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Functional-Integral Approach to Dual-Resonance Theory

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It is shown that dual-resonance amplitudes can be expressed in terms of rudimental amplitudes defined by functional integrals which correspond to transition amplitudes of quantum-mechanical systems of strings with imaginary time. The equivalence between the path-integral and operator formulation of quantum mechanics is used to establish the connection between this approach and the usual operator approach. The factorization of rudimental amplitudes is studied to obtain the Feynman-like rules for dual-resonance amplitudes. This allows us to express N-Reggeon vertices in terms of rudimental amplitudes, and to determine the propagator, which is shown to be the usual spurious-free twisted propagator. N-loop orientable diagrams are calculated. In general, the functional integrals considered can be calculated by solving appropriate Neumann's boundary-value problems of corresponding bounded Riemann surfaces. This provides a generalization of the analog model to the case of external Reggeons which are described by extended momentum distributions on the boundaries.

I. INTRODUCTION

The dual-resonance theory[1] has been advanced considerably through the factorization[2] of the multiparticle Veneziano amplitudes, and the establishment of the operator formulation.[3] There are still some important formal problems left, such as unitarity of the S matrix and its renormalization.[4] In the operator formulation the resonance structure and factorization of the dual-resonance amplitudes are evident, but a complicated manipulation is required to prove the duality properties of the amplitudes.[5] Recently, an interesting development has been obtained by Alessandrini[6] and Lovelace.[7] Starting from the operator formalism, they showed that the dual-resonance amplitudes can be written in terms of automorphic functions, thus demonstrating Nielsen's conjecture.[8] However, their proof is somewhat lengthy, because in the operator formalism automorphic functions do not come in naturally.

Some time ago, the functional formulation of the theory was presented[9] and shown to be equivalent to the operator formulation. In the functional formulation the crossing symmetry and duality of scattering amplitudes are evident, although the resonance structure is not. So, to some extent, these two formulations are complementary. It is important to know whether one can simplify the complicated machinery of one formulation by using the other formulation instead. As an example, the automorphic functions come in naturally in the functional formulation, as was shown in Ref. 9 for particular cases.

In a recent note,[10] it has been shown that the dual-resonance amplitudes in the functional formulation can be regarded as the higher-order limit of ordinary semiplanar diagrams with some approximations. The surfaces of semiplanar diagrams correspond to the Riemann surfaces on which the functional integration variables $\Phi_\mu(x, y)$ are defined. Since the factorization of a Feynman diagram is made simply by cutting internal lines appropriately, the factorization of a dual-resonance amplitude in the functional formulation may be done by slicing the corresponding Riemann sheet. Once the factorization properties are established, the basic elements such as Reggeon propagators and Reggeon vertices are obtained, and one can generate higher-order multiloop amplitudes. In this paper we discuss those problems. We shall show that our method of factorization is automatically spurious-free, contrary to the usual operator method, since we obtain directly the spurious-free twisted propagator.[11]

We first notice that in order to factorize a semiplanar Feynman diagram one must cut an infinite number of internal lines in the higher-order limit. Since the state of a number $n\to\infty$ of internal scalar lines is specified by a momentum function $p_\mu(\xi)$ $(0\le\xi\le\pi)$ in the sense that $p_\mu(j\pi/n)$ is the momentum of the jth line, the sum over the intermediate

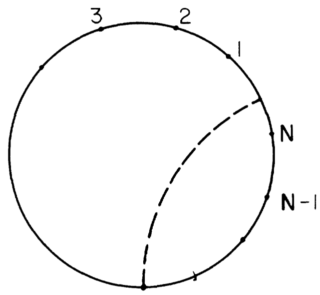

FIG. 1. D for the Koba-Nielsen N-particle Veneziano amplitude.

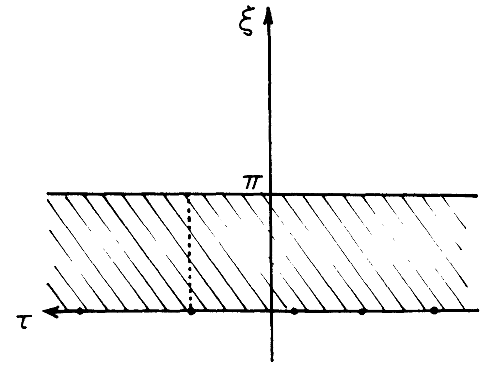

FIG. 2. D for the N-particle Veneziano amplitude in the operator formulation.

states is expressed by a functional integration over $p_\mu(\xi)$. Therefore, the factorization of a dual amplitude can be expressed as a functional average over two functionals of $p_\mu(\xi)$.

Next let us remark that the functional integral for a dual amplitude given in Ref. 9 is invariant under conformal transformations. A unit-disk domain has been used to obtain the Koba-Nielsen[12] N-particle amplitude (Fig. 1), while the shape of the domain relevant to the operator formalism is a strip[13] shown in Fig. 2. The factorization in the operator formulation is made by inserting the completeness relation $\sum_\lambda |\lambda\rangle\langle\lambda| = 1$ between operators. This corresponds (see Ref. 9) to the broken line of Fig. 2 which maps to an orthogonal arc of a circle drawn on Fig. 1. Therefore, the factorization of multiparticle amplitudes is symmetrically made by slicing the unit disk by various arcs which orthogonally intersect the unit circle.

As has been shown in Ref. 9, the functional integrals of dual-resonance amplitudes can be evaluated using the method of Neumann's boundary-value problem to express the integral in terms of energy integrals[14] which we call *rudimental amplitudes*. They are functionals of momentum-distribution functions $p_\mu(\xi)$ on the boundary. The factorization of a dual-resonance amplitude should therefore be expressed as a functional integral of a product of two rudimental amplitudes of two connected regions over the momentum distribution at the common boundary of those regions.

In the path-integral formulation of quantum mechics, one relates functional integrals to matric elements of the corresponding operators. Using the same method, we establish the relation between functional and operator methods. Since, as shown in Ref. 9, the corresponding time is pure imaginary, some different features appear, as compared with the case of the usual quantum mechanics.

In Sec. II we define rudimental amplitudes, and discuss their properties. In particular, we establish the slicing rule. Moreover, we establish the connection with the operator method, and this allows us to generalize the slicing rule to the case of overlapping domains.

In Sec. III we write the multiparticle Veneziano amplitudes in terms of rudimental amplitudes. We also introduce the N-Reggeon amplitudes and discuss their properties.

In Sec. IV the expressions for the Reggeon propagator and for the 3-Reggeon vertex are given. We then show that by the Feynman rules for dual-resonance amplitudes we can obtain the multiparticle Veneziano and the N-Reggeon amplitudes introduced in Sec. III.

Finally in Sec. V, we construct the planar-loop diagrams, and this leads very simply to the introduction of automorphic functions associated with the corresponding surfaces with handles.

II. RUDIMENTAL AMPLITUDES

It has been shown in Ref. 9 that the N-particle Veneziano amplitude can be expressed as a functional average of

$$\exp\left(i\sqrt{\pi}\sum_{i=1}^{N}k_i\cdot\Phi(x_i,y_i)\right)$$

over $\Phi_\mu(x,y)$ on a unit disk, where $Z_i = x_i + iy_i$ is a point on the boundary of the disk. In this section, we generalize the expression to the case of an arbitrary boundary with arbitrary momentum distributions on it, and investigate its properties. In Sec. II A we define them in terms of functional integrations, and in Sec. II B we relate them to matrix elements of operators.

A. Functional Method

Let us first consider a bounded simply connected domain D. We assume that its boundary contains m line segments γ_i $(i=1,\ldots,m)$, which are defined by the following set of conformal transformations g_i $(i=1,\ldots,m)$:

$$g_i : \Sigma_0 \to \gamma_i\,, \tag{2.1}$$

where Σ_0 represents the upper half of the unit circle,

$$\Sigma_0 = \{Z \in \Sigma_0 : Z = \sigma_0[\xi] \equiv e^{i\xi};\ 0<\xi<\pi\}\,. \tag{2.2}$$

Let us next define the rudimental amplitude in D, with momentum distributions p_i $(i=1,\ldots,m)$, by the following functional average:

$$\begin{bmatrix} p_1\ p_2 \cdots p_m \\ g_1\ g_2 \cdots g_m \end{bmatrix} = \left\langle \exp\left(i\sqrt{2\pi} \sum_i \int_0^\pi d\xi\, p_i(\xi)\cdot \Phi(g_i\sigma_0[\xi])\right)\right\rangle_D\,, \tag{2.3}$$

where $\langle\cdots\rangle_D$ is a functional average defined by

$$\langle A(\Phi)\rangle_D = \frac{1}{n_0}\int\cdots\int \mathfrak{D}^{(4)}\Phi(x,y) A(\Phi)\exp\left(\int_D dxdy\, \mathcal{L}(\Phi)\right)\,. \tag{2.4}$$

In order to avoid confusion we adopt the convention that for any conformal transformation, say h, such that $Z\to Z'$, we write $Z'=h[Z]$; on the other hand, $h(x)$ denotes a conformal transformation which depends upon a parameter x.

In Eq. (2.4) one has

$$n_0 = [(2\pi)^4\delta^4(0)]^{-1}\int\cdots\int \mathfrak{D}^{(4)}\Phi(x,y)\exp\left(\int_D dxdy\, \mathcal{L}\right), \tag{2.5}$$

$$\mathcal{L}(\Phi) = -\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]\,. \tag{2.6}$$

Throughout this paper we use the Euclidian metric assuming that the Wick rotation can be performed at the end of the calculations.

Next, let us consider a curve Σ which is obtained from Σ_0 by a conformal transformation Λ:

$$\{Z\in\Sigma\,;\ Z=\sigma[\xi]\equiv\Lambda\sigma_0[\xi]\}\,. \tag{2.7}$$

We analogously define a general amplitude with momentum distributions on Σ by replacing σ_0 and Σ_0 by σ and Σ, respectively, in Eq. (2.3).

Let us next define the Neumann's function[15] of D as the function which satisfies the following conditions:

$$N(Z,Z') = \ln|Z-Z'| + N'(Z,Z')\,, \tag{2.8}$$

$$\nabla_Z{}^2 N'(Z,Z') = 0\,, \tag{2.9}$$

and for Z on the boundary

$$\frac{\partial N(Z,Z')}{\partial n_Z} = \text{const}\,, \tag{2.10}$$

where n_Z is the normal to the boundary.

As has been shown in Ref. 9, the functional average can be evaluated by the following change of variables:

$$\Phi(x,y) = \Phi'(x,y) - i\frac{1}{\sqrt{2\pi}}\sum_i \int_0^\pi d\xi\, p_i(\xi) N(Z, g_i\sigma[\xi])\,. \tag{2.11}$$

Since $2\iint_D dxdy\,\mathcal{L}(\Phi)$ is a Dirichlet integral, i.e.,

$$-2\iint_D dxdy\,\mathcal{L}(\Phi) = \sum_{\mu=1}^4 D(\Phi_\mu,\Phi_\mu)\,, \tag{2.12}$$

with

$$D(\Phi,\chi) = \iint_D dxdy\left(\frac{\partial\Phi}{\partial x}\frac{\partial\chi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\chi}{\partial y}\right),$$

and since

$$D(\Phi_\mu(Z), N(Z,\zeta)) = -2\pi\Phi_\mu(\zeta)\,, \tag{2.13}$$

we obtain the following expression for the rudimental amplitude Eq. (2.3):

$$\begin{bmatrix} p_1\ p_2 \cdots p_m \\ g_1\ g_2 \cdots g_m \end{bmatrix}_\Sigma = (2\pi)^4\delta^{(4)}\left(\sum_i \int_0^\pi d\xi\, p_i(\xi)\right) \times\exp\left(\frac{1}{2}\sum_{ij}\int_0^\pi d\xi\int_0^\pi d\xi'\, p_i(\xi)p_j(\xi') \times N(g_i\sigma[\xi], g_j\sigma[\xi'])\right)\,. \tag{2.14}$$

The rudimental amplitudes defined in Eq. (2.3) and Eq. (2.4) have the following important properties, which can easily be verified from their definition:

(1) Cyclic symmetry,

$$\begin{bmatrix} p_1\ p_2 \cdots p_m \\ g_1\ g_2 \cdots g_m \end{bmatrix}_\Sigma = \begin{bmatrix} p_2 \cdots p_m\ p_1 \\ g_2 \cdots g_m\ g_1 \end{bmatrix}_\Sigma\,. \tag{2.15}$$

(2) Conformal invariance,

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ gg_1 & gg_2 & \cdots & gg_m \end{bmatrix}_\Sigma = \begin{bmatrix} p_1\ p_2 \cdots p_m \\ g_1\ g_2 \cdots g_m \end{bmatrix}_\Sigma\,, \tag{2.16}$$

where g is any conformal transformation.

(3) Reference curve-invariance,

$$\begin{bmatrix} p_1\ p_2 \cdots p_m \\ g_1\ g_2 \cdots g_m \end{bmatrix}_{\Sigma_0} = \begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ \Lambda g_1\Lambda^{-1} & \Lambda g_2\Lambda^{-1} & \cdots & \Lambda g_m\Lambda^{-1} \end{bmatrix}_\Sigma\,. \tag{2.17}$$

As discussed in Sec. I, a rudimental amplitude can be sliced into two rudimental amplitudes. Next we establish a theorem relevant to this problem.

Slicing rule. Let us consider two simply connected domains D_1 and D_2 which are connected by a common boundary γ (see Fig. 3). Let g be a conformal transformation such that g: $\Sigma \to \gamma$ and $p(\xi)$ be a momentum distribution on Σ. A rudimental amplitude of D ($D = D_1 \cup D_2$) is expressed as an average of a product of rudimental amplitudes of D_1 and D_2 over $p(\xi)$:

$$\begin{bmatrix} p_1 \cdots p_m \; p_{m+1} \cdots p_n \\ g_1 \cdots g_m \; g_{m+1} \cdots g_n \end{bmatrix}_\Sigma = n_0 \int \cdots \int \mathfrak{D}^{(4)} p(\xi) \begin{bmatrix} p_1 \cdots p_m \; p \\ g_1 \cdots g_m \; g \end{bmatrix}_\Sigma \begin{bmatrix} -p \; p_{m+1} \cdots p_n \\ g \; g_{m+1} \cdots g_n \end{bmatrix}_\Sigma . \tag{2.18}$$

The proof we give here is a heuristic one which also shows the relation of two functional measures $\mathfrak{D}^{(4)}p(\xi)$ and $\mathfrak{D}^{(4)}\Phi(\xi)$. Let $\prod_\xi \delta^{(4)}(\Phi(g\sigma[\xi]))$ be a δ functional such that

$$\int \cdots \int \mathfrak{D}^{(4)}\Phi(g\sigma[\xi]) \prod_\xi \delta^{(4)}(\Phi(g\sigma[\xi])) = 1 .$$

Let us define the measure $\mathfrak{D}^{(4)}p(\xi)$ such that

$$\prod_\xi \delta^{(4)}(\Phi(g\sigma[\xi])) = \int \cdots \int \mathfrak{D}^{(4)} p(\xi) \exp\left(\sqrt{2\pi}\, i \int_0^\pi d\xi\, p(\xi) \cdot \Phi(g\sigma[\xi])\right). \tag{2.19}$$

One obtains

$$\int \cdots \int \mathfrak{D}^{(4)}\Phi'(g\sigma(\xi)) \int \cdots \int \mathfrak{D}^{(4)} p(\xi) \exp\left(\sqrt{2\pi}\, i \int_0^\pi d\xi \{p(\xi)\Phi(g\sigma(\xi)) - p(\xi)\Phi'(g\sigma[\xi])\}\right) = 1 . \tag{2.20}$$

Inserting this expression into the integral (2.3), and factorizing it into two parts belonging to D_1 and D_2, respectively, by identifying $\mathfrak{D}^{(4)}\Phi'(g\sigma[\xi])$ as a part of the D_2 functional integration and noting that n_0 is independent of D, provided it is simply connected, we obtain Eq. (2.18).

From now on we shall use the following convention:

$$n_0 \int \cdots \int \mathfrak{D}^{(4)} p(\xi) = \int dp . \tag{2.21}$$

In order to use the rule for sewing two rudimental amplitudes, one must use the conformal-invariance property (2.16) in general, so that the two sewn domains have a common segment of boundaries.

B. Operator Expression

In the operator formalism of dual-resonance amplitudes one considers a Hilbert space generated by an infinite number of creation and annihilation operators, and a pair of coordinate and momentum operators. The commutation relations are given by

$$[a_\mu(n), a_\nu^\dagger(m)] = \delta_{\mu\nu}\delta_{nm}, \tag{2.22}$$

$$\mu, \nu = 1, 2, 3, 4, \text{ and } m, n = 1, 2, 3, \ldots$$

and

$$[x_\mu, p_\nu] = i\delta_{\mu\nu}, \tag{2.23}$$

respectively. As in Ref. 9, we introduce the following field operator $\Phi_\mu(\xi)$ and its canonical conjugate momentum $\pi_\mu(\xi)$ in this space:

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \left\{ x_\mu + \sum_{n=1}^\infty \left(\frac{2}{n}\right)^{1/2} \cos(n\xi)[a_\mu(n) + a_\mu^\dagger(n)] \right\}, \tag{2.24}$$

$$\pi(\xi) = \frac{1}{\sqrt{2\pi}} \left\{ 2p_\mu + \sum_{n=1}^\infty \frac{1}{i}\sqrt{2n}\cos(n\xi)[a_\mu(n) - a_\mu^\dagger(n)] \right\}, \tag{2.25}$$

where the parameter ξ is restricted to $0 < \xi < \pi$. This field can be associated with a free string[13,16] of length π. By taking the time to be pure imaginary, the system is described by Lagrangian Eq. (2.6).

Let us consider a Heisenberg field operator de-

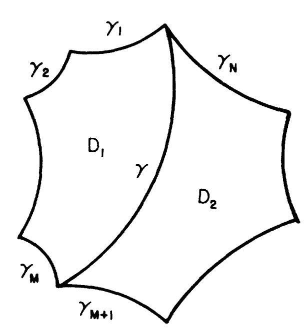

FIG. 3. Slicing of $D_1 + D_2 - \gamma$ into D_1 and D_2.

fined by

$$\Phi(\xi,\nu)=e^{-\nu H}\Phi(\xi)e^{\nu H}, \tag{2.26}$$

where H is the usual Hamiltonian given by

$$H=p^2+\sum_{n=1}^{\infty} n a^\dagger(n)a(n). \tag{2.27}$$

This Hamiltonian can be derived from Lagrangian (2.6) with the usual canonical formalism.

We may regard field (2.26) as a function of a complex variable $Z=\exp(\nu+i\xi)$ and of its complex conjugate $\overline{Z}$. The explicit form of $\Phi(Z,\overline{Z})$ is given by

$$\Phi_\mu(Z,\overline{Z})=\frac{1}{\sqrt{2\pi}}\Big(x_\mu-p_\mu\ln(Z\overline{Z})+\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}\big[(Z^n+\overline{Z}^n)a_\mu(n)+(Z^{-n}+\overline{Z}^{-n})a_\mu^\dagger(n)\big]\Big). \tag{2.28}$$

For a given Möbius transformation g on the complex Z plane, we consider the corresponding operator $O(g)$ acting in the Hilbert space of states such that[17]

$$O(g)\Phi_\mu(Z,\overline{Z})O(g)^{-1}=\Phi_\mu(g[Z],g[\overline{Z}]). \tag{2.29}$$

We can construct the corresponding generators from the Lagrangian by using Noether's theorem and the canonical quantization. Let us consider an infinitesimal transformation δg such that

$$\delta g[Z]=Z+\delta Z\equiv x_1+ix_2+\delta x_1+i\delta x_2.$$

Then using Noether's theorem we obtain the corresponding generator

$$F(\delta g)=\int_{\Sigma_0} d\sigma_i T_{ij}\delta x_j, \tag{2.30}$$

where

$$T_{ij}=\frac{\partial\Phi}{\partial x_i}\frac{\partial\mathfrak{L}}{\partial(\partial\Phi/\partial x_j)}-\delta_{ij}\mathfrak{L}. \tag{2.31}$$

Let $L_{(n)}$ be the generator of the infinitesimal transformation $Z\to Z'$ such that

$$Z'^n=Z^n+n\epsilon, \tag{2.32}$$

where n is an integer. Then we get

$$L_{(n)}=\int_0^\pi d\xi\left\{\tfrac{1}{2}\cos(n\xi)\left[\pi^2+\left(\frac{\partial\Phi}{\partial\xi}\right)^2\right]-i\sin(n\xi)\pi\frac{\partial\Phi}{\partial\xi}\right\}. \tag{2.33}$$

Using Eq. (2.24) and Eq. (2.25), we can verify that

$$:L_{(0)}:=H \tag{2.34}$$

and $L_{(1)}$, $L_{(-1)}$, and $:L_{(0)}:$ are the usual generators[18] of SL(2, R). Furthermore, $L_{(n)}$, for all n, consists of the generators of Virasoro's transformations.[19]

One can construct $O(g)$ by exponentiating these generators. We may regard $O(g)$, $g\in G$, as a representation (nonunitary) of G, where G is the group of Möbius transformations $[G\sim \mathrm{SL}(2,C)]$.

Since the operators $\Phi(\xi)$ $(0<\xi<\pi)$ commute among themselves, they can be diagonalized simultaneously to lead to the following representation:

$$\Phi(\xi)|\varphi\rangle_{\Sigma_0}=\varphi(\xi)|\varphi\rangle_{\Sigma_0}, \tag{2.35}$$

where $\varphi(\xi)$ is a real function of ξ. Since $\Phi(\xi)$ can be considered as the operator $\Phi(Z,\overline{Z})$ on the upper half of the unit circle Σ_0, we write Σ_0 to the side of the ket. We then consider a state such that $\Phi(Z,\overline{Z})$ on Σ is diagonal, where Σ is given by Eq. (2.7):

$$\Phi(Z,\overline{Z})|\varphi\rangle_\Sigma=\varphi(\xi)|\varphi\rangle_\Sigma \text{ if } Z=\sigma(\xi). \tag{2.36}$$

Obviously,

$$|\varphi\rangle_\Sigma\equiv|\varphi\rangle_{\Lambda\Sigma_0}=O(\Lambda)|\varphi\rangle_{\Sigma_0}.$$

In the Appendix we show that a matrix element of $O(g)$ $[g\in \mathrm{SL}(2,R)]$ is expressed in terms of a functional integration:

$$\langle\varphi|O(g)|\varphi'\rangle_{\Sigma_0}=\left\langle\prod_\xi\delta(\varphi(\xi)-\Phi(g^{-1}\sigma_0[\xi]))\prod_\xi\delta(\varphi'(\xi)-\Phi(\sigma_0[\xi]))\right\rangle. \tag{2.37}$$

The domain D in the functional integration is shown on Fig. 4. We restrict ourselves, for the moment, to the case where g is such that our definition of rudimental amplitude makes sense, namely, we assume that Σ_0 and $g\Sigma_0$ do not cross. The general case will be considered later on.

By taking the functional Fourier transform of Eq. (2.37), we obtain the following rudimental amplitude:

$$\iint\mathfrak{D}\varphi\mathfrak{D}\varphi'\exp\left(i\sqrt{2\pi}\int_0^\pi\int_0^\pi d\xi d\xi'[p(\xi)\varphi(\xi)+p'(\xi)\varphi'(\xi')]\right)\langle\varphi|O(g)|\varphi'\rangle_{\Sigma_0}=\begin{bmatrix}p & p'\\ 1 & g\end{bmatrix}_{\Sigma_0}.$$

Let us next introduce

$$|p,g\rangle_\Sigma=\iint\mathfrak{D}\varphi\exp\left(i\sqrt{2\pi}\int d\xi\, p(\xi)\varphi(\xi)\right)O(g)|\varphi\rangle_\Sigma. \tag{2.38}$$

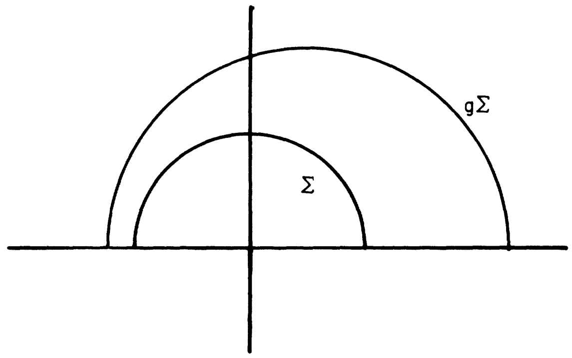

FIG. 4. D for the functional integral (2.37).

We would like to express the rudimental amplitude (2.37) as a transition probability amplitude between two states in such a way as to satisfy the three properties (2.15), (2.16), and (2.17). First, notice that (2.37) cannot be written as $\langle p,1|p',g\rangle_{\Sigma_0}$ since this would contradict the conformal-invariance property (2.16). The reason for this is that the representation $O(g)$ is not unitary in the ordinary metric of the Fock spaces.

Therefore, we introduce bra vectors ${}_\Sigma\langle p,g\|$ and ${}_\Sigma\langle\varphi\|$ which satisfy

$$ {}_\Sigma\langle p,g'g\| = {}_\Sigma\langle p,g\|O(g')^{-1}\,, \tag{2.39}$$

$$ {}_\Sigma\langle p,1\| \equiv \int \mathfrak{D}\varphi \exp\left(i\sqrt{2\pi}\int d\xi\, p(\xi)\cdot\varphi(\xi)\right) {}_\Sigma\langle\varphi\|\,, $$
$$ {}_\Sigma\langle\varphi\| = {}_{\Sigma_0}\langle\varphi|O(\Lambda)^{-1}\,. \tag{2.40}$$

With this definition one gets

$$\langle p,g\|p',g\rangle_\Sigma = \prod_\xi \delta(p(\xi)+p'(\xi))\,, \tag{2.41}$$

where the δ functional is normalized in such a way that

$$\int dp\,|p,g\rangle_\Sigma\,{}_\Sigma\langle -p,g\| = 1\,. \tag{2.42}$$

Using (2.16), (2.17), (2.37), (2.39), and (2.40), one can now show that

$$\begin{bmatrix} p & p' \\ g & g'\end{bmatrix}_\Sigma \equiv \langle p,g\|p',g'\rangle_\Sigma\,. \tag{2.43}$$

It follows from our previous results that

$$\langle p,1\|p',g\rangle_\Sigma = \langle p',g\|p,1\rangle_\Sigma\,, \tag{2.44}$$

$$\langle p,1\|p',g\rangle_\Sigma = \langle p',g'\|p'g'g\rangle_\Sigma\,, \tag{2.45}$$

and

$$\langle p,g_1\|p',g_2\rangle_{\Sigma_1} = \langle p,\Lambda g_1\Lambda^{-1}\|p',\Lambda g_2\Lambda^{-1}\rangle_{\Sigma_2 = \Lambda\Sigma_1}\,. \tag{2.46}$$

Property (2.44) indicates that the metric introduced by the symbol $\langle\ \|\ \rangle$ is such that $\langle\ \|$ is linear rather than antilinear as it is in quantum mechanics.[20] This unusual convention is necessary because Eq. (2.44) is just the cyclic symmetry (2.15) of the rudimental amplitude.

Properties (2.45) and (2.46) correspond to the conformal invariance (2.16), and to the reference-curve invariance (2.17) of rudimental amplitudes, respectively.

As we already emphasized, (2.37) defines the matrix element of $O(g)$ only when the rudimental amplitude exists. Moreover, we have defined the sewing of two rudimental amplitudes only in as much as they do not overlap. On the other hand, the operator $O(g)$ is defined more generally by Eq. (2.29), and it follows from Eq. (2.42) that

$$\int dp \langle q_1,1\|O(g_1)|p,1\rangle_\Sigma \langle -p,1\|O(g_2)|q_2,1\rangle_\Sigma = \langle q_1,1\|O(g_1g_2)|q_2,1\rangle_\Sigma$$

even if the corresponding sewing of rudimental amplitudes cannot be done, or if those operators cannot be defined by rudimental amplitudes. Formulas (2.29) and (2.42) allow us therefore to generalize the results of Sec. II A.

In particular, let us apply Eq. (2.42) to the case of two rudimental amplitudes with domains D_1 and D_2 such that $D_1\subset D_2$ (see Fig. 5). One can see that the matrix element of $O(g_1g_2)$ corresponds to the rudimental amplitude obtained by integrating only at the points of D_2 which are outside D_1. In this way, for example, one gives a meaning to the equation $OO^{-1}=1$ in this generalized sewing rule.

Assume now that

$$\langle q_1,1\|O(g_1)|p,1\rangle_\Sigma$$

is such that Σ and $g_1\Sigma$ intersect. Let us call D_1

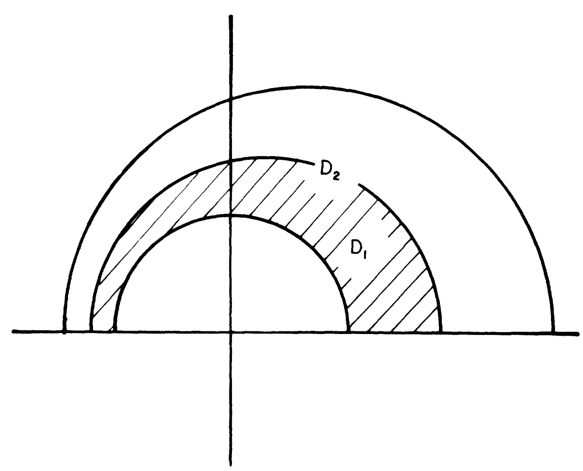

FIG. 5. Sewing of two rudimental amplitudes with domains D_1 and D_2 such that $D_1\subset D_2$.

the domain shown on Fig. 6(a), which is the natural extension of the case considered in Sec. II A. If

$$\langle p, 1 \| O(g_2) | q_2, 1\rangle_\Sigma$$

corresponds to an ordinary rudimental amplitude [Fig. 6(b)],

$$\langle q_1, 1 \| O(g_1 g_2) | q_2, 1\rangle$$

is given by the functional integral obtained by integrating over the points of $D_1 \cup D_2$ which are outside $D_1 \cap D_2$. Indeed this domain is bounded by the curves $g_1\Sigma$ and $g_2^{-1}\Sigma$ as shown in Fig. 6(c).

We now introduce multi-Reggeon states by

$$|p_1, g_1; p_2, g_2; \ldots; p_n, g_n\rangle_\Sigma = \int dp\, |p, g_0\rangle_\Sigma \begin{bmatrix} -p & p_1 & \cdots & p_n \\ g_0 & g_1 & \cdots & g_n \end{bmatrix}_\Sigma \tag{2.47}$$

and

$$_\Sigma\langle p_1 | g_1; p_2, g_2; \ldots; p_n, g_n \| = \int dp \begin{bmatrix} -p & p_1 & \cdots & p_n \\ g_0 & g_1 & \cdots & g_n \end{bmatrix}_\Sigma {}_\Sigma\langle p, g_0 \| . \tag{2.48}$$

Using the completeness property (2.42), we can easily show that definitions (2.47) and (2.48) do not depend on g_0.

Using the conformal invariance property (2.16), we can show that

$$O(g) | p_1, g_1; \ldots; p_n, g_n\rangle_\Sigma = | p_1, gg_1; \ldots; p_n, gg_n\rangle_\Sigma , \tag{2.49}$$

$$_\Sigma\langle p_1, g_1; \ldots; p_n, g_n \| O(g)^{-1} = {}_\Sigma\langle p_1, gg_1; \ldots; p_n, gg_n \| . \tag{2.50}$$

We can also prove that

$$\begin{bmatrix} p_1 & \cdots & p_n \\ g_1 & \cdots & g_n \end{bmatrix}_\Sigma = \langle p_{k_1}, g_{k_1}; p_{k_2}, g_{k_2}; \ldots; p_{k_l}, g_{k_l} \| p_{k_{l+1}}, g_{k_{l+1}}; \ldots; p_{k_n}, g_{k_n}\rangle_\Sigma ,$$

where $k_1, \ldots, k_n$ is any cyclic permutation of $1, \ldots, n$, and l is an arbitrary integer between 0 and n.

The slicing rule of Sec. II A is now a simple consequence of Eq. (2.42), which here also allows us to generalize the sewing procedure with essentially the rule that one integrates functionally at the points of the union between the two sewn domains which are outside their intersection.

In the usual operator formalism one uses the matrix $\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which satisfies

$$\Gamma \tilde{g} \Gamma = g^{-1} , \tag{2.51}$$

where g and $\tilde{g}$ are related by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{2.52}$$

For $g \in SL(2, R)$ one can show that

$$O(\tilde{g}) = O^\dagger(g) . \tag{2.53}$$

So, if there exists an operator $O(\Gamma)$ which satisfies the analogous operator equation

$$O(\Gamma) O^\dagger(g) O(\Gamma) = O(g)^{-1} \tag{2.54}$$

for all $g \in SL(2, R)$, we could write

$$\langle p, g \| = \langle p, g | O(\Gamma) . \tag{2.55}$$

However, $O(\Gamma)$ does not exist in this Hilbert space. This can be shown by noticing that from

$$O(\Gamma) \Phi(Z, \bar{Z}) O(\Gamma) | \varphi\rangle_{\Sigma_0} = \Phi\left(\frac{1}{Z}, \frac{1}{\bar{Z}}\right) | \varphi\rangle_{\Sigma_0} , \tag{2.56}$$

it follows that

$$O(\Gamma) | \varphi\rangle_{\Sigma_0} = | \varphi\rangle_{\Sigma_0} . \tag{2.57}$$

Since the set of states $|\varphi\rangle_{\Sigma_0}$ is complete, it follows that $O(\Gamma) = 1$, which is a contradiction.

III. MULTIPARTICLE VENEZIANO AMPLITUDES AND N-REGGEON AMPLITUDE

In this section we first write the N-particle Veneziano amplitude in terms of a rudimental amplitude defined in Sec. II, in such a form as to be factorized by using the slicing rule. We also discuss the N-Reggeon amplitude.

Let us consider N points on a unit circle (see Fig. 1), which divide the circle into N segments γ_i ($i = 1, \ldots, N$). Consider a set of Möbius transformations such that

$$g_i : \Sigma \to \gamma_i , \tag{3.1}$$

$$g_i \sigma[0] = Z_i , \quad g_i \sigma[\pi] = Z_{i+1} . \tag{3.2}$$

Then the N-particle Veneziano amplitude [(A11) of Ref. 9] can be written as

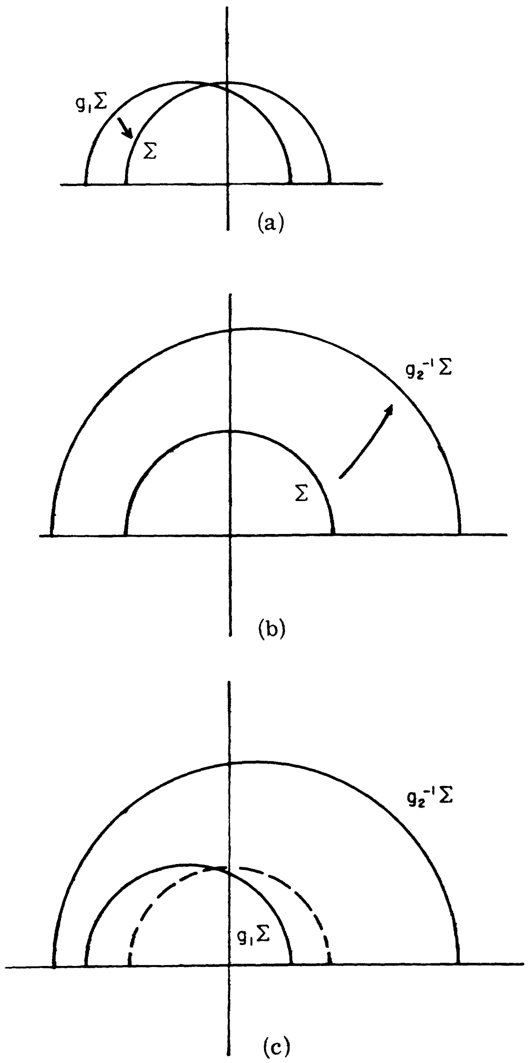

FIG. 6. Sewing of two rudimental amplitudes with domains D_1 and D_2 in the case when the boundaries of D_1 intersect: (a) Boundaries of D_1: Σ and $g_1\Sigma$ which intersect. (b) Domain D_2. (c) Domain of the sewn rudimental amplitude.

$$(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{n} k_i\right)V_N(k_1,\ldots,k_N)$$
$$=\int\cdots\int d\mathcal{V}(Z_1,\ldots,Z_N)\prod_{i=1}^{N}|Z_i-Z_{i-1}|^{\alpha_0}$$
$$\times\lim_{\epsilon\to 0}\prod_{i=1}^{N}E(g_i)^{-1}\begin{bmatrix}k_1(\xi) & k_2(\xi) & \cdots & k_N(\xi)\\ g_1 & g_2 & \cdots & g_N\end{bmatrix}_{\Sigma}, \tag{3.3}$$

where

$$k_i(\xi)=k_i\rho_\epsilon(\xi), \tag{3.4}$$

$$Z_j=e^{i\theta_j}, \tag{3.5}$$

$$E(g_i)=\exp\left(\alpha_0\int_0^\pi d\xi\int_0^\pi d\xi'\rho_\epsilon(\xi)\rho_\epsilon(\xi')\times\ln|g_i\sigma[\xi]-g_i\sigma[\xi']|\right), \tag{3.6}$$

and $d\mathcal{V}(Z_1,\ldots,Z_N)$ is a Möbius-transformation-invariant measure[21] defined by

$$d\mathcal{V}(Z_1,\ldots,Z_N)$$
$$=\left(\prod_{i=1}^{N-1}d\theta_i\right)\frac{|Z_1-Z_N||Z_N-Z_{N-1}||Z_1-Z_{N-1}|}{\prod_{i=1}^{N}|Z_i-Z_{i-1}|}. \tag{3.7}$$

The $\rho_\epsilon(\xi)$ is a smearing function which we assume to have the following property:

$$\lim_{\epsilon\to 0}\rho_\epsilon(\xi)=\delta(\xi), \tag{3.8}$$

so that

$$E(g)=E(1)\left(\left|\frac{\partial g\sigma[\xi]}{\partial\sigma[\xi]}\right|_{\xi=0}^{\alpha_0}+O(\epsilon)\right). \tag{3.9}$$

As a preparation for the factorization of the amplitude let us briefly discuss Möbius transformations.

A Möbius transformation

$$Z\to Z'=g[Z]=\frac{aZ+b}{cZ+d} \tag{3.10}$$

is represented by the following 2×2 unimodular complex matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}. \tag{3.11}$$

Let Z_1, Z_2, and Z_3 be a set of three points on the complex plane which are transformed from a given set of three points, say Y_1, Y_2, and Y_3 by a transformation g. As is well known, g is uniquely specified by Z_1, Z_2, and Z_3 for a fixed set of Y_1, Y_2, and Y_3. Following Lovelace, we denote it by

$$g=\begin{bmatrix}Y_1 & Y_2 & Y_3\\ Z_1 & Z_2 & Z_3\end{bmatrix}. \tag{3.12}$$

From now on we choose the Y's to be 1, -1, and 0, respectively; and we specify the curve Σ to be Σ_0. Geometrically g is represented by the figure shown in Fig. 7.

It is convenient to use all these representations – Eq. (3.11), Eq. (3.12), and Fig. 7 – in the following discussions, so we tabulate in Fig. 8 those transformations that will be used frequently in this section.

The transformation g_i is defined by Eqs. (3.1) and (3.2), and can be expressed as

$$g_i=\tilde{g}_i\tau,\quad \tilde{g}_i=\begin{bmatrix}1 & -1 & 0\\ Z_i & Z_{i+1} & \tilde{Z}_i\end{bmatrix},\quad \tau=\begin{bmatrix}1 & -1 & i\\ 1 & -1 & 0\end{bmatrix}, \tag{3.13}$$

where $\tilde{Z}_i$ is an arbitrary point on the unit circle which will be fixed later. The transformation τ maps the real axis onto the unit circle keeping 1 and -1 fixed.

We factor out the Z dependence of $E(g_i)$ by using Eq. (3.9):

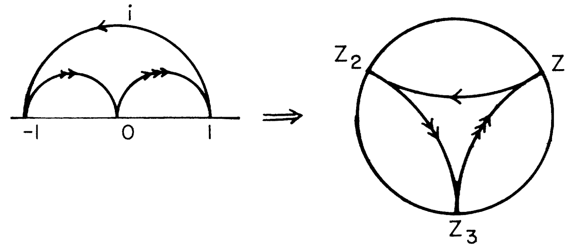

FIG. 7. Geometrical representation of $\begin{bmatrix}1 & -1 & 0\\ Z_1 & Z_2 & Z_3\end{bmatrix}$.

$$E(g_i)=\frac{1}{2^{\alpha_0}}\frac{|Z_{i+1}-Z_i|^{\alpha_0}|Z_i-\tilde{Z}_i|^{\alpha_0}}{|Z_{i+1}-\tilde{Z}_i|^{\alpha_0}}E(1). \tag{3.14}$$

Then inserting (3.14) into (3.3), we obtain

$$(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{N}k_i\right)V_N(k_1,\ldots,k_N)=\lim_{\epsilon\to 0}[2^{\alpha_0}/E(1)]^N\int\cdots\int d\mathfrak{V}(Z_1,\ldots,Z_N)\prod_{i=1}^{N}\frac{|Z_{i+1}-\tilde{Z}_i|^{\alpha_0}}{|Z_i-\tilde{Z}_i|}\begin{pmatrix}k_1 & \cdots & k_N\\ g_1 & \cdots & g_N\end{pmatrix}_{\Sigma_0}. \tag{3.15}$$

From the results of this section and of Sec. IV, it will follow that the N-Reggeon amplitude is the generalization of (3.15) obtained by dropping condition (3.4) and taking general extended distributions $p_i(\xi)$ for all Reggeons. Namely, we shall prove that the N-Reggeon amplitude is written

g	$\dfrac{aZ+b}{cZ+d}$	$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$	$\begin{bmatrix}1 & -1 & 0\\ Z_1 & Z_2 & Z_0\end{bmatrix}$
r	$-\dfrac{1}{Z}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{bmatrix}1 & -1 & 0\\ -1 & -1 & \infty\end{bmatrix}$
τ	$\dfrac{Z-i}{1-iZ}$	$\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$	$\begin{bmatrix}1 & -1 & 0\\ 1 & -1 & -i\end{bmatrix}$
κ	$\dfrac{1+Z}{1-3Z}$	$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}$	$\begin{bmatrix}1 & -1 & 0\\ -1 & 0 & 1\end{bmatrix}$
$h(s)$	$\dfrac{Z+\tanh(1/2s)}{1+Z\tanh(1/2s)}$	$\begin{pmatrix}\cosh(1/2s) & \sinh(1/2s)\\ \sinh(1/2s) & \cosh(1/2s)\end{pmatrix}$	$\begin{bmatrix}1 & -1 & 0\\ 1 & -1 & \tanh(1/2s)\end{bmatrix}$

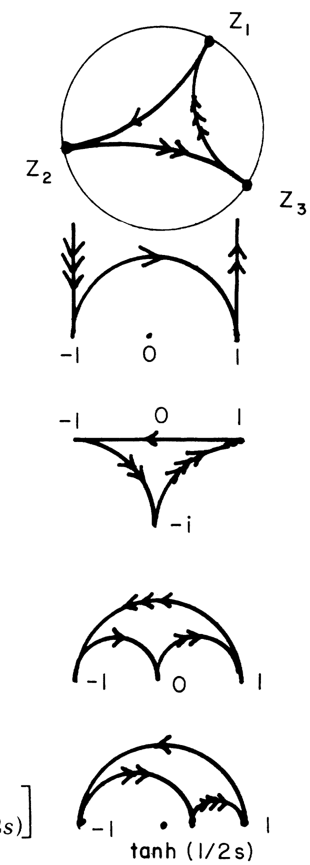

FIG. 8. Table of the Möbius transformations.

$$(2\pi)^4\delta^{(4)}\left(\sum_i \int d\xi\, p_i(\xi)\right) R_N(p_1(\xi),\ldots,p_N(\xi)) = \iint d\mathcal{V}(Z_1,\ldots,Z_N)\prod_{i=1}^{N}\left(\frac{|Z_{i+1}-\tilde{Z}_i|}{|Z_i-\tilde{Z}_i|}\right)^{\alpha_0}\begin{pmatrix} p_1 & \cdots & p_N \\ g_1 & \cdots & g_N\end{pmatrix}_{\Sigma_0}. \quad (3.16)$$

In this picture the ith Reggeon is described by an extended distribution along the unit circle between Z_i and Z_{i+1}. In the particular case of a distribution reduced to a δ function, at Z_i Eq. (3.16) is reduced to Eq. (3.15) when the appropriate limit is taken after multiplying by factor $[2^{\alpha_0}/E(1)]^N$. Lowest-lying external particles are therefore represented by external momenta located at the corresponding Koba-Nielsen points. This last feature is well known in the analog model.[8] A representation of general Reggeons by extended momentum distributions has been proposed by Alessandrini.[22] However, he takes distributions extended from Z_{i-1} to Z_{i+1}, instead of Z_i to Z_{i+1} as we have here, so that in his picture distributions overlap, and as we shall see this is not satisfactory from the point of view of factorization.

Formula (3.16) involves N arbitrary parameters $\tilde{Z}_i$ ($i=1,\ldots,N$). Let us investigate how it actually depends on them. For this we make use of the following identity:

$$g_i = \begin{bmatrix} 1 & -1 & 0 \\ Z_i & Z_{i+1} & \hat{Z}_i \end{bmatrix} \tau h(\beta),$$
$$\beta = \ln|(Z_{i+1}, \tilde{Z}_i, Z_i, \hat{Z}_i)|, \quad (3.17)$$

where $h(\beta)$ is defined in Fig. 8:

$$h(\beta) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & \tanh\frac{1}{2}\beta \end{bmatrix}.$$

In general we denote cross ratios by

$$(Z_a, Z_b, Z_c, Z_d) = \frac{(Z_a - Z_b)(Z_c - Z_d)}{(Z_a - Z_d)(Z_c - Z_b)}. \quad (3.18)$$

Moreover, one can write

$$\left|\frac{Z_{i+1}-\tilde{Z}_i}{Z_i-\tilde{Z}_i}\right|^{\alpha_0} = \left|\frac{Z_{i+1}-\hat{Z}_i}{Z_i-\hat{Z}_i}\right|^{\alpha_0} e^{\beta\alpha_0}. \quad (3.19)$$

Therefore, changing $\tilde{Z}_i$ into $\hat{Z}_i$ in formula (3.16) amounts to multiplying the ith Reggeon line by $e^{\beta\alpha_0}h(\beta)$. As we shall see this corresponds to a gauge transformation, and therefore all dependence on $\tilde{Z}_i$ disappears when (3.16) is taken between physical states, or multiplied by spurious-free propagator, as we shall always do.

Assume now that we change all Z_i and $\tilde{Z}_i$ by the same Möbius transformation g; then one can write

$$\begin{bmatrix} 1 & -1 & 0 \\ g[Z_i] & g[Z_{i+1}] & g[\tilde{Z}_i] \end{bmatrix} = g\begin{bmatrix} 1 & -1 & 0 \\ Z_i & Z_{i+1} & \tilde{Z}_i \end{bmatrix}. \quad (3.20)$$

From formula (2.16) it follows that the functional integral does not change. Moreover the integration volume of (3.16) and the factor which is at the power α_0 are Möbius-invariant. Therefore, the integrand of (3.16) is Möbius-invariant. Thus the N-Reggeon amplitude (3.16) is independent, up to gauge transformations, of the three fixed points chosen when one integrates over the Z variables.

In practice two different ways of choosing the $\tilde{Z}_i$ are of interest, depending on the particular property of the amplitude one is interested in.

1. Cyclic Symmetry. Formula (3.16) will be cyclic-symmetric if the $\tilde{Z}_i$ are chosen in a cyclically symmetric way. An example of such a choice is

$$\tilde{Z}_i = Z_{i-1}. \quad (3.21)$$

As we shall see, this corresponds to Lovelace's N-Reggeon amplitude.[23] However, with this choice, factorization is true only up to gauge transformations, which we shall discuss in Sec. V.

2. Factorization. We shall show how to factorize (3.16) in any tree configuration. This will be done by appropriate choices of $\tilde{Z}_i$ which will be such that, in general, cyclic symmetry is only true up to gauge transformations.

First let us change the integration variable Z_i to variables which are Möbius-invariant and more appropriate for factorization. From (3.17) it follows that if the following equation is satisfied,

$$\begin{bmatrix} 1 & -1 & 0 \\ Z_2 & Z_1 & Z_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} h(s)r, \quad (3.22)$$

then the parameter s is expressed by

$$s = \ln|(Z_4, Z_2, Z_3, Z_1)|, \quad (3.23)$$

where r is defined in Fig. 8: $r[Z] = -1/Z$.

We try to express the integration variables $Z_2,\ldots,Z_{N-2}$ of integral (3.3) in terms of the three fixed points Z_1, Z_N, Z_{N-1} and $N-3$ cross ratios as follows. Choose one of the integration variables – say Z_l – and express it by a similar expression, as (3.22). Explicitly it is given by

$$Z_l = \begin{bmatrix} 1 & -1 & 0 \\ Z_1 & Z_{N-1} & Z_N \end{bmatrix} h(s_l) r[0]. \quad (3.24)$$

Then we connect the relevant four points Z_1, Z_{N-1}, Z_N, Z_l in terms of the circular arcs orthogonal to the unit circle, as shown in Fig. 9, indicating that Z_l is expressed in terms of the other corner points of the circular quadrilateral $[1, N-1, N, l]$ and its cross ratio. Next choose another point Z_m and express it in terms of Z_1, Z_{N-1}, Z_l, and a

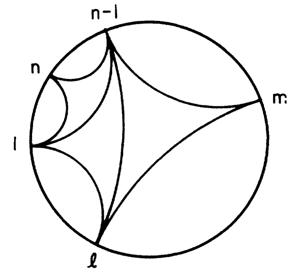

FIG. 9. Triangulation of D in terms of circular arcs.

cross ratio using the circular quadrilateral $[m, N-1, 1, l]$ (see Fig. 9):

$$Z_m = \begin{bmatrix} 1 & -1 & 0 \\ Z_l & Z_{N-1} & Z_1 \end{bmatrix} h(s_m) r [0] , \tag{3.25}$$

$$Z_m = \begin{bmatrix} 1 & -1 & 0 \\ Z_1 & Z_{N-1} & Z_N \end{bmatrix} h(s_l) r \kappa^{-1} h(s_m) r [0] , \tag{3.26}$$

where we used (3.22) and κ is defined in Fig. 8:

$$\kappa = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} .$$

$$\begin{bmatrix} 1 & -1 & 0 \\ Z_l & Z_{N-1} & Z_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ Z_{N-1} & Z_1 & Z_l \end{bmatrix} \kappa^{-1} .$$

Repeating this procedure, we can express all Z_i successively in terms of s_i.

There are as many ways of expressing them as there are possible triangulations of Fig. 1 in terms of orthogonal circular arcs. An example corresponding to multi-Regge exchange is given by Fig. 10. In a given triangulation, each triangle represents a vertex. To each arc common to two triangles will be associated a propagator whose integration variable is the s variable computed in the way described previously from the rectangle obtained by joining the two neighboring triangles.

We shall therefore change the integration variables from Z_i to s_i in (3.16) in order to obtain a factorized form.

For a given triangulation let us choose $\tilde{Z}_i$ of (3.16) to the opposite corner of the arc $(Z_i Z_{i+1})$ of the triangle which contains $(Z_i Z_{i+1})$.

In the case of the triangulation of Fig. 10, we have

$$\begin{aligned} &\tilde{Z}_i = Z_{N-1}, \quad i = N, 1, \ldots, N-3 \\ &\tilde{Z}_{N-2} = Z_{N-3} , \\ &\tilde{Z}_{N-1} = Z_1 ; \end{aligned} \tag{3.27}$$

$$\begin{aligned} &Z_i = \begin{bmatrix} 1 & -1 & 0 \\ Z_1 & Z_{N-1} & Z_N \end{bmatrix} f_i [1] , \\ &g_i = \begin{bmatrix} 1 & -1 & 0 \\ Z_1 & Z_{N-1} & Z_N \end{bmatrix} f_i \tau , \end{aligned} \tag{3.28}$$

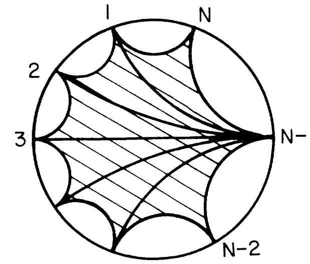

FIG. 10. The triangulation corresponding to multi-Regge exchange.

where the f_i's are given by

$$\begin{aligned} &f_1 = h(s_2) r \kappa , \\ &f_i = f_{i-1} \kappa h(s_{i-1}) r \kappa , \quad i = 2, \ldots, N-2 \\ &f_{N-2} = f_{N-3} \kappa , \\ &f_{N-1} = \kappa , \\ &f_N = \kappa^{-1} . \end{aligned} \tag{3.29}$$

We then compute the Jacobian using the following partial derivative formula for s_i which is defined by Eq. (3.23)

$$\frac{\partial \theta_i}{\partial s_i} = \left| \frac{\partial Z_i}{\partial s_i} \right| = \frac{|Z_i - Z_1| \, |Z_i - Z_{N-1}|}{|Z_{N-1} - Z_1|} . \tag{3.30}$$

We obtain

$$\begin{aligned} \prod_{i=2}^{N-2} ds_i &= \frac{|Z_1 - Z_N| \, |Z_N - Z_{N-1}| \, |Z_1 - Z_{N-1}|}{\prod_{i=1}^{N} |Z_i - Z_{i-1}|} \prod_{i=2}^{N-2} d\theta_i \\ &= d\mathcal{V}(Z_1, \ldots, Z_N) , \end{aligned} \tag{3.31}$$

which is obviously Möbius-invariant.

The factor

$$\prod_{i=1}^{N} \frac{|Z_{i+1} - \tilde{Z}_i|}{|Z_i - Z_i|}$$

in the integrand (3.16) is then expressed in terms of the cross ratios s_i, and we finally obtain

$$\begin{aligned} &(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{N} \int p_i d\xi \right) R_n(p_1 \ldots, p_N) \\ &\quad = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=2}^{N-2} (ds_i e^{\alpha_0 s_i}) \begin{bmatrix} p_1 & \cdots & p_N \\ g_1 & \cdots & g_N \end{bmatrix}_{\Sigma_0} \end{aligned} \tag{3.32}$$

Apart from the factor $e^{\alpha_0 s_i}$ the integrand of (3.32) depends upon s_i through the dependence of the g_i's on those variables.

We have obtained (3.32) using a particular triangulation, but one can verify that the same formula is true in all cases with the appropriate definitions of g_i.

As we shall see in Sec. IV, Eq. (3.32) is factorized directly by using the slicing rule of Sec. II, without performing any gauge transformation. On the other hand, this expression becomes cyclically symmetric after rather complicated gauge transformations are multiplied.

IV. FEYNMAN-LIKE RULES OF DUAL-RESONANCE AMPLITUDES

If we use the slicing rule along the arcs in a given triangulation discussed in Sec. III, we can express multiparticle Veneziano amplitudes in terms of the three-Reggeon vertex and the Reggeon propagator. In this section we present the expression of these in terms of rudimental amplitudes and show that the multiparticle Veneziano amplitudes are constructed from them by Feynman-like rules.

These rules simply consist of:

(1) drawing a Feynman-like diagram,[24]

(2) writing down factors corresponding to the vertices and propagators of the diagram, and

(3) integrating (functionally in this case) over internal Reggeon momenta.

A. Reggeon Propagator

The Reggeon propagator is given by the following matrix element[25] of the operator P:

$$\langle p \| P | p' \rangle = \int_{-\infty}^{+\infty} ds\, e^{\alpha_0 s} \begin{bmatrix} p & p' \\ 1 & h(s)r \end{bmatrix}_{\Sigma_0}$$

$$= \int_{-\infty}^{+\infty} ds\, e^{\alpha_0 s} \langle p \| O(h(s)r) | p' \rangle_{\Sigma_0} . \quad (4.1)$$

Using (2.44), (2.45), and the equality

$$rh(s)r = h(-s) = h^{-1}(s) , \quad (4.2)$$

we can prove that $\langle p \| P | p' \rangle$ is symmetric under the exchange of p and p'.

The operator P is invariant when multiplied by the gauge-transformation operator

$$G(u) = e^{\alpha_0 u} O(h(u)) , \quad (4.3)$$

which follows from the identity

$$h(u)h(s) = h(u+s) . \quad (4.4)$$

In order to apply the gauge transformation to the propagator from the right, we must multiply by $e^{\alpha_0 u} O(h(-u))$ because, in general,

$$\int dp \langle q \| O_1 | -p \rangle \langle p' \| O_2 | p \rangle = \langle p' \| O_2 O_1^{-1} | q \rangle .$$

This relation combined with Eq. (4.2) shows that Eq. (4.4) is invariant by gauge transformations applied on both sides.

B. 3-Reggeon Vertex

We define the 3-Reggeon vertex as the following rudimental amplitude:

$$\Gamma(p_1, p_2, p_3) = \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & \kappa & \kappa^{-1} \end{bmatrix}_{\Sigma_0} . \quad (4.5)$$

The explicit form of the transformation κ is given in Fig. 8, from which one can derive

$$\kappa^2 = \kappa^{-1} . \quad (4.6)$$

Using (2.15), (2.16), and (4.5), we can prove the cyclic symmetry of Γ:

$$\Gamma(p_1, p_2, p_3) = \Gamma(p_2, p_3, p_1) . \quad (4.7)$$

Indeed (4.5) corresponds to a special case of the N-Reggeon vertex we gave in Eq. (3.16) with $\tilde{Z}_i = Z_{i-1}$.

The sewing of the vertices and propagators is done according to our results of Sec. II. Using relations similar to (3.29), one can verify that for any dual configuration this leads to an expression identical to (3.16) except that for all Reggeon legs $\tilde{g}_i$ appears instead of g_i [see formula (3.13)]. One thus obtains what we call a N-Reggeon vertex where the functional integral is performed over a domain bounded by arcs of circles orthogonal to the unit circle (shaded area of Fig. 10).

As it is clear from their construction, those N-Reggeon vertices can also be used as building blocks. On the other hand, the N-Reggeon amplitude (3.16) will be obtained if one multiplies $O(\tau)$ on all Reggeon distributions. This is equivalent to changing the reference curve Σ_0 to the segment $[-1, +1]$ on the real axis.[26]

As shown in Sec. III, all operators are transformed by the corresponding similarity transformation, and this leads to (3.16) where the domain of the functional integral is the unit disk. As already mentioned in Sec. III, the N-particle Veneziano amplitude (3.15) is obtained by taking the appropriate limit of (3.16).

Using the identity

$$\frac{1}{E(1)} \begin{bmatrix} k & \cdots \\ 1 & \cdots \end{bmatrix} = \frac{1}{E(h(s))} \begin{bmatrix} k & \cdots \\ h(s) & \cdots \end{bmatrix},$$

one can prove that Eq. (3.15) is indeed gauge-invariant.

The N-Reggeon vertex defined by using Σ_0 is suitable for a geometrical description of sewing. However, (3.16) can also be used with the modified propagator $O(\tau)\,P\,O(\tau^{-1})$. Let us remark that the gauge operator is not changed since

$$\tau h(\gamma) \tau^{-1} = h(\gamma) .$$

V. CONNECTION WITH THE USUAL OPERATOR FORMALISM

In this section, we establish the connection of our previous results with the usual operator formalism.[27] In particular, we shall show that the symmetric N-Reggeon amplitude (3.16) with the choice (3.21) corresponds to the N-Reggeon vertex of Lovelace.[23]

Up to now, we have described each Reggeon in term of a distribution of momentum on the upper-half unit circle Σ_0. This is natural from the point of view of Ref. 9 (see also Sec. II B) since the corresponding time ν appeared to be $\nu=-\ln|z|$, and Σ_0 thus corresponds to describing the states at the fixed time $\nu=0$. However, explicit connection with the usual operator formalism requires using instead of Σ_0 a half circle Σ, orthogonal to the real axis, going through the points 0 and 1. More precisely, we connect the points of Σ_0 and the points of Σ through the transformation

$$\Lambda=\begin{bmatrix}\infty & 0 & 1\\ 0 & 1 & -1\end{bmatrix}\Rightarrow\begin{pmatrix}0 & 1\\ -2 & 1\end{pmatrix}. \tag{5.1}$$

As discussed in Sec. II, all operators in this new description are deduced from the expressions given in Sec. IV through the similarity transformation

$$O_\Sigma=O(\Lambda)^{-1}O_{\Sigma_0}O(\Lambda)\ . \tag{5.2}$$

In particular, the 2×2 matrices corresponding to r and τ are

$$R\equiv\Lambda^{-1}r\Lambda=\begin{bmatrix}\infty & 0 & 1\\ \frac{1}{2} & 1 & 0\end{bmatrix}\Rightarrow\begin{pmatrix}-1 & 1\\ -2 & 1\end{pmatrix}, \tag{5.3}$$

$$T\equiv\Lambda^{-1}\tau\Lambda=\begin{bmatrix}\infty & 0 & 1\\ \frac{1}{2}(1+i) & 0 & 1\end{bmatrix}\Rightarrow\begin{pmatrix}i & 0\\ 1+i & -1\end{pmatrix}. \tag{5.4}$$

It is also easy to check that

$$O(\Lambda)^{-1}O(h(\gamma))O(\Lambda)=e^{-\gamma W}\ , \tag{5.5}$$

where

$$W=L_{(0)}-L_{(-1)}\ , \tag{5.6}$$

and $L_{(n)}$ is given by Eq. (2.33).

Therefore the gauge transformation $e^{\gamma\alpha_0}O(h(\gamma))$, which we introduced, just becomes the standard gauge operator $e^{\gamma(\alpha_0-W)}$ of the usual operator formalism. This shows that our propagator which was found to be invariant by $e^{\gamma\alpha_0}O(h(\gamma))$ does not propagate the spurious states. Indeed, one has

$$O(\Lambda)^{-1}\int_{-\infty}^{+\infty}ds\, e^{\alpha_0 s}\,O(h(s)r)\,O(\Lambda)=\int_0^1 dx(1-x)^{\alpha_0-1}x^{-\alpha_0-1}O(D(x))\ , \tag{5.7}$$

where we have made the change of variable

$$e^s=\frac{1-x}{x}\ ,$$

and where

$$D(x)=\begin{bmatrix}0 & 1 & \infty\\ 1 & 0 & x\end{bmatrix}\Rightarrow\begin{pmatrix}x & -x\\ 1 & -x\end{pmatrix}. \tag{5.8}$$

Our propagator coincides with the standard twisted spurious-free propagator[11]; indeed one can write

$$D(x)=d(x)\Gamma\ , \tag{5.9}$$

where

$$d(x)=\begin{bmatrix}0 & 1 & \infty\\ x & 0 & 1\end{bmatrix}$$

is the Möbius transformation which corresponds to the operator

$$x^{L(0)}\Omega(1-x)^W\ , \tag{5.10}$$

and Ω is the twisted operator which corresponds to the matrix

$$\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}. \tag{5.11}$$

It is remarkable that, with the method of factorization we have developed, the twisted propagator is automatically spurious-free. This is not the case in the usual approach.

Finally, it is convenient, following Lovelace, to choose the points $\tilde{z}_i$ to be z_{i-1} which is one of the cyclically symmetric choices we discussed in Sec. III. In this case the Reggeon distribution is associated with the transformations

$$V_i=\Lambda^{-1}g_i\Lambda=\begin{bmatrix}\infty & 0 & 1\\ z_{i-1} & z_i & z_{i+1}\end{bmatrix} \tag{5.12}$$

which are the ones introduced by Lovelace.[23] As an illustration let us discuss the sewing of two symmetric N-Reggeon vertices. The notations are displayed on Fig. 11 and the two rudimental amplitudes to be sewn are

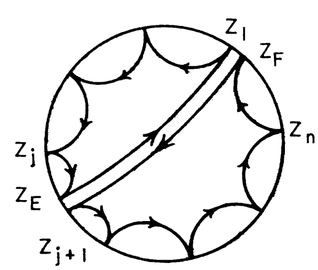

FIG. 11. Sewing of two symmetric multi-Reggeon vertices.

$$\langle p_1, \tilde{V}_1; p_2, V_2; \ldots; p_j, V_j \| p_E, V_E\rangle_\Sigma ,$$

and

$$\langle p_F, V_F \| p_{j+1}, \tilde{V}_{j+1}; p_{j+2}, V_{j+2}; \ldots; p_N, V_N\rangle_\Sigma ,$$

where V_i is given by Eq. (5.12) and

$$\tilde{V}_1 = \begin{bmatrix} \infty & 0 & 1 \\ z_E & z_1 & z_2 \end{bmatrix}, \quad V_E = \begin{bmatrix} \infty & 0 & 1 \\ z_j & z_E & z_1 \end{bmatrix},$$

$$\tilde{V}_{j+1} = \begin{bmatrix} \infty & 0 & 1 \\ z_F & z_{j+1} & z_{j+2} \end{bmatrix}, \quad V_F = \begin{bmatrix} \infty & 0 & 1 \\ z_{j+1} & z_F & z_n \end{bmatrix}. \tag{5.13}$$

In order to sew, one inserts the propagator and integrates functionally over the intermediate Reggeon distribution. Taking into account the completeness property (2.42), we can write

$$\langle p_1, \tilde{V}_1; \ldots; p_j, V_j \| p_{j+1}, \tilde{V}_{j+1}; \ldots; p_N, V_N\rangle_\Sigma = \int dp_E dp_F \langle p_1, \tilde{V}_1; \ldots; p_j, V_j \| p_E, V_E\rangle_\Sigma \langle -p_E, V_E \| -p_F, V_F\rangle_\Sigma \langle p_F, V_F \| p_j, \tilde{V}_{j+1}; \ldots; p_N, V_N\rangle_\Sigma . \tag{5.14}$$

In order that $\langle -p_E, V_E \| -p_F, V_F\rangle$ be identified with the matrix element of the propagator (5.8) the end points of the two momentum distributions should coincide. This is so because $D(x)$ leaves invariant the points 0 and 1. Therefore, one should have

$$z_E = z_{j+1}, \quad z_F = z_1 . \tag{5.15}$$

In this case one can check that indeed

$$\langle -p_E, V_E \| -p_F, V_F\rangle_\Sigma = \langle -p_E \| -p_F D(x)\rangle_\Sigma ,$$

since

$$D(x) = V_E^{-1} V_F \tag{5.16}$$

if

$$x = (z_N, z_{j+1}, z_1, z_j) . \tag{5.17}$$

Condition (5.15) was obtained by Lovelace[23] in the operator formalism without giving any clear interpretation.[28]

In order to obtain the N-Reggeon symmetric vertex we have to change $\tilde{V}_1$ and $\tilde{V}_{j+1}$, respectively. As we have shown in Sec. III, this is achieved if the Reggeons p_1 and p_{j+1} in Eq. (5.14) are multiplied by the appropriate gauge transformations so that Z_E in $\tilde{V}_1$ becomes Z_N, and Z_F in $\tilde{V}_{j+1}$ becomes Z_j, respectively.

Finally, as far as the integration over the Z variables is concerned, it is convenient to choose Z_1, Z_j, Z_E and Z_F, Z_{j+1}, Z_N as fixed in the two sewn vertices. After sewing, one changes variables from $(Z_2, \ldots, Z_{j-1}, x, Z_{j+2}, \ldots, Z_{N-1})$ to $(Z_2, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_{N-1})$. It is easily checked that this leads to the Möbius-invariant integration volume $d\mathcal{V}(Z_1, \ldots, Z_N)$ with Z_1, Z_j, Z_N kept fixed. Moreover all terms to the power α_0 combine to give the appropriate factor of the N-Reggeon vertex.

As we already discussed, our propagator is symmetric, and therefore is invariant, by the same gauge transformation $e^{(\alpha_0 - W)\gamma}$ on both sides. This is in contrast with the usual twisted propagator which is left invariant if one multiplies by $e^{\gamma(\alpha_0 - W)}$ on the left and by $e^{\gamma(\alpha_0 - W^\dagger)}$ on the right. This feature of the operator formalism is not satisfactory, because right- and left-hand sides, which play the same role, should be treated symmetrically.

VI. MULTILOOP DIAGRAMS

The orientable diagrams involve the symmetric N-Reggeon vertex and the twisted propagator.[29]

Let us first discuss the sewing of the Reggeons p_i and p_j in the vertex $\langle p_1, V_1; \ldots \| \ldots; p_N, V_N\rangle_\Sigma$. According to our previous discussions one should compute

$$R^{(1)}_{N-2} = \int dp_i dp_j \langle p_1, V_1; \ldots; p_i, V_i; \ldots \| \ldots; p_j, V_j; \ldots; p_N, V_N\rangle_\Sigma \langle -p_i \| O(D(x)) | -p_j\rangle_\Sigma , \tag{6.1}$$

which can be written as

$$R^{(1)}_{N-2} = \int dp_j \langle p_1, V_1; \ldots; -p_j V_i D(x); \ldots \| \ldots; p_j, V_j; \ldots; p_N, V_N\rangle_\Sigma .$$

This functional integral is computed by going back to the explicit functional expression of $\langle \ldots \| \ldots\rangle$ and using Eq. (2.19), one gets

$$R^{(1)}_{N-2} = \left\langle \exp\left(\sum_{l\neq i, l\neq j} i\sqrt{2\pi} \int_0^\pi d\xi\, p_l(\xi) \cdot \Phi(V_l \sigma[\xi]) \right) \prod_{z\in\Sigma} \delta(\Phi(V_i D(x)[z]) - \Phi(V_j[z])) \right\rangle . \tag{6.2}$$

The integral is carried out in the domain bounded by the curves $C_l = V_l\Sigma$. The symbol δ means that one integrates over the function Φ with the boundary condition that it takes the same value on the curves C_i and C_j at points $z_i \in C_i$ and $z_j \in C_j$ such that

$$z_i = V_i D(x) V_j^{-1} z_j . \tag{6.3}$$

More generally, after sewing r pairs of Reggeons of indices $i_1, j_1; \ldots ; i_r, j_r$, one obtains[30]

$$R_{N-2r}^{(r)} = \left\langle \exp\left(i\sqrt{2\pi} \sum_{l\neq i_1,\ldots,i_N;\, l\neq j_N,\ldots,j_N} \int_0^\pi d\xi\, p_l(\xi)\cdot\Phi(V_l\sigma[\xi])\right) \prod_{k=1,\ldots,r}\prod_{z\in\Sigma} \delta(\Phi(V_{i_k}D(x_k)[z]) - \Phi(V_{j_k}[z]))\right\rangle . \tag{6.4}$$

One can interpret the δ functional as expressing the fact that the functional integration is carried out on the surface obtained from the original surface by identifying all pairs of boundaries C_{i_k}, C_{j_k} through the transformation

$$S^k = V_{i_k} D(x_k) V_{j_k}^{-1} . \tag{6.5}$$

The transformations S^k are the generators of the automorphy group of the surface which has r handles.

In order to transform the functional integral, let us first go back to the case where the boundary of the surface is a smooth curve. This is done if one multiplies by T all unsewn Reggeon distributions, and the final result is the functional integrals computed on the surface $\mathcal{S}^{(r)}$ shown on Fig. 12, where C_{i_k} and C_{j_k} are identified through S^k.

Let us now introduce the Neumann function $N^{(r)}$ of the surface $\mathcal{S}^{(r)}$, namely, the function which satisfies conditions (2.8) and (2.9), and which has a constant normal derivative on the boundaries of $\mathcal{S}^{(r)}$. The functional integral is transformed in the way recalled in Sec. II [see (2.14)], and one gets

$$R^{(r)} = K^{(r)}(2\pi)^4\delta^{(4)}\left(\frac{1}{\sqrt{2\pi}}\sum_l \int_0^\pi p_l(\xi)d\xi\right)\exp\left(\frac{1}{2}\sum_{l,m}\int_0^\pi d\xi\int_0^\pi d\xi' p_l(\xi)\cdot p_m(\xi')N^{(r)}(g_l\sigma[\xi], g_m\sigma[\xi'])\right), \tag{6.6}$$

where $K^{(r)}$ reads

$$K^{(r)} = \frac{1}{n_0}\iint \mathfrak{D}\Phi\left[\prod_{k=1,\ldots,r}\prod_{z\in\Sigma}\delta(\Phi(V_{i_k}D(x_k)[z]) - \Phi(V_{j_k}[z]))\exp\left(\int dxdy\,\mathcal{L}\right)\right]. \tag{6.7}$$

The Neumann function $N^{(r)}$ can be expressed in terms of Poincaré series.[31] This has been studied in detail by Alessandrini.[6]

In formula (6.6) one recognizes the exponential of the heat generated by the current distributions p_l on the surface $\mathcal{S}^{(r)}$. As one sees, the results of the analog model are proved very simply in the functional method where the Neumann function is naturally introduced through (2.11). In the usual operator formalism one has to collect a lot of terms in order to reconstruct that function after rather tedious computations.[6,7]

K is not given by the analog model. Let us notice that K can be obtained from $R^{(r)}$ by setting all p_l be zero. In order to compute K, it is better to go back to the operator expression for $R^{(r)}$, since in this limit it becomes simpler; the reason for this is that the vertex operators involving external Reggeons reduce to 1. The general expression of $K^{(r)}$ has been calculated.[32]

VII. CONCLUSION

It is clear that the formalism we have developed in this paper is somewhat more abstract than the usual operator formalism. However, it seems to be useful because the two-dimensional geometry introduced here is the natural framework to describe the properties of dual diagrams. In particular one is led rather simply to the formulation of the analog model in terms of energies associated with momentum distribution attached to general surfaces. Moreover, as already emphasized in the Introduction, this approach is strongly suggested by the fact that dual diagrams can be considered as limits of very dense semiplanar Feynman diagrams.

In this paper we did not consider nonorientable diagrams. It is well known that this case is more

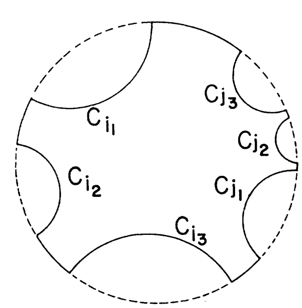

FIG. 12. The surface $\mathcal{S}^{(r)}$.

complicated. In fact, there is no simple spurious-free untwisted propagator. Also, the introduction of the corresponding Neumann's function requires the doubling of the original surface.[6] In our formalism we shall have to extend our definition of rudimental amplitudes to the case where the momentum distributions are not all running in the same direction. This problem is now under investigation. We think that, in the same way as in the case of orientable loop diagrams which were found to be automatically spurious-free, our method may turn out to be more powerful than the usual one.

Finally, let us remark that the N-Reggeon vertex we have obtained is dual if spurious states are removed from the external legs. If one thus multiplies the twisted propagators to every external Reggeon leg, one may define the off-shell dual Green's functions of the dual theory. This would be very important for understanding renormalization and introduction of currents in this theory.

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APPENDIX

In this appendix we present a proof of (2.37) using a method[33] which provides some insight to the measure of functional integration. Since, as mentioned in the text, the slicing rule is equivalent to the completeness condition of the operator method, and since $O(g)$ is expressed as a product of an infinite number of infinitesimal transformations, it is sufficient to prove relation (2.37) for an infinitesimal transformation.

Let δg be an infinitesimal transformation which maps a point $\sigma_0[\xi]=x_1(\xi)+ix_2(\xi)$ on Σ_0 to a point $\delta g\sigma_0[\xi]=x_1'(\xi)+ix_2'(\xi)$ on $\Sigma'=\delta g\Sigma_0$. Then the equation to be proved is

$$\langle \varphi|O(\delta g)|\varphi'\rangle_{\Sigma_0}=\left\langle \prod_\xi \delta(\varphi(\xi)-\Phi(\sigma_0[\xi]))\prod_\xi \delta(\varphi'(\xi)-\Phi(\delta g\sigma_0[\xi]))\right\rangle$$
$$\equiv n_0^{-1}\int\cdots\int_{\varphi'}^{\varphi}\mathfrak{D}\Phi(x,y)\exp\iint dx\,dy\,\mathfrak{L}(x,y)\,. \tag{A1}$$

To first order in δg the functional integral of (A1) is approximated by the exponential of the action, where $\mathfrak{L}$ is computed by replacing $\partial\Phi/\partial x_i$ by finite differences. Let

$$\delta x_i(\xi)=x_i'(\xi)-x_i(\xi) \tag{A2}$$

and $d\sigma_i$ be the surface element of Σ_0 defined by

$$d\sigma_i=\epsilon_{ij}\frac{\partial x_j}{\partial\xi}d\xi\,, \tag{A3}$$

where ϵ_{ij} is given by

$$\epsilon_{ij}=-\epsilon_{ji},\quad \epsilon_{12}=1\,. \tag{A4}$$

The functional integral is then written as

$$\frac{1}{A}\exp\int_{\Sigma_0}d\sigma_i\delta x_i\mathfrak{L}\,, \tag{A5}$$

where A^{-1} is the measure to be used in the integration. The Lagrangian in (A5) can be expressed in terms of $\varphi(\xi)$ and $\varphi'(\xi)$ as follows. Because of the δ functional in (A1), we have

$$\Phi(x')=\Phi(x)+\frac{\partial\Phi}{\partial x_i}\delta x_i=\varphi'(\xi)\,,$$
$$\Phi(x)=\varphi(\xi)\,, \tag{A6}$$

so that

$$\frac{\partial\Phi}{\partial x_i}=\epsilon_{ij}\frac{(\partial x_j/\partial\xi)[\varphi(\xi)-\varphi'(\xi)]+\delta x_j(\partial\varphi/\partial\xi)}{(d\sigma_i/d\xi)\delta x_i}\,. \tag{A7}$$

Inserting (A7) into the Lagrangian, we obtain

$$\int\mathfrak{L}d\sigma_i\delta x_i=-\frac{1}{2}\int_0^\pi d\xi\left\{\left(\frac{d\sigma_i}{d\xi}\delta x_i\right)^{-1}\left[\varphi-\varphi'+\frac{\partial\varphi}{\partial\xi}\frac{\partial x_i}{\partial\xi}\delta x_i\right]^2+\left(\frac{d\sigma_i}{d\xi}\delta x_i\right)\left(\frac{\partial\varphi}{\partial\xi}\right)^2\right\}\,, \tag{A8}$$

where we have used $(\partial x_i/\partial\xi)^2 = 1$.

We then use the following identity:

$$\exp\left\{-\frac{1}{2}\int_0^\pi d\xi\left(\frac{d\sigma_i}{d\xi}\delta x_i\right)^{-1}\left[\varphi-\varphi'+\frac{\partial\varphi}{\partial\xi}\frac{\partial x_i}{\partial\xi}\delta x_i\right]^2\right\}$$
$$=C\int\mathfrak{D}p(\xi)\exp\int_0^\pi d\xi\left\{-\pi p^2(\xi)\frac{d\sigma_i}{d\xi}\delta x_i+i\sqrt{2\pi}\,p(\xi)\left[\varphi-\varphi'+\frac{\partial\varphi}{\partial\xi}\frac{\partial x_i}{\partial\xi}\delta x_i\right]\right\}$$
$$=C\exp\left\{-\frac{1}{2}\int_0^\pi d\xi\left[\pi(\xi)^2\frac{d\sigma_i}{d\xi}\delta x_i-2i\frac{\partial\varphi}{\partial\xi}\pi(\xi)\frac{\partial x_i}{\partial\xi}\delta x_i\right]\right\}\prod_\xi\delta(\varphi(\xi)-\varphi'(\xi)),\tag{A9}$$

where

$$C^{-1}=\int\mathfrak{D}p(\xi)\exp\left(-\pi\int_0^\pi d\xi\,p^2(\xi)\frac{d\sigma_i}{d\xi}\delta x_i\right),\tag{A10}$$

and

$$\pi(\xi)=-i\frac{\delta}{\delta\varphi(\xi)}.\tag{A11}$$

Then we obtain the following expression for (A5):

$$\frac{n_0}{AC^{-1}}\exp\left(-\frac{1}{2}\int_0^\pi d\xi\left\{\frac{d\sigma_i}{d\xi}\delta x_i\left[\pi^2(\xi)+\left(\frac{\partial\varphi}{\partial\xi}\right)^2\right]-2i\frac{\partial\varphi}{\partial\xi}\pi(\xi)\frac{\partial x_i}{\partial\xi}\delta x_i\right\}\right)\langle\varphi\,|\,\varphi'\rangle,\tag{A12}$$

where we have used the normalization

$$\langle\varphi\,|\,\varphi'\rangle=n_0\prod_\xi\delta(\varphi(\xi)-\varphi'(\xi))\tag{A13}$$

which is consistent with all the definitions in the text.

Thus if we choose $A=C$, we obtain (A1) from (A12), provided that the exponent of (A12) is equal to (2.30). This last fact can easily be checked using the following relations:

$$\pi(\xi)=i\frac{d\sigma_i}{d\xi}\frac{\partial\mathfrak{L}}{\partial(\partial\Phi/\partial x_i)}=-i\frac{d\sigma_i}{d\xi}\frac{\partial\Phi}{\partial x_i},$$
$$\frac{\partial\Phi}{\partial\xi}=\frac{\partial x_i}{\partial\xi}\frac{\partial\Phi}{\partial x_i}.\tag{A14}$$

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[25]From now on, we write $|p\rangle$ instead of $|p,1\rangle$.

[26]One can also use more general transformations given by $\left[\begin{smallmatrix} 1 & -1 & i \\ 1 & -1 & \alpha \end{smallmatrix}\right]$, where α is any complex number. One gets a domain with a different shape.

[27]See Refs. 3, 5, 11, 18 and, in general, Ref. 1.

[28]Let us remark that condition (5.15) does not agree with Alessandrini's description of the kth Reggeon as a current distribution located between Z_{k-1} and Z_{k+1} (see Ref. 22). Indeed with this choice the two sewn Reggeon distributions do not match.

[29]We only consider orientable diagrams. The general case will be dealt with in a subsequent paper.

[30]There are subtle questions of double counting when one integrates over the Z's and the x's. We do not discuss them here. One of us (B.S.) is grateful to V. Alessandrini and D. Amati for discussions on this particular point.

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