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Relativistic Two-Variable Expansions for Three-Body Decay Amplitudes

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Three-body decays $1 \to 2+3+4$ are considered using a frame of reference analogous to the c.m. system for scattering. The physical decay region is mapped onto an O(4) sphere, so that the decay amplitude $f(\alpha, \theta)$ is a function on this sphere (depending only on two of the three angles α, θ, and ϕ). The amplitude is then expanded in terms of the basis functions of O(4) and we obtain two-variable expansions, in which all the dependence on the kinematic parameters is explicitly displayed in special functions. These expansions make it possible to treat decays and scattering on the same footing, in that they are intimately related to O(3, 1) expansions of scattering amplitudes, considered previously. Some analyticity properties are built in, so that each partial wave has the correct behavior at the threshold $(m_3+m_4)^2$ and pseudothreshold $(m_1-m_2)^2$. The expansions make it possible to perform a kinematically (or group-theoretically) motivated harmonic analysis of Dalitz-plot distributions for $K\to 3\pi$ and $\eta \to 3\pi$ decays (the results will be presented separately).

I. INTRODUCTION

The purpose of this paper is to suggest a new method of treating general decay processes of the type

$$1 \to 2+3+4 \quad (1)$$

by making use of certain two-variable expansions, based on the representation theory of the group O(4). The method is an extension of one developed in a series of previous articles mainly for two-body scattering.[1-8] The aim of the method, both for scattering and decays, is to present a reaction theory in terms of explicit two-variable (or, more generally, multivariable) expansions of the corresponding amplitudes. All the dependence on the kinematic parameters (energies, angles, momentum transfers, etc.) is displayed explicitly in known functions, whereas all the dynamics are transferred to the expansion coefficients. The method can thus be considered to be a generalization of standard partial-wave analysis, where the dependence on one variable only, the scattering angle, is explicit. Since this program is quite general, it must be based only on very general and well-established principles of scattering theory, like Lorentz invariance, analyticity, crossing, unitarity, etc.

The method used to generate two-variable expansions for two-body scattering of spinless particles was the following. The scattering amplitude $f(s,t)$ is considered as a function $M(p_1,p_2,p_3,p_4)$ of all four particle momenta, subject to the restrictions imposed by the conservation laws, mass-shell conditions, and Lorentz invariance. It is then possible to choose a convenient frame of reference by fixing the momenta of three of the particles, after which the amplitude can be considered to be a function of the one remaining four-momentum only. In the case of zero-spin particles, only two of the three independent coordinates of the chosen momentum p are essential, so the scattering amplitude is a function of two variables only, as it should be. However, these variables are now interpreted as coordinates of the momentum p; thus the amplitude $f(s,t)$ is a function of a point on the hyperboloid $p^2=m^2$ (the mass shell). If the coordinates are chosen correctly, then the point p runs over the entire upper sheet of the hyperboloid $p^2=m^2$, as the Mandelstam variables s and t run through the physical region of a given channel. Given a function $f(p)$ of a point on a homogeneous manifold, it is a simple matter to expand it in terms of the corresponding spherical functions, i.e., in terms of the basis functions of the irreducible representations of the group of motions of the manifold. In our case the group of motions is the homogeneous Lorentz group O(3, 1) and the described method provides the required two-variable expansions.

When actually deriving the expansions we encounter the following ambiguities: (1) We can choose an arbitrary ("convenient") frame of reference. (2) We must choose coordinates on the hyperboloid. (3) We must choose a specific basis for the group representations. These three choices are interrelated and the solution of the problem becomes essentially unique if we add some physical requirements. Thus, if we wish to make full use of Lorentz invariance by incorporating all the single-variable expansions appearing in the literature as "little-group expansions,"[9-11] we are led to three quite definite two-variable expansions, corresponding to the reduction of O(3, 1) to different subgroups G in the reduction chain $O(3,1)\supset G \supset O(2)$. If we wish to incorporate the O(3) little-group expansion of standard partial-wave analysis, we choose O(3) to play the role both of the subgroup G in the reduction (which determines the choice of a basis and the choice of coordinates on the hyperboloid[1-3, 7, 8]), and of the little-group of a timelike vector, namely, the total energy-momentum p_1+p_2, the standardization of which $[p_1+p_2=(\sqrt{s},0,0,0)]$ determines the center-of-mass system. Similarly, the O(2, 1) expansion of Regge-pole theory is obtained by identifying the subgroup G with O(2, 1) and letting this O(2, 1) also figure as the little group of a spacelike vector – the momentum transfer, the standardization of which $[p_1-p_3=(0,0,0,\sqrt{-t})]$ leads to the brick-wall frame of reference. In both cases we obtain a standard little-group expansion in one variable and the group O(3, 1) supplies a further expansion of the corresponding partial-wave amplitude.

Similarly, choosing the group G to be the Euclidean group E_2 we obtain the E_2 little-group expansion for lightlike momentum transfer ($t=0$ in non-equal-mass reactions) and a generalization of this expansion for $t\neq 0$, supplemented by an integral representation of the corresponding partial-wave amplitude.

The motivation for the whole approach using two-variable expansions is to achieve a greater degree of separation between kinematics and dynamics than is provided by the little-group expansions. This should enable us to make dynamical assumptions about the behavior of the expansion coefficients, leading to "predictions" for high-energy behavior, angular distributions, etc. Further, the separation should enable us to impose some of the general principles of scattering theory in a general manner ("as kinematics"[5, 6]). Finally, a successful application of the two-variable expansions would make it possible to perform model-independent phenomenological fits to experimental data of a more extensive character than using only the single-variable expansions. Thus, the expansions based on the $O(3,1)\supset O(3)\supset O(2)$ chain are suitable for performing energy-dependent phase-shift analysis, i.e., describing angular distributions at different energies in terms of one set of parameters. The $O(3,1)\supset O(2,1)\supset O(2)$ expansions, on the other hand, are suitable for supplementing the complex-angular-momentum description of high-energy scattering by an explicit parametrization of the momentum-transfer dependence of the amplitudes.

In order to actually perform such phenomenological fits, it is still necessary to overcome certain difficulties related to the noncompactness of the group O(3, 1). To avoid these problems we first consider a problem that is mathematically simpler and is of separate physical interest, namely, two-variable expansions of decay amplitudes. The big simplification is due to the fact that the physical decay region is finite, so that it can be mapped onto a manifold homogeneous with respect to a compact group, namely, O(4). Since this group is compact, the two-variable expansions will involve only sums and no integrals, which makes the actual treatment of data a simple matter.

In Sec. II we consider the kinematics of three-body decays and obtain the decay amplitude as a function on an O(4) sphere. In Sec. III we discuss some relevant results on the representation theory of O(4) (most, if not all, of which are contained in the literature). In Sec. IV we present and discuss the two-variable expansions and compare them with other treatments of three-body decays. In Sec. V we discuss their possible applications to processes like $K\to 3\pi$ or $\eta\to 3\pi$ decays.

An actual application of the suggested formalism to $K\to 3\pi$ decays has been performed with encouraging results and will be published separately.

Let us note here that two-variable expansions inside the Mandelstam triangle have been considered previously, not, however, for actual decay amplitudes, but for scattering amplitudes in a nonphysical region (when all four masses $m_1=m_2=m_3=m_4$ are equal).[4, 12, 13]

II. THREE-BODY DECAY AMPLITUDE AS A FUNCTION ON A SPHERE

Consider the decay $1\to 2+3+4$, where all particles have zero spin and let the particle momenta and masses satisfy

$$p_1+p_2+p_3+p_4=0,\quad m_1\geqslant m_2+m_3+m_4, \tag{2}$$

and the mass-shell condition $p_i^2=p_{i0}^2-\vec{p}_i^{\,2}=m_i^2$ for all four particles ($i=1,\ldots,4$).

The Mandelstam variables

$$s=(p_1+p_2)^2,\quad t=(p_1+p_3)^2,\quad u=(p_1+p_4)^2, \tag{3}$$

in the decay region satisfy[14]

$$\begin{aligned} &stu-K^2(as+bt+cu)\geqslant 0,\\ &K^2=m_1^2+m_2^2+m_3^2+m_4^2=s+t+u, \end{aligned} \tag{4}$$

with

$$K^3a=(m_1^2m_2^2-m_3^2m_4^2)(m_1^2+m_2^2-m_3^2-m_4^2),$$
$$K^3b=(m_1^2m_3^2-m_2^2m_4^2)(m_1^2-m_2^2+m_3^2-m_4^2),$$
$$K^3c=(m_1^2m_4^2-m_2^2m_3^2)(m_1^2-m_2^2-m_3^2+m_4^2),$$

and

$$(m_3+m_4)^2\leqslant s\leqslant(m_1-m_2)^2,$$
$$(m_2+m_4)^2\leqslant t\leqslant(m_1-m_3)^2, \qquad (5)$$
$$(m_2+m_3)^2\leqslant u\leqslant(m_1-m_4)^2.$$

In order to apply the same approach as for particle scattering we introduce a center-of-mass-like frame of reference for the decay (see Fig. 1). In this frame particle 1 decays in flight, particle 2 emerges with momentum $\vec{p}_2=\vec{p}_1$, and particles 3 and 4 emerge with $\vec{p}_4=-\vec{p}_3$ and $|\vec{p}_3|$ such that energy is conserved. Using spherical coordinates, choosing 0_{13} as the scattering plane and putting $\vec{p}_3=-\vec{p}_4\|0_3$, we can write the individual momenta as

$$p_1=m_1(\cosh a_1,\sinh a_1\sin\theta,0,\sinh a_1\cos\theta),$$
$$p_2=-m_2(\cosh a_2,\sinh a_2\sin\theta,0,\sinh a_2\cos\theta),$$
$$p_3=-m_3(\cosh a_3,0,0,\sinh a_3), \qquad (6)$$
$$p_4=-m_4(\cosh a_4,0,0,-\sinh a_4).$$

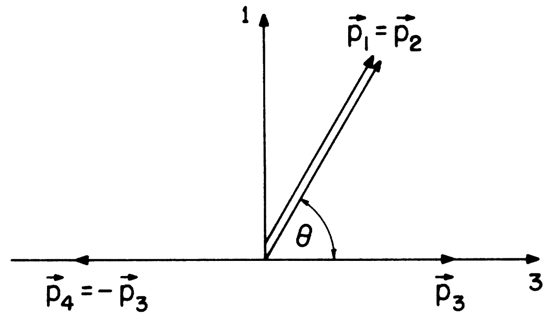

FIG. 1. The c.m.-like frame of reference for decays; $|\vec{p}_1|=|\vec{p}_2|=m_1\sinh a_1=m_2\sinh a_2$; $|\vec{p}_3|=|\vec{p}_4|=m_3\sinh a_3=m_4\sinh a_4$.

The conservation laws imply:

$$m_1\cosh a_1=m_2\cosh a_2+m_3\cosh a_3+m_4\cosh a_4,$$
$$m_1\sinh a_1=m_2\sinh a_2, \qquad (7)$$
$$m_3\sinh a_3=m_4\sinh a_4.$$

Choosing $a_1\equiv a$, and θ as independent variables, (7) can easily be solved to express a_2, a_3, and a_4 in terms of a. The decay amplitude is then $F(s,t)=M(p_1,p_2,p_3,p_4)=f(a,\theta)$, i.e., a function of the coordinates of one single momentum p_1.

In terms of the Mandelstam variables (3) we can readily obtain

$$\zeta\equiv\cosh a=\frac{s+m_1^2-m_2^2}{2m_1\sqrt{s}} \qquad (8)$$

$$z\equiv\cos\theta=\frac{2s(t-m_1^2-m_3^2)+(s+m_1^2-m_2^2)(s+m_3^2-m_4^2)}{\{[-s+(m_1+m_2)^2][-s+(m_1-m_2)^2][-s+(m_3+m_4)^2][-s+(m_3-m_4)^2]\}^{1/2}}\,. \qquad (9)$$

Formulas (8) and (9) are precisely the same kinematic relations that determine the variables used previously when treating scattering in the c.m.[3-8]. For scattering, the range of the variables s and t is such that $1\leqslant\zeta<\infty$, $-1\leqslant z\leqslant1$, i.e., $0\leqslant a<\infty$, $0\leqslant\theta\leqslant\pi$ so that the point p_1 covers the whole hyperboloid $p_1^2=m_1^2$ (if we add a cyclic angle ϕ). For a particle decay, however, s, t, and u are in a finite region, restricted by the boundary (4) and satisfying (5). The range of the new variables for a decay is

$$1\leqslant\zeta\leqslant\frac{(m_3+m_4)^2+m_1^2-m_2^2}{2m_1(m_3+m_4)},\quad -1\leqslant z\leqslant1, \qquad (10)$$

so that

$$0\leqslant a\leqslant a_{\max},\quad 0\leqslant\theta\leqslant\pi.$$

Thus, the physical decay region does not get mapped onto an entire hyperboloid, but only onto a finite "cup" close to the vertex of the hyperboloid $p_1^2=m_1^2$. Such a region is certainly not convenient for writing expansions in terms of the representations of O(3,1), since this group acts transitively on the whole upper sheet. Besides, one point on the boundary of the physical decay region, the singular point $s=(m_3+m_4)^2$, where $\cos\theta$ changes from $+1$ to -1, gets mapped onto an entire circle on the hyperboloid $a=a_{\max}$, $0\leqslant\theta\leqslant\pi$. If such a mapping is used, analyticity properties of $f(s,t)$ must be imposed artificially, this leading to various difficulties, like kinematical constraints on partial-wave amplitudes. To get rid of these difficulties we make use of the finiteness of the decay region to perform a further mapping of the physical region from the hyperboloid onto a sphere, this being similar to going into the Euclidean region of momentum space. The simplest way of doing this is simply to construct a parallel mapping of our section of the hyperboloid onto a hemisphere having the same radius as the maximal circle on the hyperboloid (see Fig. 2). This can be done by performing the mapping parallel to the axis of the hyperboloid onto a hemisphere with its center at (0, 0, 0, 0). Instead of momentum space let us consider relativistic velocity space, defining relativistic velocity as $v = p/m$. Thus, in velocity space we have

$$v = (\cosh a, \sinh a \sin\theta \cos\phi, \sinh a \sin\theta \sin\phi, \sinh a \cos\theta),$$
$$0 \leq a \leq a_{\max}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \tag{11}$$

where $\cosh a$ and $\cos\theta$ are given by relations (8) and (9). We project the point v on the hyperboloid onto point v_s on the sphere ("Euclidean velocity space"), satisfying

$$v_s = R(\cos\beta, \sin\beta \sin\theta \cos\phi, \sin\beta \sin\theta \sin\phi, \sin\beta \cos\theta),$$
$$0 \leq \beta \leq \tfrac{1}{2}\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \tag{12}$$

Since the mapping is parallel to the timelike axis the angles θ and ϕ in (11) and (12) are the same and we have

$$R \sin\beta = \sinh a. \tag{13}$$

In particular the radius of the sphere is

$$R = \sinh a_{\max} = \frac{\{[(m_1 + m_2)^2 - (m_3 + m_4)^2][(m_1 - m_2)^2 - (m_3 + m_4)^2]\}^{1/2}}{2m_1(m_3 + m_4)}. \tag{14}$$

Using (8), (13), and (14) we finally obtain the coordinates of point v_s on the sphere as $\cos\theta$, given by (9), and

$$\cos\beta = \left(\frac{s[m_1^2 - m_2^2 + (m_3 + m_4)^2]^2 - (m_3 + m_4)^2(s + m_1^2 - m_2^2)^2}{s[(m_1 + m_2)^2 - (m_3 + m_4)^2][(m_1 - m_2)^2 - (m_3 + m_4)^2]}\right)^{1/2}. \tag{15}$$

However, the singular point $s = (m_3 + m_4)^2$ is now mapped onto the equator of the sphere and this still leads to difficulties. We prefer to map both singular points in s, namely the "threshold" $s = (m_3 + m_4)^2$ and "pseudothreshold" $s = (m_1 - m_2)^2$, into the two poles of the O(4) sphere. This is easily achieved by introducing a new variable

$$\alpha = 2\beta, \tag{16}$$

given in terms of the Mandelstam variable s as

$$\cos\alpha = 1 - \frac{[(m_1 + m_2)^2 - s][(m_1 - m_2)^2 - s]}{2m_1^2 R^2 s}, \tag{17}$$

with $0 \leq \alpha \leq \pi$.

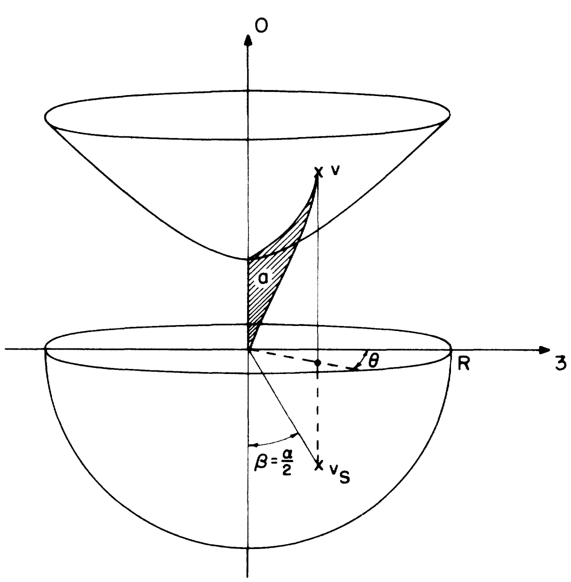

FIG. 2. Mapping from hyperboloid onto sphere. $R = \sinh a_{\max}$; $\sinh a = R \sin\frac{1}{2}\alpha$; $0 \leq a \leq a_{\max}$, $0 \leq \alpha \leq \pi$, $0 \leq \theta \leq \pi$.

To each point on the Dalitz plot for the decay (1) we can ascribe coordinates α and θ, given by expressions (9) and (17) (the coordinate ϕ determines the position of the decay plane and the amplitude cannot depend on it). From the Dalitz plot we can reconstruct the decay amplitude $f(\alpha, \theta)$. This is now a function defined on an entire sphere and it can be expanded in terms of the irreducible representations of the group O(4), rather than of the homogeneous Lorentz group O(3, 1), which provides expansions of scattering amplitudes. The O(4) expansions are given in Sec. IV and we shall see that the correct threshold behavior, occurring in the new variables at $\alpha = 0$ and $\alpha = \pi$, is ensured automatically by our choice of variables.

III. SOME RESULTS ON THE REPRESENTATIONS OF O(4)

The group O(4) has received extensive treatment in the literature.[15–17] To define our notation and conventions we shall give a few results relevant for our purposes. Readers who are only interested in the resulting expansion may wish to skip to Sec. IV.

A. The Algebra and its Invariants

Using the notations

$$L_i = \tfrac{1}{2}\epsilon_{ikl}M_{kl}, \quad A_i = M_{i4}, \tag{18}$$

for the six generators of O(4), we can write the commutation relations as

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, A_j] = i\epsilon_{ijk}A_k,$$
$$[A_i, A_j] = i\epsilon_{ijk}L_k.$$

The local isomorphism between O(4) and O(3)×O(3) is demonstrated by putting

$$J_i = \tfrac{1}{2}(L_i + A_i), \quad \tilde{J}_i = \tfrac{1}{2}(L_i - A_i), \tag{19}$$

yielding

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [\tilde{J}_i, \tilde{J}_j] = i\epsilon_{ijk}\tilde{J}_k,$$
$$[J_i, \tilde{J}_k] = 0.$$

The two invariant operators (Casimir operators) of O(4) can be written as

$$\begin{aligned} I_1 &= \vec{L}^2 + \vec{A}^2 = 2(\vec{J}^2 + \vec{\tilde{J}}^2),\\ I_2 &= \vec{L}\cdot\vec{A} = \vec{J}^2 - \vec{\tilde{J}}^2, \end{aligned} \tag{20}$$

and their eigenvalues can be written as

$$\begin{aligned} I_1 &= 2[j(j+1) + \tilde{j}(\tilde{j}+1)] \equiv n(n+2) + \nu^2,\\ I_2 &= j(j+1) - \tilde{j}(\tilde{j}+1) \equiv (n+1)\nu, \end{aligned} \tag{21}$$

so that

$$n = j + \tilde{j}, \quad \nu = j - \tilde{j}. \tag{22}$$

Since j and $\tilde{j}$ can only be positive integers or half-integers, we obtain that n and ν are integer or half-integer and

$$n \geq |\nu|. \tag{23}$$

Among the different possible parametrizations of a group element two are particularly useful. One corresponds to the O(3)×O(3)⊃O(2)×O(2) reduction of O(4):

$$g = e^{i\alpha_1 J_3} e^{i\alpha_2 J_2} e^{i\alpha_3 J_3} e^{i\alpha_4 \tilde{J}_3} e^{i\alpha_5 \tilde{J}_2} e^{i\alpha_6 \tilde{J}_3}, \tag{24}$$

the other corresponds to the O(4)⊃O(3)⊃O(2) reduction

$$g = e^{i\psi_1 L_3} e^{i\theta_1 L_2} e^{i\phi_1 L_3} e^{i\alpha A_3} e^{i\theta_2 L_2} e^{i\phi_2 L_3} \tag{25}$$

and is particularly suitable for our expansions, being intimately related to the O(3,1)⊃O(3)⊃O(2) reduction of the homogeneous Lorentz group.

B. The Basis Functions for Irreducible Representations

The group O(4) is compact; thus all its irreducible representations are finite-dimensional, unitary, and can be labeled by the two numbers ν and n.

We shall make use of two different sets of basis functions, corresponding, respectively, to the reductions O(3)× O(3)⊃O(2)×O(2) and O(4)⊃O(3) ⊃O(2).

The first set will simply consist of products of basis functions of O(3) and can be written as

$$|jm\tilde{j}\tilde{m}\rangle = Y_{jm}(\theta,\phi)Y_{\tilde{j}\tilde{m}}(\tilde{\theta},\tilde{\phi}). \tag{26}$$

The O(4)⊃O(3)⊃O(2) basis functions are eigenfunctions of the operators I_1, I_2, $\vec{L}^2$, and L_3 and can be defined to be

$$|\nu nLM\rangle = \sum_{m,\tilde{m}} (jm\tilde{j}\tilde{m}|LM)|jm\tilde{j}\tilde{m}\rangle, \tag{27}$$

where $(jm\tilde{j}\tilde{m}|LM)$ are Clebsch-Gordan coefficients of O(3). Note that the definition (27) not only ensures the proper transformation properties of $|\nu nLM\rangle$ but also contains a phase convention.

Indeed, the phase convention in (26) is such that the matrix elements of all the generators J_i and $\tilde{J}_i$ are real and satisfy

$$H_\mu|jm\tilde{j}\tilde{m}\rangle = [j(j+1)]^{1/2}(jm1\mu|jm+\mu)|jm+\mu\tilde{j}\tilde{m}\rangle, \tag{28}$$
$$\tilde{H}_\mu|jm\tilde{j}\tilde{m}\rangle = [\tilde{j}(\tilde{j}+1)]^{1/2}(\tilde{j}\tilde{m}1\mu|\tilde{j}\tilde{m}+\mu)|jm\tilde{j}\tilde{m}+\mu\rangle,$$

where

$$\begin{aligned} H_0 &= J_3, \quad H_{\pm 1} = \mp(\tfrac{1}{2})^{1/2}(J_1 \pm iJ_2),\\ \tilde{H}_0 &= \tilde{J}_3, \quad \tilde{H}_{\pm 1} = \mp(\tfrac{1}{2})^{1/2}(\tilde{J}_1 \pm i\tilde{J}_2). \end{aligned} \tag{29}$$

Putting

$$L_\mu = H_\mu + \tilde{H}_\mu, \quad A_\mu = H_\mu - \tilde{H}_\mu, \tag{30}$$

and using (27) and (28), we can use standard angular momentum theory to obtain

$$L_\mu|\nu nLM\rangle = [L(L+1)]^{1/2}(LM1\mu|LM+\mu)|\nu nLM+\mu\rangle,$$

$$A_\mu|\nu nLM\rangle \tag{31}$$
$$= \sum_k (-1)^k R^{n\nu}_{L+k}(LM1\mu|L+kM+\mu)|\nu nL+kM+\mu\rangle$$

with

$$\begin{aligned} R^{n\nu}_{L-1} &= \left(\frac{[(n+1)^2 - L^2](L^2-\nu^2)}{L(2L-1)}\right)^{1/2},\\ R^{n\nu}_{L} &= \frac{(n+1)\nu}{[L(L+1)]^{1/2}},\\ R^{n\nu}_{L+1} &= -\left(\frac{[(n+1)^2-(L+1)^2][(L+1)^2-\nu^2]}{(L+1)(2L+3)}\right)^{1/2}. \end{aligned} \tag{32}$$

Thus, in the "canonical" basis (27) the phase convention is such that all matrix elements of L_μ and A_μ are real.

We are actually interested in the O(4) spherical harmonics, i.e., in basis functions $|\nu nLM\rangle$ realized as functions on an O(4) sphere.

Let us introduce spherical coordinates, putting

$$x_1 = \sin\alpha \sin\theta \cos\phi, \quad x_3 = \sin\alpha \cos\theta,$$
$$x_2 = \sin\alpha \sin\theta \sin\phi, \quad x_4 = \cos\alpha. \tag{33}$$

Consider the quasiregular representation[18]

$$g \to T_g, \quad T_g f(x) = f(g^{-1}x), \tag{34}$$

where $f(x)$ belongs to the Hilbert space of functions over the sphere, satisfying

$$\int |f(\alpha, \theta, \phi)|^2 \sin^2\alpha \sin\theta \, d\alpha d\theta d\phi < \infty. \tag{35}$$

It is a simple matter to find the generators of O(4) as differential operators on the sphere:

$$\begin{aligned}
L_1 &= i\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi, \frac{\partial}{\partial\phi}\right),\\
L_2 &= i\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi}\right),\\
L_3 &= -i\frac{\partial}{\partial\phi},\\
A_1 &= i\left(\sin\theta\cos\phi \frac{\partial}{\partial\alpha} + \cot\alpha\cos\theta\cos\phi \frac{\partial}{\partial\theta} - \cot\alpha \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\
A_2 &= i\left(\sin\theta\sin\phi \frac{\partial}{\partial\alpha} + \cot\alpha\cos\theta\sin\phi \frac{\partial}{\partial\theta} + \cot\alpha \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\
A_3 &= i\left(\cos\theta \frac{\partial}{\partial\alpha} - \cot\alpha\sin\theta\frac{\partial}{\partial\theta}\right).
\end{aligned} \tag{36}$$

Writing down the operators I_1, L^2, and L_3 as differential operators, noticing that $I_2 = 0$ identically, so that the only representations realized in this space are those with $\nu = 0$, and solving the equations

$$\begin{aligned}
I_1\Phi_{NLM}(\alpha, \theta, \phi) &= n(n+2)\Phi_{NLM}(\alpha, \theta, \phi),\\
L^2\Phi_{NLM}(\alpha, \theta, \phi) &= L(L+1)\Phi_{NLM}(\alpha, \theta, \phi),\\
L_3\Phi_{NLM}(\alpha, \theta, \phi) &= M\Phi_{NLM}(\alpha, \theta, \phi),
\end{aligned} \tag{37}$$

we can check that the normalized spherical functions satisfying (31) are

$$\begin{aligned}
|0NLM\rangle = \Phi_{NLM}(\alpha, \theta, \phi) &= e^{-i(\pi/2)L}\frac{2^{L+1/2}\Gamma(L+1)}{2\pi}\left[(2L+1)\frac{(L-M)!}{(L+M)!}\frac{(N+1)\Gamma(N-L+1)}{\Gamma(N+L+2)}\right]^{1/2}\\
&\quad\times(\sin\alpha)^L C_{N-L}^{L+1}(\cos\alpha)P_L^M(\cos\theta)e^{iM\phi}.
\end{aligned} \tag{38}$$

Here $P_L^M(\cos\theta)$ and $C_{N-L}^{L+1}(\cos\alpha)$ are Legendre and Gegenbauer polynomials, respectively. An equivalent form, more similar to the one figuring in O(3, 1) expansions, is

$$|0NLM\rangle = \Phi_{NLM}(\alpha, \theta, \phi) = e^{-i(\pi/2)L}\left[\frac{2L+1}{4\pi}\frac{(L-M)!}{(L+M)!}\frac{\Gamma(N+L+2)(N+1)}{\Gamma(N-L+1)}\right]^{1/2}\frac{1}{(\sin\alpha)^{1/2}}P_{1/2+N}^{-L-1/2}(\cos\alpha)P_L^M(\cos\theta)e^{im\phi}. \tag{39}$$

C. Spherical Functions and Finite - Transformation Matrices

Making use of the local isomorphism between O(4) and O(3)×O(3), it is quite simple to obtain the matrix elements of finite transformations, i.e., the O(4) D functions. We do not need their explicit form but only the relation between the D functions and the spherical functions (38). Consider again the quasiregular representation (34). By definition, we have

$$\Phi_{NLM}(g^{-1}x) = T_g\Phi_{NLM}(x) = \sum_{L'M'} D^{0N}_{L'M',LM}(g)\Phi_{NL'M'}(x). \tag{40}$$

Consider the point $x_0 = (0, 0, 0, 1)$, i.e., put $\alpha = 0$ in (33). The function $\Phi_{NLM}(0, \theta, \phi)$ cannot depend on θ and ϕ, since these angles have no meaning for $\alpha = 0$. Indeed we have

$$\Phi_{NLM}(0,\theta,\phi)=\delta_{L0}\delta_{M0}\frac{N+1}{\sqrt{2}\,\pi}\,. \tag{41}$$

Substituting (41) into (40) we have

$$\Phi_{NLM}(g^{-1}x_0)=\frac{N+1}{\sqrt{2}\,\pi}\,D^{0N}_{00,LM}(g).$$

Putting

$$g=e^{iL_3\rho_1}e^{iL_2\rho_2}e^{iL_3\rho_3}e^{iA_3\alpha}e^{iL_2\theta}e^{iL_3(\pi-\phi)}\,, \tag{42}$$

with ρ_1, ρ_2, and ρ_3 arbitrary, we have

$$\Phi_{NLM}(\alpha,\theta,\phi)=\frac{N+1}{\sqrt{2}\,\pi}\,D^{0N}_{00,LM}(g). \tag{43}$$

D. The Clebsch - Gordan Series

Let us define the Clebsch-Gordan coefficients for O(4) by the relations

$$|\nu_1 n_1 L_1 M_1\rangle|\nu_2 n_2 L_2 M_2\rangle=\sum_{\nu,n,L,M}\begin{pmatrix}\nu_1 n_1 & \nu_2 n_2 & \nu n\\ L_1 M_1 & L_2 M_2 & LM\end{pmatrix}|\nu n LM\rangle, \tag{44}$$

$$|\nu n LM\rangle=\sum_{L_1,M_1,L_2,M_2}\begin{pmatrix}\nu_1 n_1 & \nu_2 n_2 & \nu n\\ L_1 M_1 & L_2 M_2 & LM\end{pmatrix}|\nu_1 n_1 L_1 M_1\rangle|\nu_2 n_2 L_2 M_2\rangle. \tag{45}$$

Our notations are such as to stress the analogy with the O(3, 1) case.[19] Equation (27) and its inverse make it possible to express the O(4) Clebsch-Gordan coefficients in terms of purely O(3) quantities and we obtain

$$\begin{pmatrix}\nu_1 n_1 & \nu_2 n_2 & \nu n\\ L_1 M_1 & L_2 M_2 & LM\end{pmatrix}=[(2L_1+1)(2L_2+1)(n+\nu+1)(n-\nu+1)]^{1/2}(L_1M_1L_2M_2|LM)\begin{Bmatrix}\frac{1}{2}(n_1+\nu_1) & \frac{1}{2}(n_1-\nu_1) & L_1\\ \frac{1}{2}(n_2+\nu_2) & \frac{1}{2}(n_2-\nu_2) & L_2\\ \frac{1}{2}(n+\nu) & \frac{1}{2}(n-\nu) & L\end{Bmatrix}, \tag{46}$$

where the curly brackets denote a 9-J symbol (for definitions and properties see, e.g., Ref. 20).

Let us note that $\nu_1=\nu_2=0$ does not imply $\nu=0$ in (46), i.e., the direct product of two "degenerate" representations with $\nu_i=0$ will contain general "nondegenerate" representations. However, a special case of interest when $\nu_1=\nu_2=0$ does imply $\nu=0$ is

$$\begin{pmatrix}0n_1 & 0n_2 & \nu n\\ 00 & 00 & LM\end{pmatrix}=\delta_{\nu 0}\delta_{L0}\delta_{M0}\left[\frac{n+1}{(n_1+1)(n_2+1)}\right]^{1/2}. \tag{47}$$

Performing a general O(4) transformation on both sides of (45) and making use of the orthogonality relations for the basis functions, we obtain

$$D^{\nu_1 n_1}_{L_1'M_1',L_1M_1}(g)\,D^{\nu_2 n_2}_{L_2'M_2',L_2M_2}(g)=\sum_{\nu n LML'M'}\begin{pmatrix}\nu_1 n_1 & \nu_2 n_2 & \nu n\\ L_1 M_1 & L_2 M_2 & LM\end{pmatrix}\begin{pmatrix}\nu_1 n_1 & \nu_2 n_2 & \nu n\\ L_1' M_1' & L_2' M_2' & L'M'\end{pmatrix}D^{\nu n}_{L'M',LM}(g). \tag{48}$$

Using (43), (47), and (48) we obtain an expansion formula for products of spherical functions (evaluated at the same point):

$$\Phi_{nlm}(\alpha,\theta,\phi)\Phi_{n'l'm'}(\alpha,\theta,\phi)$$

$$=\frac{1}{\sqrt{2}\,\pi}[(n+1)(n'+1)(2l+1)(2l'+1)]^{1/2}\sum_{NLM}(N+1)^{1/2}(lml'm'|LM)\begin{Bmatrix}\frac{1}{2}n & \frac{1}{2}n & l\\ \frac{1}{2}n' & \frac{1}{2}n' & l'\\ \frac{1}{2}N & \frac{1}{2}N & L\end{Bmatrix}\Phi_{NLM}(\alpha,\theta,\phi). \tag{49}$$

Note that the 9-J symbol in (49) is equal to zero unless $n+n'+N+l+l'+L$ is even.

IV. TWO - VARIABLE EXPANSIONS OF DECAY AMPLITUDES

Let us now consider the decay amplitude $f(s,t)$ as a function $f(\alpha,\theta)$ of a point on an O(4) sphere, independent of the azithumal angle ϕ. Expanding in terms of the functions (38) satisfying the normalization condition

$$\int_0^\pi \sin^2\alpha\,d\alpha \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\,\Phi_{nlm}(\alpha,\theta,\phi)\Phi^*_{n'l'm'}(\alpha,\theta,\phi) = \delta_{nn'}\delta_{ll'}\delta_{mm'},$$

we obtain

$$f(\alpha,\theta) = \sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{nl}\,e^{-i(\pi/2)l}\,\frac{2^{l+1/2}\Gamma(l+1)}{2\pi}\left[(2l+1)\frac{(n+1)\Gamma(n-l+1)}{\Gamma(n+l+2)}\right]^{1/2}(\sin\alpha)^l C^{l+1}_{n-l}(\cos\alpha)P_l(\cos\theta), \tag{50}$$

$$a_{nl} = e^{i(\pi/2)l}\,2^{l+1/2}\Gamma(l+1)\left[(2l+1)\frac{(n+1)\Gamma(n-l+1)}{\Gamma(n+l+2)}\right]^{1/2}\int_0^\pi \sin^2\alpha\,d\alpha\int_0^\pi \sin\theta\,d\theta f(\alpha,\theta)(\sin\alpha)^l C^{l+1}_{n-l}(\cos\alpha)P_l(\cos\theta). \tag{51}$$

If particles 3 and 4 are identical we must have

$$f(s,t,u) = f(s,u,t), \quad \text{i.e.,} \quad f(\alpha,\theta) = f(\alpha,\pi-\theta), \tag{52}$$

so that $a_{nl}=0$ for odd values of l. The same symmetry (52) results from C invariance in $\eta\to 3\pi$ decay where the π^0 is particle 2. For kaons, CP conservation and the $\Delta I=\frac{1}{2}$ rule also relate the amplitudes for various decays (see Ref. 21).

If particles 2, 3, and 4 are all identical (e.g., $K^0\to\pi^0\pi^0\pi^0$) then $f(s,t,u)$ must be invariant under arbitrary permutations of s, t, and u. This is somewhat more difficult to ensure; however, use can be made of the O(4) partial-wave crossing matrices derived in a previous publication.[4]

Expansion (50) of the decay amplitude can be used directly to fit experimental data, considering the a_{nl} to be arbitrary complex numbers. It is, however, convenient to also have an expression for the decay probability directly, i.e., an expansion of the square modulus $|f(\alpha,\theta)|^2$.

Using (50) and (49) such an expansion can be readily obtained. Indeed writing (50) in the form

$$f(\alpha,\theta) = \sum_{n,l} a_{nl}\,\Phi_{nl}(\alpha,\theta), \tag{53}$$

we have

$$|f(\alpha,\theta)|^2 = \sum_{n,l,n',l'} a_{nl}\,a^*_{n'l'}\,\Phi_{nl}\,\Phi^*_{n'l'} = \sum_{n,l,n',l'} e^{i\pi l'} a_{nl}\,a^*_{n'l'}\Phi_{nl}\Phi_{n'l'}.$$

Thus

$$|f(\alpha,\theta)|^2 = \sum_{N=0}^{\infty}\sum_{L=0}^{N} b_{NL}\Phi_{NL}(\alpha,\theta), \tag{54}$$

with

$$b_{NL} = \frac{(N+1)^{1/2}}{\sqrt{2}\,\pi}\sum_{n,l,n',l'}[(n+1)(n'+1)(2l+1)(2l'+1)]^{1/2}(l0l'0|L0)\begin{Bmatrix}\frac{1}{2}n & \frac{1}{2}n & l\\ \frac{1}{2}n' & \frac{1}{2}n' & l'\\ \frac{1}{2}N & \frac{1}{2}N & L\end{Bmatrix} e^{i\pi l'} a_{nl}a^*_{n'l'}. \tag{55}$$

The relation (55) immediately supplies useful selection rules. Thus $b_{NL}\neq 0$ only if

$$l+l'+L=\text{even}, \quad n+n'+N=\text{even}, \quad |l-l'|\le L\le l+l', \quad |n-n'|\le N\le n+n', \quad 0\le L\le N. \tag{56}$$

In particular, if l and l' are even, then L must also be even. The amplitudes b_{NL} are of course real numbers. The correct threshold behavior of our partial waves is ensured by the presence of the factor $(\sin\alpha)^l$ in (50), since

$$\sin\alpha = \frac{\{[(m_1+m_2)^2-s][(m_1-m_2)^2-s][s-(m_3+m_4)^2][(m_1{}^2-m_2{}^2)^2-(m_3+m_4)^2 s]\}^{1/2}}{2m_1{}^2(m_3+m_4)R^2 s} \tag{57}$$

vanishes correctly for $s=(m_1-m_2)^2$ and $s=(m_3+m_4)^2$ [the additional zero of $\sin\alpha$ at $s=(m_1{}^2-m_2{}^2)^2/(m_3+m_4)^2$ has no physical meaning, since it lies in the physical scattering region, where we use the original O(3,1) mapping].

In a separate publication[22] we apply the presented method, in particular, expansions (50) and (54) to treat the Dalitz-plot distribution of $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ decays[23]. In doing this we must naturally restrict ourselves to a finite number of terms in each sum, e.g., by taking $0\le n\le n_0$ for some fixed n_0 and considering all acceptable values of l. The relation between the amplitudes a_{nl} in (50) and b_{NL} in (54) for low-lying values of n and l is given in Appendix A.

Let us now discuss the relation between our analysis and the usual one in terms of Dalitz variables (see, e.g., Refs. 21 and 23).

The Dalitz variables x and y are introduced when considering a three-body decay in the rest frame of the decaying particle. Let the kinetic energies of the produced particles 2, 3, and 4 in the rest frame of particle 1 be T_k ($k=2,3,4$) and introduce

$$Q=T_2+T_3+T_4=m_1-m_2-m_3-m_4. \quad (58)$$

Then we define

$$x=\sqrt{3}\,\frac{T_3-T_4}{Q}, \quad y=\frac{3T_2-Q}{Q}. \quad (59)$$

Our variables α and θ can easily be related to x and y (see Fig. 3):

$$\cos\alpha=1-\frac{2Q}{3R^2}\,\frac{(1+y)[6m_2+Q(1+y)]}{3(m_1-m_2)^2-2m_1Q(1+y)}, \quad (60)$$

$$\begin{aligned}\cos\theta=&\{Q(1+y)[6m_2+Q(1+y)][(m_1-m_2)^2-(m_3+m_4)^2-\tfrac{2}{3}m_1Q(1+y)][(m_1-m_2)^2-(m_3-m_4)^2-\tfrac{2}{3}m_1Q(1+y)]\}^{-1/2}\\ &\times\{-[(m_1-m_2)^2-\tfrac{2}{3}m_1Q(1+y)][6m_3+Q(2+\sqrt{3}\,x-y)]\\ &+[3(m_1-m_2)-Q(1+y)][(m_1-m_2)^2+m_3^2-m_4^2-\tfrac{2}{3}m_1Q(1+y)]\}. \end{aligned} \quad (61)$$

The most common way of treating, say, $K\to 3\pi$ decays is to expand the amplitude in a power series in x and y:

$$f(s,t)=\sum_{m,k=0}^{\infty}R_{km}x^my^k. \quad (62)$$

Let us compare this expansion with the O(4) expansion (50) which we write as

$$f(s,t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}a_{nl}\,\Phi_{nl}(\alpha,\theta).$$

Making use of the inverse formula (51), we obtain

$$a_{nl}=\sum_{m,k=0}^{\infty}I_{nl}^{km}R_{km}, \quad (63)$$

with

$$I_{nl}^{km}=2^{l+1/2}l!\left[\frac{(2l+1)(n+1)(n-l)!}{(n+l+1)!}\right]^{1/2}e^{i(\pi/2)l}\int_0^{\pi}d\alpha(\sin\alpha)^{l+2}C_{n-l}^{l+1}(\cos\alpha)y^k\int_0^{\pi}x^mP_l(\cos\theta)\sin\theta\,d\theta. \quad (64)$$

Similarly, making use of the fact that

$$R_{km}=\frac{1}{m!k!}\,\frac{\partial^{m+k}}{\partial x^m\partial y^k}f(s,t)\bigg|_{x=y=0},$$

we find that

$$R_{km}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}J_{km}^{nl}a_{nl}, \quad (65)$$

with

$$J_{km}^{nl}=\frac{l!}{m!k!}\,\frac{2^l}{\sqrt{2}\,\pi}\left[\frac{(2l+1)(n+1)(n-l)!}{(n+l+1)!}\right]^{1/2}\frac{\partial^{m+k}}{\partial x^m\partial y^k}\sin^l\alpha\,C_{n-l}^{l+1}(\cos\alpha)P_l(\cos\theta)\bigg|_{x=y=0}. \quad (66)$$

The "overlap coefficients" I_{nl}^{km} and J_{km}^{nl} can easily be calculated explicitly and we discuss them further in Appendix B. In any case, it is obvious that the relation between the two types of expansions is not trival and that a finite number of terms in one expansion corresponds to an infinite number in the other.

In particular, if we use the linear approximation for $f(s,t)$ (as is often done when presenting decay data),

$$f(s,t)=R_{00}+R_{10}y, \quad (67)$$

then the nonzero terms in the O(4) expansion will be

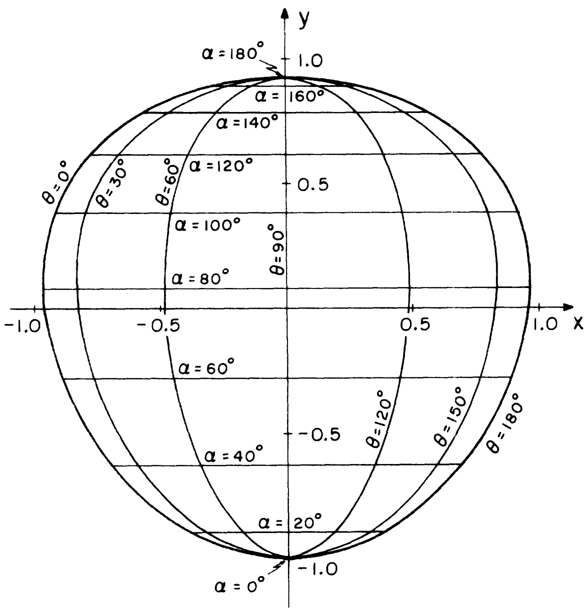

FIG. 3. Dalitz plot for $K\to 3\pi$ decay parametrized in terms of the Dalitz variables (x,y) and the O(4) variables (α,θ). The boundary of the physical decay region is $\cos\theta=\pm 1$.

$$a_{00}=I^{00}_{00}R_{00}+I^{10}_{00}R_{10},$$
$$a_{n0}=I^{10}_{n0}R_{10},\quad 1\leqslant n\leqslant\infty. \tag{68}$$

The first few coefficients I^{k0}_{n0} and J^{n0}_{k0} have been evaluated numerically for $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ decays and are presented in Tables I and II.

V. CONCLUSIONS

Let us summarize the contents of this paper. Two variable expansions previously developed for scattering amplitudes,[1-8] have been modified to be applicable to general three-body decay processes (1) involving particles of spin zero. The physical decay region is mapped onto a three-dimensional sphere in Euclidean velocity space, so that the decay amplitude is a function of the variables α and θ, given by expressions (9) and (17). The O(4) -group expansion of $f(\alpha,\theta)$ is then provided by formula (50), or alternatively, the decay probability $|f(\alpha,\theta)|^2$ is expanded using formulas (54) and (55).

Important features of this method are:

(1) Scattering and decays are treated on the same footing, the only difference being that the expansions for scattering must be provided by the noncompact group O(3, 1), since the physical region is infinite.

(2) The fact that the expansions are explicit in both kinematic parameters makes it possible to represent the total information contained in the Dalitz plot in terms of a few parameters – the O(4) amplitudes a_{nl} in (50). These amplitudes should then be sensitive to the actual dynamics of the decay, i.e., to questions like CP violations in $K\to 3\pi$ decays, the $\Delta I=\frac{1}{2}$ rule in nonleptonic weak decays, etc.

(3) The partial-wave amplitudes have the correct threshold and pseudothreshold behavior built in explicitly, since the basis functions contain a factor$(\sin\alpha)^l$ vanishing as $[s-(m_3+m_4)^2]^{l/2}$ and $[(m_1-m_2)^2-s]^{l/2}$, respectively.

(4) A "fringe benefit" obtained by using the O(4) expansions (50) is that the statistical errors in the expansion coefficients a_{nl} are (at least in a somewhat idealized case) uncorrelated, since the functions Φ_{nl} are orthonormal when integrated over the physical decay region only. Indeed, the phenomenological coefficients a_{nl} are obtained by minimizing the χ^2 function

$$\chi^2=\sum_{k=0}^{K}\left|\frac{f(\alpha_k,\theta_k)-\sum a_{nl}\Phi_{nl}(\alpha_k,\theta_k)}{\Delta_k}\right|^2, \tag{69}$$

where the k summation is over all experimental points (or over the centers of each bin in the Dalitz plot, or some other such quantities, depending on how the data are presented). Here Δ_k is the experimental error in $f(\alpha_k,\theta_k)$. The n, l sums are over finite numbers of terms. The errors on the coefficients are uncorrelated if the inverse error matrix

TABLE I. The overlap coefficients I^{k0}_{n0} expressing the O(4) expansion coefficients a_{n0} in terms of the power-series coefficients R_{k0} for $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ decays.

k \ n	0	1	2	3	4	5	6	7	8	9	10
0	4.443	0	0	0	0	0	0	0	0	0	0
1	0.767	−2.027	−0.294	−0.050	−0.010	−0.002	⋯	⋯	⋯	⋯	⋯
2	1.077	−0.426	0.890	0.267	0.066	0.016	0.004	0.001	⋯	⋯	⋯
3	0.318	−0.973	0.091	−0.405	−0.185	−0.060	−0.017	−0.005	−0.001	⋯	⋯
4	0.498	−0.261	0.647	0.069	0.205	0.177	0.047	0.016	0.005	0.001	⋯
5	0.161	−0.567	0.109	−0.389	−0.112	−0.119	−0.074	−0.034	−0.013	−0.004	−0.001
6	0.283	−0.157	0.458	0.007	0.231	0.104	0.077	0.048	0.024	0.010	0.004

TABLE II. The overlap coefficients J_{k0}^{n0} expressing the power-series coefficients R_{k0} in terms of the O(4) expansion coefficients a_{n0} for $K^{\pm}\to\pi^{\pm}\pi^{\pm}\pi^{\mp}$ decays.

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	0.225	0	0	0	0	0	0	0	0	0	0
1	0.114	−0.446	−0.137	−0.034	−0.008	−0.002	...	...	...	...	...
2	−0.168	−0.452	0.747	0.510	0.209	0.072	0.023	2.027	0.002	0.001	...
3	−0.199	0.550	1.513	−0.890	−1.280	−0.780	−0.357	−0.141	−0.051	−0.017	−0.006
4	0.067	1.124	−0.952	−4.267	−0.089	2.370	2.198	1.315	0.637	0.271	0.106
5	0.232	−0.115	−4.265	−0.075	10.110	4.857	−2.516	−4.824	−3.887	−2.316	−1.163
6	0.051	−1.643	−1.119	12.725	7.959	−19.283	−19.100	−3.195	7.413	9.318	7.017

$$H_{nl,n'l'}=\frac{1}{2}\frac{\partial^2\chi^2}{\partial a_{nl}\partial a_{n'l'}} \quad (70)$$

is diagonal. We have

$$H_{nl,n'l'}=\sum_k \frac{\Phi_{nl}(\alpha_k,\theta_k)\Phi_{n'l'}(\alpha_k,\theta_k)}{\Delta_k^{\,2}}. \quad (71)$$

Consider now the ideal case of a very large number of measurements [$K\to\infty$ in (69)]. We can replace the sums over experimental points by integrals over the Dalitz plot (the physical decay region), so that

$$H_{nl,n'l'}=\int_0^\pi\int_0^\pi \frac{\Phi_{nl}(\alpha,\theta)\Phi_{n'l'}(\alpha,\theta)}{\Delta^2(\alpha,\theta)}\sin^2\alpha\, d\alpha\,\sin\theta\, d\theta. \quad (72)$$

However, if we assume that the errors are uniform over the Dalitz plot $\Delta(\alpha,\theta)=\Delta=\text{const}$, we can make use of the orthogonality properties of Φ_{nl} to obtain

$$H_{nl,n'l'}=\frac{e^{i\pi l}}{2\pi}\frac{1}{\Delta^2}\delta_{nn'}\delta_{ll'}, \quad (73)$$

so that the errors are indeed uncorrelated.

Important questions which should be settled by an actual treatment of data, e.g., for $K\to3\pi$ and $\eta\to3\pi$ decays are the following: How fast do the O(4) expansions converge for physical amplitudes? How stable are they with respect to the cutoff n_0? How sensitive are the expansion coefficients a_{nl} with respect to interesting features of the decay? How do the O(4) expansions compare with other treatments? Some answers to these questions are contained in Ref. 22.

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APPENDIX A: REMARKS ON THE DECAY PROBABILITY

If a decay amplitude is expanded using formula (53), then the decay probability will be represented by (54). The general connection between b_{NL} and a_{nl} is given by (55). The amplitudes a_{nl} are in general complex, whereas the b_{NL} are real. In practice, we always consider a finite number of terms only. Since the phase of $f(\alpha,\theta)$ is arbitrary, we can fix an over-all phase for the coefficients a_n , e.g., by putting a_{00} = real, positive.

Let us consider the case when we cut off the sums in (53) at $n_0=2$, i.e., consider only $a_{00}\geq 0$ and a_{10}, a_{20}, and a_{22}, which are complex. Further, assume that particles 3 and 4 are identical so that l and l' are even. The selection rules (56) show that these a_{nl} contribute to b_{NL} with $0\leq N\leq 4$, $0\leq L\leq N$. Explicitly evaluating the $3j$ and $9j$ symbols in (55), we find:

$$\begin{aligned}
b_{00}&=\frac{1}{\sqrt{2}\,\pi}(|a_{00}|^2+|a_{10}|^2+|a_{20}|^2+|a_{22}|^2),\\
b_{10}&=\frac{\sqrt{2}}{\pi}\,\mathrm{Re}\,a_{10}(a_{00}^*+a_{20}^*),\\
b_{20}&=\frac{1}{\sqrt{2}\,\pi}(|a_{10}|^2+|a_{20}|^2-\tfrac{1}{2}|a_{22}|^2+2\,\mathrm{Re}\,a_{00}a_{20}^*),\\
b_{22}&=\frac{1}{2\pi}(-|a_{22}|^2+2\sqrt{2}\,\mathrm{Re}\,a_{00}a_{22}^*-\sqrt{2}\,\mathrm{Re}\,a_{20}a_{22}^*),\\
b_{30}&=\frac{\sqrt{2}}{\pi}\,\mathrm{Re}\,a_{10}a_{20}^*,\\
b_{32}&=\frac{1}{\pi}\,\mathrm{Re}\,a_{10}a_{22}^*,\\
b_{40}&=\frac{1}{\sqrt{2}\,\pi}(|a_{20}|^2+\tfrac{1}{10}|a_{22}|^2),\\
b_{42}&=\frac{1}{2\pi}\frac{1}{\sqrt{35}}(|a_{22}|^2+7\sqrt{2}\,\mathrm{Re}\,a_{20}a_{22}^*),\\
b_{44}&=\frac{9}{5\pi}\frac{1}{\sqrt{7}}|a_{22}|^2.
\end{aligned} \quad (A1)$$

It follows from the above relations that a three-parameter fit to $f(\alpha, \theta)$ (the parameters being $a_{00} \geq 0$, $\mathrm{Re}\,a_{10}$, and $\mathrm{Im}\,a_{10}$) corresponds to a simple three-parameter fit to $|f(\alpha, \theta)|^2$ (the parameters are b_{00}, b_{10}, and b_{20}). On the other hand, a seven-parameter fit to $f(\alpha, \theta)$ corresponds to a nine-parameter fit to the decay probability [the parameters $b_{00}, \ldots, b_{44}$ of (A1)], with two (nonlinear) constraints on the b_{NL}. These constraints can readily be obtained from Eqs. (A1), however they are too inelegant to merit publication.

APPENDIX B: RELATION BETWEEN O(4) AND DALITZ VARIABLE EXPANSIONS

The overlap coefficients, I_{nl}^{km} and J_{km}^{nl}, relating the two different types of expansions, have been defined in Sec. IV [see formulas (64) and (66)]. Let us first discuss I_{nl}^{km}. In order to perform the integrations in (64) we first express x and y as functions of $\zeta = \cos\alpha$ and $z = \cos\theta$. We then expand x^m and y^k and $(1-\zeta)^{l/2}$ in powers of ζ and z, so that finally the only integrals actually needed are[24]

$$\int_{-1}^{1} z^a P_l(z)dz = [1+(-1)^{l+a}]\frac{\sqrt{\pi}}{2^{a+1}}\,\frac{\Gamma(a+1)}{\Gamma(\frac{1}{2}(l+a+3))\,\Gamma(\frac{1}{2}(-l+a+2))} \quad (\mathrm{Re}\,a > -1) \tag{B1}$$

and

$$\int_{-1}^{1}(1-\zeta^2)^{l+1/2}\zeta^b\, C_{n-l}^{l+1}(\zeta)d\zeta = [1+(-1)^{n-l+b}]\frac{\pi(n+l+1)!\,\Gamma(b+1)}{2^{2l+2+b}\,l!\,\Gamma(\frac{1}{2}(b-n+l)+1)\,\Gamma(\frac{1}{2}(b+n+l)+2)(n-l)!} \quad [\mathrm{Re}(b-n+l) > -1]. \tag{B2}$$

Expressions (60) and (61) can be inverted, to give

$$y = \frac{1}{2m_1 Q}\Big\{-6m_1m_2 - 2m_1Q - 3m_1^{\,2}R^2 + 3m_1^{\,2}R^2 + 3m_1^{\,2}R^2\cos\alpha + 3m_1m_2\left[(2+R^2)\left(2+\frac{m_1^{\,2}R^2}{m_2^{\,2}}\right)\left(1-\frac{R^2}{2+R^2}\cos\alpha\right)\left(1-\frac{m_1^{\,2}R^2/m_2^{\,2}}{2+m_1^{\,2}R^2/m_2^{\,2}}\cos\alpha\right)\right]^{1/2}\Big\}, \tag{B3}$$

and a somewhat more complicated expression for x as a function of α and θ.

We have obtained a general expression for I_{nl}^{km}, which is, however, quite cumbersome, containing one infinite sum and many finite ones. If we only consider a special case when $m=0$, then we obtain a somewhat simpler result, namely,

$$\begin{aligned} I_{nl}^{k0} = {} & \delta_{l0}(-1)^{k+n}\sqrt{2}\,\pi(n+1)k!\left(\frac{2}{R}\right)^{2n}\left(\frac{6m_2+2Q+3m_1R^2}{2Q}\right)^k\left(\frac{m_2}{m_1}\right)^{2n} \\ & \times\sum_{a=0}^{\infty}\sum_{b=0}^{k}\sum_{c=0}^{b}\sum_{p}(-1)^c\frac{(2p-n)!\,\Gamma(c/2+1)}{(k-b)!\,c!\,(b-c)!\,(2p-a-b+c-n)!\,(2a+b-c+n-2p)!\,(p+1)!\,(p-n)!\,\Gamma(c/2-a+1)} \\ & \times 2^{2a+b-c-4p}(3m_1)^b\left(\frac{m_1^{\,2}}{m_2^{\,2}}\right)^{2p-a-b+c/2}R^{4p}\left(1+\frac{m_1^{\,2}}{m_2^{\,2}}+\frac{m_1^{\,2}}{m_2^{\,2}}R^2\right)^{2a+b-c+n-2p} \\ & \times\left[(2+R^2)\left(2+\frac{m_1^{\,2}}{m_2^{\,2}}R^2\right)\right]^{c/2-a}(6m_2+2Q+3m_1R^2)^{-b}. \end{aligned} \tag{B4}$$

Expression (B4) was actually used to calculate the first few coefficients numerically. The program was so arranged that all terms in the finite sums were taken and the infinite sum over a was cut off as soon as the sum over the last three terms in a was less than 1% of the total value of the corresponding coefficient I_{nl}^{k0}. The results are presented in Table I.

The coefficient J_{km}^{nl} of (66) can also be calculated in general. To do this we need an expression for the nth derivative of a composite function, which can be written as

$$\frac{d^nF(\phi(t))}{dt^n} = \sum_{a=0}^{n}\sum_{b=0}^{a}\frac{(-1)^b}{b!\,(a-b)!}[\phi(t)]^b\frac{d^n}{dt^n}[\phi(t)]^{a-b}\frac{d^a}{d\phi^a}F(\phi). \tag{B5}$$

The general coefficient J_{km}^{nl} is again quite complicated, the special case $l=0$ is simple enough to be presented here. Indeed, we have

$$J^{n0}_{km} = \delta_{m0} \frac{1}{k!\sqrt{2}\pi} \frac{d^k}{dy^k} C^1_n(\cos\alpha)\bigg|_{y=0} . \tag{B6}$$

We can now directly apply (B5) with $\phi = \cos\alpha$ and $F(\phi) = C^1_n(\cos\alpha)$, $\cos\alpha$ being given by (60). The result can be presented as

$$J^{n0}_{km} = \delta_{m0} \frac{1}{k!\sqrt{2}\pi} \left(\frac{2m_1Q}{3}\right)^k \sum_{a=0}^{\min(k,n)} \sum_{b=0}^{a} \sum_{c=0}^{a-b} \sum_{d=0}^{c} \frac{(-1)^b 2^a a!\,\Gamma(-a+b+c+d+k)}{b!(a-b-c)!(c-d)!d!\,\Gamma(-a+b+c+d)}$$

$$\times \lfloor \cos\beta C^{a+1}_{n-a}(\cos\beta)\rfloor\,|_{y=0} (s)^{a-b-c-d-k}|_{y=0} A^{a-b-c} B^{c-d} C^d , \tag{B7}$$

where

$$\cos\beta|_{y=0} = 1 - \frac{2Q}{3R^2} \frac{6m_2+Q}{3(m_1-m_2)^2 - 2m_1Q}, \quad s|_{y=0} = (m_1-m_2)^2 - \tfrac{2}{3}m_1Q ,$$

$$A = -\frac{1}{2m_1{}^2R^2}, \quad B = \frac{m_1{}^2+m_2{}^2+m_1{}^2R^2}{m_1{}^2R^2}, \quad C = -\frac{(m_1{}^2-m_2{}^2)^2}{2m_1{}^2R^2} . \tag{B8}$$

Formula (B7) was used to calculate the first few values of J^{n0}_{km} numerically, as presented in Table II.

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[1]N. Ya. Vilenkin and J. A. Smorodinsky, Zh. Eksperim. i Teor. Fiz. 46, 1793 (1964) [Soviet Phys. JETP 19, 1209 (1964)].

[2]P. Winternitz and I. Friš, Yadern. Fiz. 1, 889 (1965) [Soviet J. Nucl. Phys. 1, 636 (1965)].

[3]P. Winternitz, J. A. Smorodinsky, and M. B. Sheftel, Yadern. Fiz. 7, 1325 (1968); 8, 833 (1968) [Soviet J. Nucl. Phys. 7, 785 (1968); 8, 485 (1969)].

[4]P. Winternitz, Czech. J. Phys. B19, 1589 (1969).

[5]P. Pajas and P. Winternitz, Phys. Rev. D 1, 1105 (1970).

[6]N. W. Macfadyen and P. Winternitz, J. Math. Phys. 12, 281 (1971); Phys. Rev. D 3, 1874 (1971).

[7]P. Winternitz, Rutherford Laboratory Report No. RPP/T/3, 1968 (unpublished).

[8]P. Winternitz, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Colorado Univ. Press, Boulder, Colorado, 1971), Vol. 13. (The last two papers contain numerous references to previous and related work.)

[9]H. Joos, Fortschr. Physik 10, 65 (1962) and *Lectures in Theoretical Physics*, Ref. 8, Vol. 7A.

[10]M. Toller, Nuovo Cimento 53A, 671 (1968); 54A, 295 (1968).

[11]J. F. Boyce, R. Delbourgo, A. Salam, and J. Strathdee, Trieste Report No. IC/67/9, 1967 (unpublished).

[12]A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968).

[13]A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D 1, 1773 (1970).

[14]T. W. B. Kibble, Phys. Rev. 117, 1159 (1960).

[15]A. Z. Dolginov, Zh. Eksperim. i Teor. Fiz. 30, 746 (1956) [Soviet Phys. JETP 3, 589 (1956)].

[16]L. C. Biedenharn, J. Math. Phys. 2, 433 (1961).

[17]G. Domokos, Phys. Rev. 159, 1387 (1967).

[18]N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations* (Am. Math. Soc. Math. Monograph Transl. No. 22, Providence, R.I., 1968).

[19]R. L. Anderson, R. Raczka, M. A. Rashid, and P. Winternitz, J. Math. Phys. 11, 1050 (1970); 11, 1059 (1970).

[20]D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon Press, Oxford, 1962).

[21]R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley, New York, 1969).

[22]H. R. Hicks and P. Winternitz (unpublished).

[23]W. T. Ford, P. A. Piroué, R. S. Remmel, A. J. S. Smith, and P. A. Souder, Phys. Rev. Letters 25, 1370 (1970) and private communications.

[24]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).