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Generalized Coherent States

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Using an operator technique similar to one we have used in a previous paper, we study the generalized coherent states which were introduced by Titulaer and Glauber and discussed recently by Bialynicka-Birula. Using this technique, we provide an alternative derivation of the most general coherent state and then derive a number of interesting properties, some of which do not seem to be in the literature. For example, we show that the generalized coherent states are not minimum-uncertainty packets.

I. INTRODUCTION

It has been demonstrated[1] that the coherent states, i.e., annihilation-operator eigenstates, are not the only pure states which satisfy the conditions for full coherence. There exists a larger class of states which includes the coherent states as a special case. This larger class, the generalized coherent states, gives the same results as do the ordinary coherent states for the correlation functions $G^{(n)}$. The generalized coherent states display properties differing from those of the annihilation-operator eigenstates for the case of processes which are described by matrix elements containing unequal numbers of creation and annihilation operators.[1,2]

In Sec. II we develop an operator approach to the generalized coherent states. We write these states in the form of a unitary operator acting upon a regular coherent state. This approach yields an alternate derivation of the generalized coherent states, and provides a convenient way to calculate with them.

In Sec. III we discuss some properties of a sequence of operators introduced in the process of developing the formalism. By means of these operators we show that a certain species of generalized coherent state are eigenvectors of integral powers (>1) of the annihilation operator while not being eigenvectors of the annihilation operator itself.

In Sec. V we prove that the generalized coherent states do not minimize the uncertainty product $\Delta x \Delta p$ of position and momentum as do the ordinary coherent states.

In Sec. VI we prove that the operators introduced in Sec. II are unitary, which demonstrates in the context of the present work that the generalized coherent states completely exhaust the pure states which satisfy the conditions for coherence. This fact was, of course, also demonstrated in Ref. 1.

II. DERIVATION OF THE GENERAL COHERENT STATE

Glauber and Titulaer have shown[1] that any pure state of the electromagnetic field which possesses first-order coherence may be thought of as one in which only a single mode is excited. This mode need not be a monochromatic one in general, but can be a generalized packet mode. We are interested in the most general pure state which is coherent to all orders. It is convenient to discuss this state in terms of that single mode which is excited by virtue of the state having first-order coherence. Let a and $a^\dagger$ be the annihilation and creation operators corresponding to that mode. We can now express the condition for full coherence to all orders by the following equation:

$$\langle\psi|a^{\dagger n}a^n|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle^n \quad \text{for} \quad n \geq 1. \tag{1}$$

The most general coherent state will be denoted by $|\psi\rangle$. Equation (1) is identically satisfied for $n=1$ since, as we have just stated, any single-mode state is first-order coherent.

We will look for the most general pure state satisfying (1) by writing $|\psi\rangle$ in the form

$$|\psi\rangle = T|\beta\rangle \quad \text{with} \quad T^\dagger = T^{-1}, \tag{2}$$

where $|\beta\rangle$ is a regular coherent state of the excited mode, i.e., an eigenstate of a. This approach of looking for solutions in the form of a unitary operator acting on an eigenstate of the annihilation operator seems quite useful and has been used by the author in connection with a study of minimum-uncertainty packets.[3] The coherence condition then becomes

$$\langle\beta|T^\dagger a^{\dagger n}a^n T|\beta\rangle = \langle\beta|T^\dagger a^\dagger a T|\beta\rangle^n. \tag{3}$$

Now we define a sequence of operators, T_n, related to T by means of the relation

$$a^n T = T_n a^n. \tag{4}$$

The operators T_n are of interest in several contexts and they can be calculated once T is given. We use (4), its Hermitian conjugate, and the relation $a|\beta\rangle=\beta|\beta\rangle$ in Eq. (3) to get

$$\langle\beta|T_n^\dagger T_n|\beta\rangle=\langle\beta|T_1^\dagger T_1|\beta\rangle^n, \quad \text{all} \quad n\geq 1. \tag{5}$$

So we have translated the coherence condition from a condition involving the matrix elements of known operators in unknown states to one involving matrix elements of unknown operators in known states.

It is clear that the coherence condition will be satisfied if the T_n are unitary, i.e., if $T_n^\dagger T_n=1$. We shall, at this point, assume this to be the case.

By multiplying (4) by its Hermitian adjoint and using the unitarity of the T_n, we get

$$T^\dagger a^{\dagger m}a^m T=a^{\dagger m}a^m, \quad \text{all} \quad m\geq 1. \tag{6}$$

Since T is unitary, the above equation for $m=1$ implies that T commutes with the number operator $a^\dagger a$. This means that we can represent T as an operator which is diagonal in the number-eigenstate basis, i.e.,

$$T=\sum_{n=0}^{\infty} C_n|n\rangle\langle n|. \tag{7}$$

The unitarity of T implies

$$T^\dagger T=\sum_n |C_n|^2|n\rangle\langle n|=1. \tag{8}$$

Hence we have $|C_n|^2=1$ and $C_n=e^{i\theta_n}$. So we finally have

$$T=\sum_{n=0}^{\infty} e^{i\theta_n}|n\rangle\langle n|, \tag{9}$$

where θ_n = arbitrary real number.

Writing

$$|\beta\rangle=e^{-|\beta|^2}\sum_{n=0}\frac{\beta^n}{\sqrt{n!}}|n\rangle$$

and letting $\beta=re^{i\phi}$, we get the following expression for $|\psi\rangle$:

$$|\psi\rangle=T|\beta\rangle=e^{-r^2/2}\sum_{n=0}^{\infty}e^{i(\theta_n+n\phi)}\frac{r^n}{\sqrt{n!}}|n\rangle. \tag{10}$$

This is exactly the same result obtained by Titulaer and Glauber[1] in a rather different manner. Equation (10) here coincides with Eqs. (3.16) of Ref. 3 if one writes our $\theta_n+n\phi$ as simply θ_n'. We have reproduced *all* of the states that are termed the most general coherent states in Ref. 3 within the assumption that all of the T_n are unitary. It might seem that there could be even more states which satisfy (1) than are obtained in Ref. 3, namely, those states which correspond to nonunitary T_n. However, this is not the case because, as will be proven in Sec. VI, the T_n cannot be nonunitary.

III. PROPERTIES OF THE T_n

Using the expression (7) for T, we can obtain an explicit form for T_1. Multiplying (7) by a we have

$$\begin{aligned} aT &= \sum_{s=0}^{\infty} e^{i\theta_s}\sqrt{s}\,|s-1\rangle\langle s| \\ &= \sum_{n=0}^{\infty} e^{i\theta_{n+1}}\sqrt{n+1}\,|n\rangle\langle n+1| \\ &= \left(\sum_{n=0}^{\infty} e^{i\theta_{n+1}}|n\rangle\langle n|\right)a \\ &= T_1 a. \end{aligned} \tag{11}$$

Hence we have the result

$$T_1=\sum_{n=0}^{\infty} e^{i\theta_n - 1}|n\rangle\langle n|. \tag{12}$$

In the same manner one can show that, in general, we have

$$T_l=\sum_{n=0}^{\infty} e^{i\theta_{n+l}}|n\rangle\langle n|. \tag{13}$$

From (13) we see that T_l is gotten from T by shifting the phase of $|n\rangle\langle n|$ upwards by l units. Equation (13) can be derived directly by using a relation which follows from (4) and is independent of the explicit form of T_n, namely,

$$a^l T_n = T_{n+l}\,a^l. \tag{14}$$

Another relation, also independent of the explicit form of T_n, that follows easily from (14) is

$$a^l(T_nT_{n'})=(T_{n+l}T_{n'+l})a^l, \tag{15}$$

so the product of two T_n's is of the same type as the T_n's. We can use the properties of the T_n's to study the general coherent states.

IV. PROPERTIES OF THE GENERAL COHERENT STATES (GCS)

From the form $|\psi\rangle=T|\beta\rangle$, it is clear that the GCS share the properties of the coherent states as to overcompleteness, representation of operators, etc. For example, by multiplying the relation $(1/\pi)\int|\beta\rangle\langle\beta|d^2\beta=1$ on the left by T and on the right by $T^\dagger$, we have

$$\frac{1}{\pi}\int|\psi(\beta)\rangle\langle\psi(\beta)|d^2\beta=1. \tag{16}$$

An interesting subset of the GCS is the set of states whose phases θ_n are periodic, i.e., $\theta_{n+N}=\theta_n$ for some N and all n. These states have certain properties which simplify calculations involving them. Bialynicka-Birula[2] has stated that such states may be written as a discrete superposition of N coherent states of the form $|\rho e^{i\alpha_\kappa}\rangle$ where $\alpha_\kappa=\theta+2\pi\kappa/N$ and $\kappa=1,2,\ldots,N$.

We can see how this comes about by noticing that for $\theta_{n+N}=\theta_n$ the operator

$$T_N=\sum_{n=0}^{\infty} e^{i\theta_{n+N}}|n\rangle\langle n|$$

is equal to T itself. Denote a state whose phases have periodicity N by $|\psi_N\rangle$. Then by multiplying $|\psi_N\rangle$ by a^N, we have

$$a^N|\psi_N\rangle = a^N T|\beta\rangle = T_N a^N|\beta\rangle = \beta^N T_N|\beta\rangle = \beta^N T|\beta\rangle; \tag{17}$$

so we see that $|\psi_N\rangle$ is an eigenstate of a^N with the eigenvalue β^N. Next we note that the $|\psi_N\rangle$'s are not the only eigenvectors of a^N. Any coherent state of the form $|\lambda_i\alpha\rangle$, where λ_i is one of the Nth roots of unity will be an eigenvector of a^N with eigenvalue α^N. Since the N states of this type are linearly independent, we may orthogonalize them and construct a set of N *orthogonal eigenvectors* of a^N. The $|\psi_N\rangle$ may then be expanded in terms of this orthogonal set. Since each member of the set is a linear combination of the $|\lambda_i\alpha\rangle$, so also is $|\psi_N\rangle$. The set of vectors built up out of the N eigenvectors of a^N for all N may be a useful one in computations because of its "pockets of orthogonality." The states $|\psi_N\rangle$ have the property that they maximize the modulus of the expectation value of a^N. This result is easy to prove and seems nontrivial since a^N is not self-adjoint. The proof proceeds as follows: For an arbitrary (GCS) $|\psi\rangle$, we have

$$\langle\psi|a^n|\psi\rangle = \langle\beta|T^\dagger a^n T|\beta\rangle = \langle\beta|T^\dagger T_n|\beta\rangle\beta^n. \tag{18}$$

Schwarz's inequality then implies that

$$|\langle\psi|a^n|\psi\rangle|^2 \leq |\beta|^{2n}, \tag{19}$$

with the equality holding when $T_n|\beta\rangle$ is proportional to $T|\beta\rangle$ or $T = e^{i\eta}T_n$. This implies that $|\psi\rangle = |\psi_n\rangle$. Q.E.D. Making use of Eq. (19) for $n=1$ and the triangle inequality, we can easily show that, for $x \sim (a+a^\dagger)$, we have

$$|\langle\psi|x|\psi\rangle| \leq 2|\beta|, \tag{20}$$

with the equality holding only when $|\psi\rangle$ is a coherent state, i.e., for $T_1 = T$.

V. NONMINIMALITY OF THE GENERAL COHERENT STATE

We now prove that the only pure states which satisfy the coherence condition (1) and which are also minimum-uncertainty states are the annihilation-operator eigenstates.

We define position and momentum operators as follows:

$$x = \frac{1}{\lambda\sqrt{2}}(a + a^\dagger), \qquad p = \frac{\lambda}{i\sqrt{2}}(a - a^\dagger), \tag{21}$$

where λ is an arbitrary scale factor.

It may be shown[3] that any state which minimizes the uncertainty product $\Delta x \Delta p$ is a solution of the following eigenvalue equation:

$$(C_r a + S_r a^\dagger)|\phi\rangle = \mu|\phi\rangle, \tag{22}$$

where $C_r = \cosh r$, $S_r = \sinh r$, μ is a complex number, and r is real. In Ref. 3 we have shown that any minimum-uncertainty packet (m.u.p.) can be written as

$$|\text{m.u.p.}\rangle = U_r|\alpha\rangle \equiv |r;\alpha\rangle, \tag{23}$$

where

$$U_r = \exp[\tfrac{1}{2}r(a^2 - a^{\dagger 2})] \tag{24}$$

and

$$U_r a U_r^\dagger = C_r a + S_r a^\dagger. \tag{25}$$

In view of the above facts, we see that if one of the GCS minimizes the uncertainty product it must satisfy the equation

$$(C_r a + S_r a^\dagger)T|\beta\rangle = \mu T|\beta\rangle \tag{26}$$

for some T, β, and μ. If we use (25) in Eq. (26) we get

$$U_r a U_r^\dagger T|\beta\rangle = \mu T|\beta\rangle \tag{27}$$

or, since $U_r^\dagger = U_r^{-1} = U_{-r}$,

$$a(U_{-r}T)|\beta\rangle = \mu(U_{-r}T)|\beta\rangle. \tag{28}$$

So if $T|\beta\rangle$ is to be a minimum-uncertainty packet, then $U_{-r}T|\beta\rangle$ must be an eigenvector of the annihilation operator with eigenvalue μ, i.e., it must be a regular coherent state $|\mu\rangle$,

$$U_{-r}T|\beta\rangle = |\mu\rangle. \tag{29}$$

Making use of the relations $|\mu\rangle = D(\mu)|0\rangle$, where $D(\mu)$ is the Weyl operator $\exp(\mu a^\dagger - \mu^* a)$, we see that $U_{-r}T \sim D(\mu-\beta)$ or $T \sim U_r D(\mu-\beta)$. In arriving at this result we have used the multiplication property of the Weyl operator, namely,

$$D(\alpha)D(\beta) = \exp[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)]D(\alpha+\beta).$$

Since T is diagonal in the number-state basis, the off-diagonal matrix elements of $U_r D(\mu-\beta)$ must vanish in that basis. Consider, for example, the off-diagonal matrix element

$$\langle 1|U_r D(\mu-\beta)|0\rangle = \langle 1|r;\mu-\beta\rangle. \tag{30}$$

This quantity may be obtained from Eq. (15′) of Ref. 3 and is given by[4]

$$(\beta-\mu)C_r^{-3/2}\exp\left(-\frac{|\mu-\beta|^2}{2} + \frac{(\mu-\beta)^2 S_r}{2C_r}\right).$$

This will vanish only for $\mu=\beta$, in which case we have $T \sim U_r$. But U_r is diagonal in the number basis only for $r=0$. Hence $T \sim 1$ and $T|\beta\rangle$ can only be a regular coherent state. Q.E.D.

VI. UNITARITY OF THE T_n

In this section we will show that Eq. (4), which defines the T_n, and the unitarity of T imply that T is diagonal in the number-state basis which further implies that the T_n are unitary.

First we establish the connection between the matrix elements of T and T_n in the number basis. Taking the matrix element of Eq. (4) between arbitrary number eigenstates, we have

$$\langle l|a^n T|m\rangle = \langle l|T_n a^n|m\rangle . \tag{31}$$

Using the relations

$$\begin{aligned} a^n|m\rangle &= [m(m-1)\cdots(m-n+1)]^{1/2}|m-n\rangle ,\\ \langle l|a^n &= \langle l+n|[(l+1)(l+2)\cdots(l+n)]^{1/2} \end{aligned} \tag{32}$$

in (31), we get

$$\langle r|T_n|s\rangle = \left[\frac{(r+n)!s!}{(s+n)!r!}\right]^{1/2}\langle r+n|T|s+n\rangle . \tag{33}$$

This generalizes the result of Eq. (13) which was derived under the explicit assumption that the T_n are unitary.

From Eq. (4) for $n=1$ we see that $aT|0\rangle=0$, so that we have the result that $\langle l|aT|0\rangle=0$ for arbitrary values of l. This yields the result that

$$\langle m|T|0\rangle = 0 \quad \text{for} \quad m=1,2,3,\ldots, . \tag{34}$$

Now $T^\dagger T=1$ implies that $\sum_{m=0}^{\infty}|\langle m|T|0\rangle|^2=1$ so that (34) implies $\langle 0|T|0\rangle=\lambda$, where λ is a complex number of unit modulus. In a similar way $TT^\dagger=1$ implies that $\sum_{l=0}^{\infty}|\langle 0|T|l\rangle|^2=1$. Since $|\langle 0|T|0\rangle|^2=1$, this last result implies $\langle 0|T|l\rangle=0$ for $l\geq 1$. Hence $\langle m|T|n\rangle$ is zero if either m or n is zero.

Next we make use of Eq. (14) which implies that

$$aT_n|0\rangle=0. \tag{35}$$

Taking the inner product of this relation with an arbitrary number eigenstate, we find that $\langle m|T_n|0\rangle=0$ for all values of m (and n) greater than or equal to unity. This result and Eq. (33) then imply

$$\langle r+n|T|n\rangle=0, \tag{36}$$

where $r, n=1,2,\ldots, .$

So the matrix representing T has a triangular form with all elements below the principal diagonal equal to zero. For such a matrix, unitarity implies that it is diagonal. This is easily seen to be true. Consider the matrix $[C_{ij}]$, which has the upper triangular form which we have shown T to possess and is also unitary. Unitarity of the matrix implies that its columns are orthonormal. Since the first column contains only a single element, namely C_{11}, we have $|C_{11}|^2=1$. Now the orthogonality of all the other columns to the first one implies that all the entries in the first row are zero (except C_{11}). Now we can consider the matrix obtained by deleting the first row and column of $[C_{ij}]$. It is clear that the columns of this matrix are also orthonormal, and that the first column here, too, has but one entry (C_{22}) which therefore has unit modulus. Orthogonality then, as before, implies that all the elements in the first row are zero (except C_{22}). This procedure may be continued indefinitely to show that $[C_{ij}]$ is a diagonal matrix whose elements are of unit modulus.

So we have proved that T is diagonal in the number basis and has the form $\sum_{n=0}^{\infty}e^{i\theta_n}|n\rangle\langle n|$. From this result it follows that

$$T_l=\sum_{n=0}^{\infty}e^{i\theta_{n+l}}|n\rangle\langle n|$$

and is, therefore, unitary.

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[1] U. Titulaer and R. Glauber, Phys. Rev. 145, 1041 (1965).

[2] Z. Bialynicka-Birula, Phys. Rev. 173, 1207 (1968).

[3] D. Stoler, Phys. Rev. D 1, 3217 (1970).

[4] Equations (15) and (15′) of Ref. 3 are missing a factor of $C_r^{-1/2}$ from their right-hand sides. We have remedied this in our use of them in the present work.