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Normal-Product Quantization of Currents in Lagrangian Field Theory*

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Techniques for quantizing currents in Lagrangian field theory are developed with the aid of Zimmermann's normal products. These methods greatly simplify the derivation of single-current generalized Ward identities and may be used to justify the heuristic use of formal arguments in discussing broken symmetries. Applications to the energy-momentum tensor in the A^4 model and to the current of broken orthogonal symmetry in a two-component scalar model are presented.

I. INTRODUCTION

The concept of a normal product of fields[1] has given the quantum-field theorist a powerful tool for the precise definition of currents and other observables in renormalized perturbation theory. In particular, the work of Zimmermann[1] has opened the way to the eventual development of a thouroughgoing quantization procedure in which classical currents furnished by Noether's theorem can be transformed systematically into local, covariant quantum fields satisfying appropriate conservation laws and Ward identities. The present investigation is a modest step in that direction. We shall limit ourselves here to the question of defining currents and deriving single-current generalized Ward identities in scalar theories exhibiting symmetries and broken symmetries.

The usual derivations of Ward identities[2] are plagued by a number of unsatisfactory features. Often the current and the time-ordered function are defined only in the process of obtaining the Ward identity. Certainly the latter would have more value if one used a universal notion of time-ordered function, valid for all currents. This we achieve in Sec. II. Another unsatisfactory aspect is the roundabout method of derivation: One starts from a set of formally derived equal-time commutation relations (including Schwinger terms), uses these to evaluate the divergence of a covariantized time-ordered function (naive time-ordered function plus "seagull" terms) and ends up with a generalized Ward identity which miraculously agrees with what one would have derived from the erroneous assumptions of canonical commutation relations and naive time-ordered functions.[3] Of course, one must then verify the identity in renormalized perturbation theory, suitably prescribing a subtraction procedure for making the time-ordered functions finite. We shall see in Sec. III that the use of normal products and a universal definition of time ordering greatly simplify the derivation of generalized Ward identities. The formal steps can be eliminated without resorting to cutoffs and one never encounters the Schwinger terms and seagull terms which make the usual scheme so complicated.

In Secs. IV and V are presented two applications of our techniques: Definitions of the energy-momentum tensor in the A^4 model and of the non-conserved current of broken orthogonal symmetry in a two-component scalar model. The former is particularly interesting because of attempts to show that the A^4 model exhibits broken scale invariance.[4] It is well known, by a variety of arguments,[5] that this is impossible to achieve, regardless of how much one "improves[4]" the energy-momentum tensor. The whole situation becomes quite clear in the formulation of Sec. IV. The discussion of broken orthogonal symmetry of Sec. V should be compared with the treatment of the identical problem by Symanzik.[6] The results are essentially the same, but the comparative simplicity of the approach using normal products is striking.

In order that the notation not become too unwieldy all of the discussions have been phrased in terms of basic scalar fields. However, the methods are readily generalizable to a large class of tensor and spinor models.

II. NORMAL PRODUCTS

In Lagrangian perturbation theory there is an unambiguous prescription for calculating the renormalized Green's functions to arbitrary order in the couplings once one is given the effective Lagrangian[7] ("effective" meaning "infinite counterterms omitted") as a functional of the basic fields A_i (assumed for simplicity to be spinless) of the theory. We write

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_I ,$$

where $\mathcal{L}_0$ describes the canonical free fields $A_i^{(0)}$ and $\mathcal{L}_I$ specifies the interaction. Then[8]

$$\langle 0 | T \prod_i A_i(x_i) | 0 \rangle = \text{finite part of } {}^{(0)}\!\left\langle 0 \left| T \prod_i A_i^{(0)}(x_i) \exp\left(i \int \mathcal{L}_I[A_j^{(0)}] d^4x \right) \right| 0 \right\rangle^{(0)} , \tag{2.1}$$

where the prescription for calculating the finite part is that of Bogoliubov, Parasiuk, Hepp, and Zimmermann (BPHZ).[9] It is assumed that all subtractions are to be performed at the origin in momentum space, the appropriate normalization conditions being ensured by finite counterterms in $\mathcal{L}_I$.

Assuming the reader to be familiar with the formal Feynman diagram expansion of the unsubtracted Gell-Mann–Low formula (2.1) (see Appendix A for notational conventions), let us summarize briefly the BPH subtraction procedure as formulated by Zimmermann. We associate with each type of vertex V_i a number (degree)

$$\delta_i \geq D_i = b_i + d_i ,$$

where b_i is the number of lines ending at V_i, d_i is the number of derivatives in the x-space product of fields corresponding to V_i, and D_i is the canonical operator dimension of that product. Then for any one-particle irreducible Feynman diagram G, we define

$$\delta(G) = 4 - N - \sum_i n_i (4 - \delta_i) , \tag{2.2}$$

where N is the number of external lines of G and n_i is the number of vertices of type V_i in G. The renormalized integrand R_G which replaces the integrand I_G obtained formally from the Gell-Mann–Low formula is most concisely given by Zimmermann's formula

$$R_G = \sum_{U \in \mathfrak{F}_G} \prod_{\gamma \in U} (-t_\gamma) I_G , \tag{2.3}$$

where $\mathfrak{F}_G$ is the set of all "forests" [sets of nonoverlapping proper subdiagrams γ with $\delta(\gamma) \geq 0$] of G, and t_γ denotes the Taylor operator

$$\sum_{\substack{\{\nu_1, \ldots, \nu_{N-1}\} \\ \sum \nu_i \leq \delta(\gamma)}} \left(\prod_m^{N-1} \nu_m ! \right)^{-1} \left(\prod_{i=1}^{N-1} \prod_{j=1}^{\nu_i} p_i^{\mu_{ij}} \right) \left(\prod_{k=1}^{N-1} \prod_{l=1}^{\nu_k} \frac{\partial}{\partial p_k^{\mu_{kl}}} \right)_{p_1 = \cdots = p_{N-1} = 0} ,$$

where $p_1, \ldots, p_{N-1}$ are the independent external momenta of γ. In order that Eq. (2.3) make sense, it is required that if γ_1 is contained in γ_2, then t_{γ_1} stands to the right of t_{γ_2} in the product of Taylor operators.

Normal products may be defined by a simple modification of the above formalism. For any monomial M in the basic fields A_i and their derivatives, one defines

$$\langle 0 | T N_\delta [M(x)] \prod_i A_i(x_i) | 0 \rangle = \text{finite part of } {}^{(0)}\!\left\langle 0 \left| T : M^{(0)}(x) : \prod_i A_i^{(0)}(x_i) \exp\left(i \int \mathcal{L}_I[A_j^{(0)}] \, d^4x \right) \right| 0 \right\rangle^{(0)} , \tag{2.4}$$

where $:M^{(0)}(x):$ is the same (Wick-ordered) monomial of the free fields $A_i^{(0)}$. The Feynman rules and BHPZ subtractions are as before, except that now one has a special vertex V corresponding to M. Equations (2.2) and (2.3) are as before, with $n_V = 1$ and $\delta_V = \delta \geq D_V$. The definition of normal products with separated arguments is a more delicate matter which will not concern us here.

We shall not attempt here a rigorous justification for interpreting the left-hand side of (2.4) as a time-ordered function of well-defined quantum fields. Assuming that this can be done in the sense of perturbation theory to arbitrary finite order – Zimmermann gives convincing arguments in Ref. 1 – the matrix elements of the operator-valued distribution $\mathfrak{N}_\delta[M(x)]$ corresponding to the left-hand side of (2.4) may be calculated using the reduction formula (indices suppressed)

$$\begin{aligned} {}_{\text{out}}\langle q_1, \ldots, q_k | \mathfrak{N}_\delta[M(x)] | q_1', \ldots, q_l' \rangle_{\text{in}} &= (-i)^{k+l} \prod_i^k \prod_j^l (q_i^2 - m_i^2)(q_j^2 - m_j^2) \\ &\quad \times \langle 0 | T N_\delta [M(x)] \tilde{A}(q_1) \cdots \tilde{A}(q_k) \tilde{A}(q_1') \cdots \tilde{A}(q_l') | 0 \rangle \Big|_{q_i^0 = -(\vec{q}_i^{\,2} + m_i^2)^{1/2};\ q_j'^0 = +(\vec{q}_j'^{\,2} + m_j^2)^{1/2}} . \end{aligned} \tag{2.5}$$

The reason we have not written the operator as $N_\delta[M(x)]$ is that Zimmermann's time-ordered functions are not unambiguously defined for operator normal products. The $\mathfrak{N}_\delta[P(x)]$ for polynomials $P(x)$ in the basic fields and their derivatives are not independent, being linked by certain equations of motion. Suppose $\mathfrak{N}_\delta[P_1(x)]=\mathfrak{N}_\delta[P_2(x)]$. It will not in general be true that

$$\langle 0\,|\,TN_\delta[P_1(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)\,|\,0\rangle=\langle 0\,|\,TN_\delta[P_2(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)\,|\,0\rangle\,.$$

As a simple example, let us consider $N[A\partial^2A]$ in the scalar theory with effective Lagrangian

$$\mathfrak{L}_{\text{eff}}=\tfrac{1}{2}\partial^\mu A\partial_\mu A-\tfrac{1}{2}m^2A^2-\frac{\lambda}{4!}A^4+\tfrac{1}{2}aA^2+\tfrac{1}{2}b(\partial^\mu A\partial_\mu A-m^2A^2)\,, \tag{2.6}$$

where λ is the perturbation parameter and a and b are chosen to fix the position and residue of the propagator pole at $p^2=m^2$. The formal equation of motion

$$\partial^2 A=\frac{\hat\lambda}{3!}A^3-\hat m^2A\,,\quad \hat\lambda=\frac{\lambda}{1+b}\,,\quad \hat m^2=m^2-\frac{a}{1+b}$$

may be taken inside the N_4 symbol (see Appendix B, Lemma II) as follows:

$$\langle 0\,|\,TN[A\partial^2A](x)A(x_1)\cdots A(x_N)\,|\,0\rangle=-\frac{\hat\lambda}{3!}\langle 0\,|\,TN[A^4](x)A(x_1)\cdots A(x_N)\,|\,0\rangle-\hat m^2\langle 0\,|\,TN_4[A^2](x)A(x_1)\cdots A(x_N)\,|0\rangle$$
$$-i(1+b)^{-1}\sum_{k=1}^{N}\delta(x-x_k)\langle 0\,|\,TA(x_1)\cdots A(x_N)\,|0\rangle\,. \tag{2.7}$$

Applying the reduction formula (2.5) shows that

$$\mathfrak{N}[A\partial^2A]=-\frac{\hat\lambda}{3!}\mathfrak{N}[A^4]-\hat m^2\mathfrak{N}_4[A^2]$$

even though from (2.6) we see that

$$N[A\partial^2A]\neq-\frac{\hat\lambda}{3!}N[A^4]-\hat m^2N_4[A^2]\,.$$

Clearly we have a whole equivalence class of $N_\delta[P(x)]$ corresponding to every field $\mathfrak{N}_\delta[P(x)]$. In other words, within Zimmermann's formalism there is no such thing as *the* time-ordered function $\langle 0\,|\,T\mathfrak{N}_\delta[P(x)]A(x_1)\cdots A(x_N)\,|0\rangle$: There are many, differing by δ-function terms which do not contribute to matrix elements of the field.

The above ambiguity may be resolved in a manner which parallels closely the description of formal quantum field theory in terms of canonical variables. We do not expect our basic fields A_i to satisfy canonical equal-time commutation relations with the conjugate momenta π_i defined formally from the effective Lagrangian. We do demand – this is part of what we mean by basic fields – that the normal products of the A_i, π_i, and their spatial derivatives to any order be linearly independent. There will, of course, be linear relations among normal products, namely, the equations of motion. The latter should be such (and this may be verified directly in particular models) that an arbitrary normal product of the basic fields and their derivatives may be expressed as a linear combination of normal products of the "kinematical" variables. Now, if $M(x)$ is a monomial in the A_i, π_i, and their spatial derivatives of dimension d, then we *define the canonical time-ordered function* of $\mathfrak{N}_d[M(x)]$ with fields, $A_{i_1}(x_1),\ldots,A_{i_N}(x_N)$ to be

$$\langle 0\,|\,T_c\mathfrak{N}_d[M(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)\,|\,0\rangle=\langle 0\,|\,TN_d[M(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)\,|\,0\rangle\,. \tag{2.8}$$

If $M(x)$ is an *arbitrary* monomial of dimension d in the basic fields and their derivatives, then, by application of the equations of motion and the relations between N_ϕ and N_δ, $\phi<\delta$ (see Ref. 1), one can always write for $\delta\geq d$,

$$\mathfrak{N}_\delta[M(x)]=\sum_i\mathfrak{N}_{d_i}[M_i(x)]\,,$$

where $M_i(x)$ is a monomial in the kinematical fields of dimension d_i.

As simple examples of the above scheme consider $\mathfrak{N}[A\partial^2A]$ and $\mathfrak{N}_4[A^2]$ in the A^4 theory (2.6). Applying the equation of motion (Appendix B, Lemma II) to the former yields (2.7), whereas the latter may be expanded in terms of normal products with the minimal number of subtractions:

$$N_4[A^2]=N_2[A^2]-rN_4[\partial_\mu A\partial^\mu A]-sN_4[A\partial^2A]-tN_4[A^4]\,, \tag{2.9}$$

where

$$r = -\tfrac{1}{8}\langle 0|TN_2[A^2(0)]\,\partial_\mu\tilde{A}(0)\partial^\mu\tilde{A}(0)|0\rangle^{\text{prop}'},$$

$$s = -\tfrac{1}{8}\langle 0|TN_2[A^2(0)]\,\tilde{A}(0)\partial^2\tilde{A}(0)|0\rangle^{\text{prop}'},$$

$$t = (1/4\,!)\langle 0|TN_2[A^2(0)]\,\tilde{A}(0)^4|0\rangle^{\text{prop}'},$$

where the superscript prop′ means that only proper nontrivial diagrams are included. From (2.7) and (2.9) it follows that

$$N_4[A^2] = (1 - s\hat{m}^2)^{-1}\{N_2[A^2] - rN_4[\partial_\mu A\partial^\mu A] - [t - (\lambda/3\,!)s]N_4[A^4] + (1+b)^{-1} s i A\delta/\delta A\}, \tag{2.10}$$

$$N[A\partial^2 A] = (1 - s\hat{m}^2)^{-1}\{-m^2 N_2[A^2] + r\hat{m}^2 N_4[\partial_\mu A\partial^\mu A] - [(\lambda/3\,!) - \hat{m}^2 t]N_4[A^4] - (1+b)^{-1} i A\delta/\delta A\}, \tag{2.11}$$

where the simplified notation should be clear. Dropping the δ-function terms and changing N to $\mathfrak{N}$ in Eqs. (2.10) and (2.11) completes the reduction of $\mathfrak{N}[A\partial^2 A]$ and $\mathfrak{N}_4[A^2]$ to canonical form.

Although the T_c prescription described above is perhaps the simplest and most natural scheme for defining an unambiguous time-ordered product, there may be others which would be preferred for certain applications. In particular the T_c functions of tensor fields involving second derivatives are not always covariant, although they differ from covariant Green's functions only by δ-function terms. To covariantize, one can simply go back to one of Zimmermann's N_δ. The choice will not in general be unique, of course, but this may not be harmful.

III. GENERALIZED WARD IDENTITIES

Generalized Ward identities[10] (GWI) are exact statements about Green's functions in renormalized perturbation theory which correspond formally to equal-time commutation relations (ETCR) of currents with the basic fields (current-current commutation relations are equally interesting but beyond the scope of this article). When integrated over all space-time, the generalized Ward identities of a particular current give the transformation properties of Green's functions under the (possibly broken) symmetry operations generated locally by the current. An advantage of the GWI for describing broken symmetries is that they are well defined even in cases where the ETCR are not. By defining our currents as normal products we shall be able to derive GWI's in a simple and straightforward fashion, without mention of the Schwinger terms and "seagull" terms which plague the usual derivations.

By a *generalized current* we shall mean a Hermitian tensor field of the form

$$J_\mu = \mathfrak{N}[\mathcal{O}_\mu], \tag{3.1}$$

where $\mathcal{O}_\mu$ is a covariant polynomial of dimension d in the kinematical fields (A_i, π_i, and their spatial derivatives of arbitrary order). The subscript μ refers to a derivative ∂_μ and other tensor indices have been suppressed. The generalization to the case of a linear combination of terms of possibly different dimensions will be obvious. A *generalized Ward identity* (GWI) for J_μ is a relation of the form

$$\begin{aligned}\partial^\mu\langle 0|T_c J_\mu(x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)} &= \langle 0|T_c\partial^\mu J_\mu(x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\\ &\quad+\sum_{n,k} D_{(n)}\delta^{(4)}(x-x_k)\langle 0|T_c C^{(n)}(x_k)A_{i_1}(x_1)\cdots\hat{A}_{i_k}(x_k)\cdots A_{i_N}(x_N)|0\rangle^{(t)},\end{aligned} \tag{3.2}$$

where the symbol (t) signifies that the equation is valid for both the Green's functions and the truncated Green's functions (connected diagrams only) and the $D_{(n)}$ are differentiation operators with respect to x (or unity). The caret over A_{i_k} indicates omission of that field.

To derive the GWI for a particular generalized current, we first express the left-hand side of (3.2) in terms of Zimmermann's normal products and T functions and then bring the derivative inside using Lemma I of Appendix B:

$$\begin{aligned}\partial^\mu\langle 0|T_c J_\mu(x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)} &= \partial^\mu\langle 0|TN_d[\mathcal{O}_\mu]A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\\ &= \langle 0|TN_{d+1}[\partial^\mu\mathcal{O}_\mu]A_i(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}.\end{aligned} \tag{3.3}$$

Application of the reduction formula shows that

$$\partial^\mu J_\mu(x) = \mathfrak{N}_{d+1}[\partial^\mu\mathcal{O}_\mu]. \tag{3.4}$$

Equation (3.2) then follows from (3.3) and (3.4), provided that in passing from T functions of $N_{\delta+1}[\partial^\mu\mathcal{O}_\mu]$ one picks up contact terms of the sort appearing on the right-hand side of (3.2).

By a (broken-symmetry) *current* we shall mean a generalized current J_μ satisfying a GWI (3.2) in which $C^{(0)}_{i_k}$ (corresponding to $D_{(0)} = 1$) is a linear combination of the basic fields and their derivatives, and such that $\int d^4x\,\partial^\mu J_\mu$ has negative operator dimension (sufficient condition: $\partial^\mu J_\mu$ has dimension less than 4). Then in the absence of zero-mass particles in the theory one may integrate (3.2) over all space-time to obtain

$$\sum_k \langle 0|T_c C^{(0)}_{i_k}(x_k)A_{i_1}(x_1)\cdots \hat{A}_{i_k}(x_k)\cdots A_{i_N}(x_N)|0\rangle^{(t)} = -\int d^4x \langle 0|T_c \partial^\mu J_\mu(x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)} . \tag{3.5}$$

If the current is conserved, then (3.5) with right-hand side zero expresses an invariance property of the Green's functions. If not, the right-hand side of (3.5) is a measure of the departure from perfect symmetry. By power counting,[11] this becomes negligible for the momentum-space (truncated) Green's functions in the asymptotic region where all energy-momenta are large. In other words, the short-distance behavior of the (truncated) Green's functions is asymptotically symmetric.

For the case of a conserved current J_μ and no zero-mass particles one expects from the version of Goldstone's theorem due to Kastler, Robinson, and Swieca[12] that it should be possible to construct a generalized charge operator – corresponding formally to the spatial integral of J_0 – which annihilates the vacuum and is the infinitesimal generator of the symmetry transformations. That the integrated charge density has the appropriate ETCR with the basic fields follows from the GWI, as follows. Let $f_T(t)$ be a symmetric test function on the real line which vanishes for $t > T$ and is equal to unity in a neighborhood of the origin. Let $g_R(\vec{x}) = f_R(|\vec{x}|)$. Then for $h \in \mathfrak{D}(R^4)$

$$\begin{aligned}
&\int h(y) g_R(\vec{x}) f_T(x^0)\partial^\mu_x \langle \Phi | T J_\mu(y+x)A_i(y)|\Psi\rangle d^4x\,d^4y \\
&\qquad = -\int h(y) f_T(x^0)\partial^k g_R(\vec{x})\langle \Phi | T J_k(y+x)A_i(y)|\Psi\rangle d^4x\,d^4y - \int h(y) f_T{}'(x^0) g_R(\vec{x})\langle \Phi | T J_0(y+x)A_i(y)|\Psi\rangle d^4x\,d^4y \\
&\qquad = -\int h(y) f_T(x^0)\partial^k g_R(\vec{x})\langle \Phi | J_k(y+x)A_i(y)|\Psi\rangle d^4x\,d^4y \\
&\qquad\quad -\int_0^\infty dx^0 f_T{}'(x^0)\int h(y) g_R(\vec{x})\langle \Phi | J_0(y+x)A_i(y) - A_i(y)J_0(y-x)|\Psi\rangle d^3x\,d^4y \\
&\qquad = -\int_0^\infty dx^0 f_T{}'(x^0)\int h(y) g_R(\vec{x})\langle \Phi | [J_0(y-x), A_i(y)]|\Psi\rangle .
\end{aligned} \tag{3.6}$$

In (3.6) the matrix elements are between arbitrary normalizable in and out states and are calculated using the reduction formula (2.5). The GWI (3.2) then implies that the last expression in (3.6) is equal to

$$\int dy\, h(y)\langle \Phi | C^{(0)}_i(y)|\Psi\rangle$$

for all R and T.

IV. ENERGY-MOMENTUM TENSOR IN A^4 THEORY

As the first application of normal-product quantization of currents, let us derive an expression for an energy-momentum tensor in the theory with effective Lagrangian (2.6).

Quantizing the formal energy-momentum tensor

$$\theta_{\mu\nu} = \frac{\delta \mathfrak{L}_{\rm eff}}{\delta(\partial^\mu A)}\partial_\nu A - g_{\mu\nu}\mathfrak{L}_{\rm eff} , \tag{4.1}$$

with $\mathfrak{N}_4$ normal products leads us to (up to normalization)

$$\theta_{\mu\nu} = (1+b)(\mathfrak{N}[\partial_\mu A \partial_\nu A] - g_{\mu\nu}\{\tfrac{1}{2}\mathfrak{N}[\partial_\mu A\partial^\mu A] - (\lambda/4!)\mathfrak{N}[A^4] - \tfrac{1}{2}\hat{m}^2\mathfrak{N}_4[A^2]\}) , \tag{4.2}$$

which may be reduced to canonical form using (2.10).

From Lemma I (Appendix B) and (2.10) we obtain

$$\begin{aligned}
&\partial^\mu \langle 0|T_c \theta_{\mu\nu}(x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} \\
&\qquad = (1+b)\langle 0|T\{N_5[\partial^2 A\partial_\nu A](x) + (\hat{\lambda}/3!)N_5[A^3\partial_\nu A](x) + \hat{m}^2 N_5[A\partial_\nu A](x)\}A(x_1)\cdots A(x_N)|0\rangle^{(t)} \\
&\qquad\quad -\frac{1}{2}\,\frac{s\hat{m}^2}{1-s\hat{m}^2}\sum_k \partial_\nu\delta(x-x_k)\langle 0|TA(x_1)\cdots A(x_N)|0\rangle^{(t)} .
\end{aligned}$$

Applying the equation of motion (Lemma II) then yields

$$\partial^\mu\langle 0|T_c\theta_{\mu\nu}(x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} = -i\sum_k\left\{\delta(x-x_k)\langle 0|T_cA(x_1)\cdots\partial_\nu A(x_k)\cdots A(x_N)|0\rangle^{(t)} + \frac{1}{2}\frac{s\hat{m}^2}{1-s\hat{m}^2}\partial_\nu\delta(x-x_k)\langle 0|TA(x_1)\cdots A(x_N)|0\rangle^{(t)}\right\}. \tag{4.3}$$

Similarly for the angular momentum tensor

$$M_{\mu\nu\rho} = x_\mu\theta_{\nu\rho} - x_\nu\theta_{\mu\rho}$$

we get

$$\partial^\mu\langle 0|T_cM_{\mu\nu\rho}(x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} = -i\sum_k\left\{\delta(x-x_k)\langle 0|T_cA(x_1)\cdots(x_\mu\partial_\nu - x_\nu\partial_\mu)A(x_k)\cdots A(x_N)|0\rangle^{(t)} + \frac{1}{2}\frac{is\hat{m}^2}{1-s\hat{m}^2}(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta(x-x_N)\langle 0|TA(x_1)\cdots A(x_N)|0\rangle^{(t)}\right\}. \tag{4.4}$$

Application of the reduction formula to (4.2) and (4.3) then gives the conservation of the two tensor currents, while integration over all space-time gives the translation and Lorentz invariance of the Green's functions.

For the would-be scaling current

$$D_\mu = x^\nu\theta_{\mu\nu},$$

the GWI in differential and integral form is

$$\begin{aligned}\partial^\mu\langle 0|T_cD_\mu(x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} &= \langle 0|T_c\theta_\mu{}^\mu(x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} \\ &\quad -i\left\{\sum_k\delta(x-x_k)\langle 0|TA(x_1)\cdots x_k^\nu\partial_\nu A(x_k)\cdots A(x_N)|0\rangle^{(t)} + \frac{1}{2}\frac{s\hat{m}^2}{1-s\hat{m}^2}x^\nu\partial_\nu\delta(x-x_k)\langle 0|TA(x_1)\cdots A(x_N)|0\rangle^{(t)}\right\},\end{aligned} \tag{4.5}$$

$$\begin{aligned}i\sum_k\langle 0|TA(x_1)\cdots(x_k^\nu\partial_\nu + d)A(x_k)\cdots A(x_N)|0\rangle^{(t)} &= (1+b)\int d^4x\langle 0|T(\hat{m}^2N_4[A^2](x) + (1-d)\{\hat{m}^2N_4[A^2](x) + (\lambda/3\,!)N_4[A^4](x) - N_4[\partial_\mu A\partial^\mu A](x)\})A(x_1)\cdots A(x_N)|0\rangle^{(t)}.\end{aligned} \tag{4.6}$$

In passing from (4.5) to (4.6), the identity

$$\begin{aligned}0 &= \int d^4x\langle 0|TN_4[\partial^\mu(A\partial_\mu A)](x)A(x_1)\cdots A(x_N)|0\rangle^{(t)} \\ &= \int d^4x\langle 0|T\{N_4[\partial^\mu A\partial_\mu A](x) - (\lambda/3\,!)N_4[A^4](x) - \hat{m}^2N_4[A^2](x)\}A(x_1)\cdots A(x_N)|0\rangle^{(t)} \\ &\quad -(1+b)^{-1}iN\langle 0|TA(x_1)\cdots A(x_N)|0\rangle^{(t)},\end{aligned} \tag{4.7}$$

which follows from Lemma I of Appendix B and (2.11), has been used to endow $A(x)$ with an arbitrary "dimension" d. With the help of (2.9) we see that asymptotic scale invariance would require both

$$(2-d)(r-s)\hat{m}^2 + (1-d) = 0 \quad\text{and}\quad (2-d)t\hat{m}^2 - (1-d)(\lambda/3\,!) = 0 \tag{4.8}$$

to be satisfied. These equations are consistent only with $\lambda = 0$, $d = 1$, as can be verified in an easy second-order calculation. Thus, as has been shown by other methods,[5] we do not have asymptotic scale invariance in the A^4 model ($\lambda \neq 0$) in perturbation theory.

Some authors[4] have suggested "improving" the energy-momentum tensor by adding a term formally proportional to

$$(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)A^2.$$

It was hoped that D_μ defined using the "new improved" $\theta_{\mu\nu}$ would have a "soft" (dimension less than 4) divergence, thus leading to broken scale invariance. Within the present framework the futility of tampering with the energy-momentum tensor is evident, since no modification of $\theta_{\mu\nu}$ can possibly avoid the implications of Eq. (4.6).

V. BROKEN INTERNAL SYMMETRY

As an application of normal-product quantization to the problem of symmetry breaking, let us consider the two-component scalar model with effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\partial_\mu A_i \partial^\mu A_i - \tfrac{1}{2}m_1{}^2 A_1{}^2 - \tfrac{1}{2}m_2{}^2 A_2{}^2 - \tfrac{1}{8}g(A_iA_i)^2 + \tfrac{1}{2}a_1A_1{}^2 + \tfrac{1}{2}a_2A_2{}^2 + \tfrac{1}{2}z_{ij}\partial_\mu A_i \partial^\mu A_j - (1/4!)g_{ijkl}A_iA_jA_kA_l\,. \tag{5.1}$$

The masses m_1 and m_2 are chosen arbitrarily, as well as the coupling constant g. The a_i, z_{ij}, and g_{ijkl} are power series in g which are chosen to satisfy the conditions

$$\begin{aligned}
&\tilde{\Delta}'_{F11}{}^{-1}(m_1{}^2) = 0 = \tilde{\Delta}'_{F22}{}^{-1}(m_2{}^2)\,,\\
&(p^2 - m_1{}^2)\tilde{\Delta}'_{F11}(p^2)\Big|_{p^2=m_1{}^2} = i\,,\\
&\Gamma^{(4)}_{1122}(0,0,0,0) = -ig\,,\\
&\langle 0|T\tilde{A}_i(p_1)\tilde{A}_j(p_2)\tilde{A}_k(p_3)\tilde{A}_l(p_4)|0\rangle^{\text{prop}} = (2\pi)^4\delta(p_1+p_2+p_3+p_4)\Gamma^{(4)}_{ijkl}(p_1,p_2,p_3,p_4)\,,
\end{aligned} \tag{5.2}$$

as well as the requirement of asymptotic invariance of the (truncated) Green's function under "isospin" rotations. Symmetry breaking of this sort has been discussed at length by Symanzik[6] using different techniques. The reader will note that normal-product quantization considerably simplifies the derivation of the generalized Ward identities.

Let us define a class of generalized currents (b and c real parameters)

$$j_\mu^{b,c}(x) = \mathfrak{N}[A_i(\tau_{ij} + b(\tau z)_{ij} + c(z\tau)_{ij})\partial_\mu A_j]\,, \tag{5.3}$$

which, by the methods of Sec. III (Lemma I and II of Appendix B), satisfy the identity

$$\begin{aligned}
\partial^\mu\langle 0|T_c j_\mu^{b,c}(x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}
&= \langle 0|TN_4[\partial^\mu\{A_i(\tau_{ij} + b(\tau z)_{ij} + c(z\tau)_{ij})\partial_\mu A_j\}](x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\\
&= \langle 0|TN_4[(m_1{}^2 - a_1 - m_2{}^2 + a_2)A_1A_2 + (b(\tau z)_{ij} + c(z\tau)_{ij})\partial_\mu A_i\partial^\mu A_j\\
&\qquad + ((b-1)(\tau z)_{ij} + c(z\tau)_{ij})A_i\partial^2 A_j - (1/3!)\tau_{im}g_{mjkl}A_iA_jA_kA_l](x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\\
&\quad -i\sum_k \delta(x-x_k)\tau_{ii_k}\langle 0|TA_{i_1}(x_1)\cdots A_i(x_k)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\,.
\end{aligned} \tag{5.4}$$

If the z_{ij} and the g_{ijkl} in (5.1) are not carefully chosen, it is clear that we shall not have broken invariance, since the terms involving N_4 normal products will not in general become asymptotically negligible. Our hope is that the coefficients can be so chosen that (5.4), integrated over all x, assumes the form

$$i\sum_k \tau_{i_k i}\langle 0|TA_{i_1}(x_1)\cdots A_i(x_k)\cdots A_{i_N}(x_N)|0\rangle^{(t)} = \int d^4x\langle 0|T_c\alpha_{ij}\mathfrak{N}_2[A_iA_j](x)A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle^{(t)}\,. \tag{5.5}$$

The crucial identity used in achieving this, easily derived from Zimmermann's formulas,[1] is

$$N_2[A_iA_j] = N_4[A_iA_j] + r_{ijkl}N_4[A_k\partial^2A_l] + s_{ijkl}N_4[\partial^\mu A_k\partial_\mu A_l] + t_{ijklmn}\int d^4x N_4[A_kA_lA_mA_n]\,, \tag{5.6}$$

where

$$\begin{aligned}
r_{ijkl} &= -\tfrac{1}{16}\langle 0|TN_2[A_i(0)A_j(0)](\tilde{A}_k(0)\partial^2\tilde{A}_l(0) + \partial^2\tilde{A}_k(0)\tilde{A}_l(0))|0\rangle^{\text{prop}'}\,,\\
s_{ijkl} &= -\tfrac{1}{8}\langle 0|TN_2[A_i(0)A_j(0)]\partial^\mu\tilde{A}_k(0)\partial_\mu\tilde{A}_l(0)|0\rangle^{\text{prop}'}\,,\\
t_{ijklmn} &= (1/4!)\langle 0|TN_2[A_i(0)A_j(0)]\tilde{A}_k(0)\tilde{A}_l(0)\tilde{A}_m(0)\tilde{A}_n(0)|0\rangle^{\text{prop}'}\,.
\end{aligned}$$

We are thus led to choose

$$\begin{aligned}
&\alpha_{11} = \alpha_{22} = 0\,,\\
&\alpha_{12} = \alpha_{21} = \tfrac{1}{2}(m_2{}^2 - a_2 - m_1{}^2 + a_1)\,,
\end{aligned} \tag{5.7}$$

which then implies, equating coefficients in (5.4) and (5.5) with (5.6),

$$\tfrac{1}{2}[\tau, z]_{kl} = \alpha_{ij}(r_{ijkl} - s_{ijkl}) \,,$$
$$(1/3\,!)\{\tau g\}_{klmn} = \alpha_{ij}\, t_{ijklmn} \,, \tag{5.8}$$

where braces denote full symmetrization in the indices. In order to simplify matters, we shall seek a solution of (5.8) which makes the entire effective Lagrangian invariant under the discrete orthogonal transformation $A_1 \to A_1$, $A_2 \to -A_2$ (the mass terms are already of this type). Then r_{ijkl}, s_{ijkl}, and t_{ijklmn} will also have this invariance property and (5.8) reduces to (assuming, without loss of generality, $z_{ij} = \{z\}_{ij}$, $g_{ijkl} = \{g\}_{ijkl}$)

$$\tfrac{1}{2}(z_{22} - z_{11}) = 2\alpha_{12}(r_{1212} - s_{1212}), \quad z_{12} = z_{21} = 0 \,,$$
$$(1/4\,!)(3g_{1122} - g_{1111}) = 2\alpha_{12} t_{121112}, \quad g_{1222} = 0 = g_{2111} \,, \tag{5.9}$$
$$(1/4\,!)(-3g_{1122} + g_{2222}) = \alpha_{12} t_{122221} \,.$$

These conditions, when combined with (5.2), completely determine the coefficients appearing in (5.1).

Now that the theory exhibiting broken symmetry has been specified, we may ask whether any of the currents $j_\mu^{b,c}$ is a broken-symmetry current in the sense of Sec. III. In other words, is there some choice of b, c for which we have the GWI

$$\partial^\mu \langle 0 | T_c\, j_\mu^{b,c}(x) A_{i_1}(x_1) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)}$$
$$= i \sum_k \delta(x - x_k) \tau_{i_k i} \langle 0 | T A_{i_1}(x_1) \cdots A_i(x_k) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)} - \langle 0 | T_c \alpha_{ij} \mathfrak{N}_2[A_i A_j](x) A_{i_1}(x_1) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)} \; ? \tag{5.10}$$

Provided that the parameters g, m_1, and m_2 are not such that $r_{1212} = s_{1212}$, the answer is yes, namely $b = 1 - c$, $c = \frac{1}{2} r_{1212}(r_{1212} - s_{1212})^{-1}$. It is interesting that this value does not correspond to that of Noether's theorem. The physical significance of this special current is far from clear, i.e., it is not obvious that it is any more closely associated with the broken orthogonal symmetry than any of the other currents which lead to the same integrated Ward identity.

VI. WHY DO NAIVE ARGUMENTS WORK?

It is an intriguing phenomenon that correct Ward identities for broken-symmetry currents can be derived using several naive (and incorrect) assumptions: naive time-ordered product, canonical commutation relations, and formal equations of motion for the basic fields and their conjugate momenta, no Schwinger terms. Why does this work so generally? How does one explain the success of Symanzik's program[6] of starting with the formally derived integrated GWI, constructing a rigorous theory in which the normalization conditions are consistent with it, and finally verifying that the integrated GWI is rigorously valid in that theory?

Within the framework developed in Secs. II and III above it is not difficult to justify Symanzik's seemingly miraculous working hypothesis. Let J_μ be a generalized current (3.1) of dimension d. The first steps in deriving the GWI for J_μ are given by (3.3):

$$\partial^\mu \langle 0 | T_c J_\mu(x) A_{i_1}(x_1) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)}$$
$$= \partial^\mu \langle 0 | T N_d[\mathcal{O}_\mu] s_{i_1}(x_1) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)}$$
$$= \langle 0 | T N_{d+1}[\partial^\mu \mathcal{O}_\mu] A_{i_1}(x_1) \cdots A_{i_N}(x_N) | 0 \rangle^{(t)} \,.$$

The next step is to apply the equation of motion in the sense of Lemma II. If one ignores the normal-product symbols, the result is exactly what one would get from the naive assumptions listed above. Those normal-product symbols cannot, of course, be ignored in making the final transition to the form (3.2). If any of the terms in $N_{d+1}[\partial^\mu \mathcal{O}_\mu]$ is of the form $N_{d+1}[M(x)]$ with $d+1$ greater than the dimension of $M(x)$, then further contact terms will arise in the reduction to canonical form, and the end result will differ from the formally derived one. On the other hand, there may be, as in the example of Sec. V, a possibility of *compensation*: The terms with too high an index $d+1$ may add up to give a term (or terms) with the minimal index, at least in the integrated form of the GWI. In this case $N_{d+1}[\partial^\mu \mathcal{O}_\mu]$ can be reduced to canonical form without picking up additional contact terms, and the "naive" integrated Ward identity will be valid. To complete our justification of Symanzik's working hypothesis, we note that compensation necessarily occurs for any J_μ describing a broken symmetry. In order that J_μ correspond to a broken symmetry, it must (for spinless fields) contain a term bilinear in the fields and their derivatives with at least one derivative. Hence $d+1 \geq 4$, and compensation is a necessary condition for broken symmetry.

One interesting result of Sec. V is that the generalized current leading to a given integrated Ward identity expressing broken symmetry is far from unique. The requirement that the generalized current itself have a "soft" divergence (normal-product index δ equal to its dimension <4) seems to pick out a single "broken-symmetry current" as defined in Sec. III. By the arguments of the preceding paragraph, such a current satisfies a "naive" Ward identity. Whether such a current always exists for theories with broken symmetry and whether it is of more than aesthetic interest remain to be investigated.

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APPENDIX A

In order to eliminate factors of 2π from most of the formulas appearing in this article, we have adopted the unsymmetric Fourier transformation:

$$F(x)=\int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot x}\tilde{F}(p)\,,$$

$$\tilde{F}(p)=\int d^4x\,e^{ip\cdot x}F(x)\,.$$

Thus, in the Feynman rules, each line contributes $i/(p^2-m^2+i\epsilon)$, there are no factors of 2π in the vertex contributions, and the integrand corresponding to a diagram G is to be integrated according to

$$(2\pi)^4\delta(p_1+p_2+\cdots+p_N)\int\frac{dk_1}{(2\pi)^4}\cdots\int\frac{dk_n}{(2\pi)^4}\,I_G(p_i,k_j)\,,$$

where $k_1, k_2, \ldots, k_n$ are the independent integration momenta ("loop momenta") and $p_1, \ldots, p_N$ are the external momenta of G. In the effective Lagrangians used above the combinatorial factors have been chosen to simplify as much as possible the Feynman rules, and usually do not conform to Zimmermann's conventions.[1]

APPENDIX B

In this Appendix are outlined the basic lemmas which allow us to take derivatives and equations of motion inside Zimmermann's normal-product symbol. We shall assume basic scalar fields A_i and an effective Lagrangian of the form (a, b, and g totally symmetric)

$$\mathcal{L}_{\text{eff}}=\tfrac{1}{2}\partial_\mu A_i\partial^\mu A_i-\tfrac{1}{2}\sum_j m_j^{\,2}A_j^{\,2}+\tfrac{1}{2}a_{ij}A_iA_j+\tfrac{1}{2}b_{ij}\partial_\mu A_i\partial^\mu A_j-(1/4\,!)g_{ijkl}A_iA_jA_kA_l\,. \tag{B1}$$

Lemma I. Let M be a monomial in the basic fields and their derivatives of degree m and dimension $\leq\delta$. Then

$$\partial_\mu\langle 0|TN_\delta[M(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle=\langle 0|TN_{\delta+1}[\partial_\mu M(x)]A_{i_1}(x_1)\cdots A_{i_N}(x_N)|0\rangle. \tag{B2}$$

Sketch of proof. If G is a diagram contributing to the left-hand side of (B2), V is the special vertex of G corresponding to $N_\delta[M]$, and in momentum space p, $k_1, \ldots, k_m$ are the four-momenta flowing into V, then conservation of energy and momentum and the rule for differentiating products imply immediately the equality of the unrenormalized integrands corresponding to G. The only thing remaining to be shown is that for an arbitrary renormalization part containing V, $t^\gamma_{\delta(\gamma)}$, the Taylor operator used to perform the BPHZ subtractions, satisfies

$$p_\mu t^\gamma_{\delta(\gamma)}F=t^\gamma_{\delta(\gamma)+1}(p_\mu F)\,, \tag{B3}$$

where

$$F=\prod_{\gamma'\in\mathfrak{F}}(-t^{\gamma'}_{\delta(\gamma')})I_\gamma$$

and $\mathfrak{F}$ is a "normal forest," a set of nonoverlapping proper subdiagrams of γ. This is readily verified. The general result then follows from (2.3) and repeated application of (B3).

Lemma II. Let M stand for A_i, ∂_0A_i ($i=1,2,3$), or an arbitrary spatial derivative of those fields. Suppose M has dimension $<\delta-3$. Then

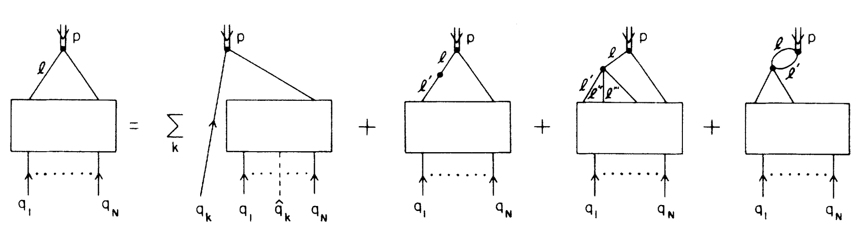

FIG. 1. Diagrams contributing to the left-hand side of (B4).

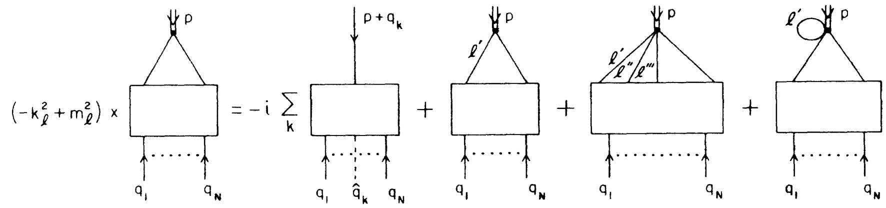

FIG. 2. Same diagrams as in Fig. 1, showing effect of Klein-Gordon operator.

$$
\begin{aligned}
&\langle 0 \,|\, TN_\delta[(\partial^2 + m_i{}^2)A_i M](x) A_{i_1}(x_1)\cdots A_{i_N}(x_N) \,|\, 0\rangle^{(t)} \\
&\qquad = \langle 0 \,|\, TN_\delta[(-(1/3\,!)g_{ijkl}A_jA_kA_l - b_{ij}\partial^2 A_j + a_{ij}A_j)M](x) A_{i_1}(x_1)\cdots A_{i_N}(x_N) \,|\, 0\rangle^{(t)} \\
&\qquad\quad - i \sum_k \delta_{i i_k}\delta(x - x_k)\langle 0 \,|\, TM(x) A_{i_1}(x_1)\cdots \hat{A}_{i_k}(x_k)\cdots A_{i_N}(x_N) \,|\, 0\rangle^{(t)} \,.
\end{aligned}
\tag{B4}
$$

Sketch of proof. As depicted in Fig. 1, the diagram contributing to the left-hand side of (B4) may be grouped into classes according to whether the line l corresponding to $\partial^2 A_i$ in the normal product connects the normal-product vertex V with (I) no vertex, (II) a 2-vertex, or (III) a 4-vertex, and in the last case according to whether there is (IIIa) no other or (IIIb) one other line between V and the same 4-vertex. There are no other possibilities.

The Klein-Gordon operator within the normal product has the effect, up to a factor $(-i)$ of contracting l to a point, as shown in Fig. 2. We now apply BPHZ and the definition of normal products. The loop diagrams are canceled by subtraction terms and the remaining diagrams give our desired result with the proper normal-product indices.

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