.

The contribution of a helicity singularity in $A_m$ can easily be evaluated by distorting the $C_m'$ contour in (2.18). Such singularities have a real part which is less than $\mathrm{Re}\,x$. A helicity pole in $m$ at $\gamma$, for example, gives an asymptotic term in $z$ of the form

$$A_1(z) \underset{z\to\infty}{\sim} z^\gamma = e^{-\alpha\gamma}.$$

To obtain a double asymptotic limit coming from a helicity pole $\gamma$ in $m$ and an angular momentum pole $\bar{L}$ in $l$ we examine (2.7). In order to fully treat the question we must also generalize the expansion in $l$ such that the $l$ contour can be moved to the right. This is done in a separate paper.[5] The basic asymptotics can still be read off from (2.6) and (2.8). If $a_{\mu,\rho}(l)$ has a simultaneous pole at $l=L$ and $\mu=-i\gamma$, the double limit is

$$A(\alpha,\xi) \underset{\alpha\to-\infty;\ |\xi|\to\infty}{\sim} e^{-\alpha\gamma}\sum_\rho d^{L*}_{-i\gamma,\rho}(\xi) \sim e^{-\alpha\gamma}(\sinh\xi)^L. \tag{2.20}$$

The limit (2.20) has been shown to give an asymptotic formula for single-particle inclusive reactions where $P_b=P_b'$ and $P_a=P_a'$. $P_a$ and $P_a'$ are two-particle clusters consisting of an imaginary and an outgoing particle. The controlling Regge singularity is related to a triple Regge vertex shown in Fig. 8, which can also be studied with the techniques of Misheloff and Landshoff and Zakrzewski.[7]

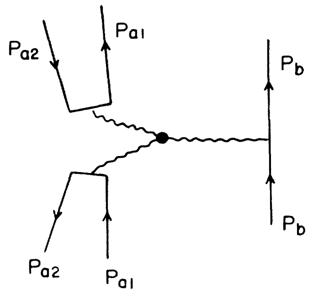


FIG. 8. Triple Regge vertex.

## III. LOCATION OF HELICITY SINGULARITIES

In order to identify helicity singularities, we must construct a two-body amplitude with two out of four squared masses negative. We do this in the simplest possible way by considering the three-body amplitude, $A$, which describes the process shown in Fig. 9. The $P_a$ cluster of Fig. 1 now consists of an incoming $P_{a1}$ and an outgoing $P_{a2}$, with

$$P_a = P_{a1} - P_{a2}. \tag{3.1}$$

The $P_a'$ cluster of Fig. 1 consists of an outgoing $P_{a1}'$ and an incoming $P_{a2}'$, with

$$P_a' = P_{a1}' - P_{a2}'. \tag{3.2}$$

In terms of our O(2, 1) parametrization, there are six variables associated with the left-hand cluster in Fig. 9, that is, with the process

$$P_{a1} + P_{a2}' \to P_{a2} + P_{a1}' + Q,$$

the mass $Q^2<0$ being the sixth of them, in addition to the usual five invariants associated with any $2\to3$ body process. We may specify these six variables in the reference frame (i) of Sec. II as $Q^2<0$, $P_a{}^2<0$, $P_a'^2<0$, and, for example, $P_{a1x}$, $P_{a1x}'$, and $P_{a1y}'$. These six variables and the O(2, 1) variables $\alpha$ and $\xi$ constitute the eight necessary to describe the $3\to3$ amplitude.

Helicity poles (or other singularities) govern the large-$\alpha$ behavior of the amplitude with all cluster variables and $\xi$ fixed; simultaneous helicity and angular momentum poles govern large $\alpha$ and $\xi$, with cluster variables fixed. We relate these asymptotic variables to conventional invariants by noting that, for large negative $\alpha$,

$$s=(P_b+P_{a1})^2 \underset{\alpha\to-\infty}{\sim} C_1 e^{-\alpha}\cosh\xi, \tag{3.3}$$

$$s'=(P_b'+P_{a1}')^2 \underset{\alpha\to-\infty}{\sim} C_2 e^{-\alpha}\cosh\xi, \tag{3.4}$$

where $C_1$ and $C_2$ are fixed, in general different, combinations of cluster variables. The invariant mass squared to the three-body system,

$$M^2=(P_b+P_{a1}-P_{a2})^2=(P_b+P_a)^2=(P_b'+P_a')^2,$$

is independent of $\alpha$, and is a linear function of

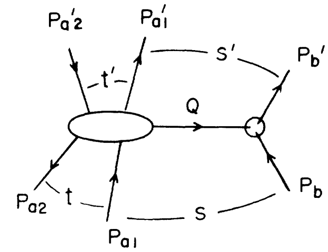


FIG. 9. Amplitude used to locate helicity singularites.

$\sinh\xi$ with cluster-variable coefficients.

Our task is thus to identify the behavior of the $3\to 3$ amplitude $A$ at fixed $M^2$, fixed cluster variables, and large values of $s$ and $s'$.

The three-body amplitude itself has a most complicated singularity structure. However, a special object of some importance is the discontinuity with respect to the variable $M^2$ of the amplitude, $\text{disc}_{M^2}A$. As shown by Mueller,[8] this may be represented by the unitarity diagram of Fig. 10, which in the forward direction ($P_{a1}=P'_{a1}$, $P_{a2}=P'_{a2}$, $P_b=P'_b$) becomes proportional to the inclusive cross section for the process $a_1+b\to a_2+\text{anything}$. For general momenta, the asymptotic behavior of the discontinuity is governed by the leading Regge pole $\gamma(t)$ in the amplitude

$$a_1+b\to a_2+\text{a state of mass } M \tag{3.5}$$

and the leading Regge pole $\gamma'(t)$ in the complex-conjugated amplitude

$$a_1'+b'\to a_2'+\text{the same state of mass } M\,; \tag{3.6}$$

the first of these is associated with the lower half of the diagram in Fig. 10, the second with the upper half. The product is then summed over all states of mass $M$ that are compatible with the quantum numbers of $a_1$, $a_2$, and $b$. The leading term is thus

$$\text{disc}_{M^2}A \underset{s,s'\to\infty}{\sim} s^{\gamma(t)}(s')^{\gamma'(t')}F(M^2,\text{cluster variables}) \tag{3.7}$$

or, in terms of the O(2, 1) variable $\alpha$,

$$\text{disc}_{M^2}A \underset{\alpha\to+\infty}{\sim} Fe^{-\alpha[\gamma(t)+\gamma'(t')]}[1+O(e^{\alpha})+\cdots]\,. \tag{3.8}$$

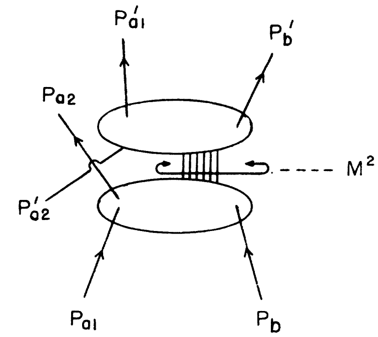


FIG. 10. Unitarity diagram for $M^2$ discontinuity.

We see from (3.8) that the leading helicity pole[9] in $\text{disc}_{M^2}A$ is at $m=\gamma(t)+\gamma'(t')$, and that there are integrally spaced subsidiary poles from $\gamma(t)+\gamma'(t')$ to $-\infty$. This set of helicity poles is associated with the function $A_1(z)$ of Eq. (2.12). The function $A_2$ will evidently have an equivalent set, going from $-\gamma(t)-\gamma'(t')$ to $+\infty$ with integer spacing. When $t$ and $t'$ are continued to timelike values for which $\gamma(t)$ and $\gamma'(t')$ take on integral values (associated with particles lying on the trajectories), the residues will vanish for $|m|>\gamma(t)+\gamma'(t')$. This is evidently required on physical grounds; mathematically, it emerges from our analysis by noting that for integral $\gamma$ and $\gamma'$ the asymptotic expansion (3.8) of the function $P_\alpha(z)P_{\alpha'}(z')$ produces a polynomial in $z$ and $z'$ with powers ranging from $z_t{}^{\gamma}z_{t'}{}^{\gamma'}$ to zero. Here $z_t$ and $z'_{t'}$ are, respectively, linear functions of $s$ and $s'$ associated with the crossed-channel analysis of the processes (3.5) and (3.6).

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