Multiparticle Partial-Wave Amplitudes and Inelastic Unitarity. I. General Formalism; Racah Coefficients for the Poincaré Group

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Multiparticle partial-wave amplitudes are defined for arbitrary scattering processes. Unitarity conditions are then imposed at arbitrary energies, resulting in nonlinear relations between the multiparticle partial-wave amplitudes. It is pointed out that the derivation of such unitarity relations is equivalent to finding the Racah coefficients for the Poincaré group. These coefficients, as with other quantities of interest in the paper, are derived for particles with arbitrary spins, masses, and boosting operations. Complications due to identical particles are also discussed.

I. INTRODUCTION

It is well known[1] that unitarity, when applied to spinless elastic scattering amplitudes, allows one to write the elastic partial-wave amplitudes $A_J(s)$ as

$$\frac{\eta_J(s)e^{2i\delta_J(s)}-1}{2i\rho},$$

where $\delta_J(s)$ is the phase shift, $\eta_J(s)\leq 1$ is the inelasticity parameter, and ρ is a phase-space factor. The arguments of the various functions are J, the total angular momentum, and $\sqrt{s}$, the invariant energy.

When the energy $\sqrt{s}$ is sufficiently large, not only new two-particle channels, but also multiparticle channels open up, all contributing to making $\eta_J(s)<1$. And just as in the case of elastic scattering amplitudes, unitarity imposes constraints on the corresponding multiparticle partial-wave amplitudes; one of the purposes of this paper will be to define multiparticle partial-wave amplitudes for arbitrary numbers of particles, all with arbitrary masses, spins, and boosts, and then derive the equations expressing the constraints which unitarity imposes on these multiparticle partial-wave amplitudes. The reason for considering arbitrary masses, spins, and boosts is not only for the sake of generality, but also that it is in some cases easier to manipulate various Wigner rotations without specifying a particular boost such as helicity. In any event the resulting equations will contain various partial-wave amplitudes related to one another via integrations weighted by appropriate phase-space factors.

It might seem that such unitarity equations would be hopelessly complicated because of the many channels which are open at arbitrary energies. But from the point of view of this paper, only four generic types of channels appear, and the many channels which in general are open can all be subsumed under these four generic types. How these four types of channels arise will be discussed in Sec. III.

The basis for the whole analysis is the construction of "symmetric" multiparticle partial-wave amplitudes,[2] built out of N-fold multiparticle states which are coupled together in such a way as to eliminate all intermediate coupling labels such as orbital or (total) spin angular momentum. Such symmetric multiparticle states are discussed in Sec. II and Appendix A, and are to be contrasted with other procedures for constructing multiparticle states, in which multiparticle states are built up in a stepwise fashion,[3] resulting in particle labels and kinematic factors which are often difficult to manipulate or eliminate. In particular, stepwise schemes become extremely cumbersome when some of the particles are identical[4]; Sec. V will discuss the advantages of the symmetric-coupling scheme, whereby the interchange of identical particles merely necessitates introducing certain Wigner D functions, whose arguments are given by relativistically invariant subenergies. Since symmetric multiparticle states are rather unfamiliar objects, Appendix A will also consider some of the properties of multiparticle helicity states, and, in particular, the phases of multiparticle helicity states will be compared with the standard phases of Jacob and Wick.[5]

Sections III and IV contain the actual derivations of the unitarity equations. For clarity of presentation, Sec. III discusses only the situation in which all particles are spinless; then having explained the technique for deriving spinless unitarity equations, Sec. IV will consider the added complication of spin and subenergy counting.

It will also become clear in Sec. III as to why knowledge of the unitarity equations is in some ways equivalent to knowledge of the Racah coefficients of the Poincaré group. For an essential problem in imposing unitarity constraints on multiparticle partial-wave amplitudes is to express the arguments of all the multiparticle partial-wave amplitudes in terms of a standard set of variables, the subenergies. Since the Racah coefficients are those coefficients which express labels of one coupling scheme in terms of another, they offer the possibility of getting the arguments of multiparticle partial-wave amplitudes into a standard form.

Now for the rotation group there is no intrinsic way of choosing one coupling scheme over another.[6] In contrast, because of the induced representation structure of the Poincaré group, there are not only stepwise coupling schemes, but also a symmetric coupling scheme, which can serve as a standard, in the sense that the Racah coefficients connecting one stepwise coupling scheme to another are obtained by reference of the stepwise coupling scheme relative to the standard symmetric scheme. The mathematics involved in finding the coefficients which express the labels of three relativistic particles, coupled in a stepwise fashion, in terms of the labels of a symmetric three particle state is carried out in Appendix B. As with the Clebsch-Gordan coefficients of the Poincaré group, these Racah coefficients of the Poincaré group turn out to involve products and integrals of Wigner rotations whose arguments are invariant subenergies.

It will become evident that one of the difficulties in reading this paper arises from an inadequate notation for rotations which have complicated arguments. Euler angles are far too cumbersome to manipulate, so what has been done is to specify a rotation as a transformation from one coordinate frame to another, and then to exploit the group properties of rotations. After all the manipulations on the rotations have been carried out the rotation can be reexpressed in terms of Euler angles. Section II will show how the various coordinate frames naturally arise and how they are used in a multiparticle partial-wave expansion.

II. MULTIPARTICLE STATES AND MULTIPARTICLE PARTIAL-WAVE AMPLITUDES

The multiparticle states which will be discussed in this section are defined so that as many quantities as possible that appear in the related partial-wave amplitudes (PWA) can be directly measured in scattering experiments. No quantities such as intermediate angular momenta and their attendant projections will appear; rather quantities such as the energies of various particles or clusters of particles and spin projections of individual particles will be considered. The variables that will be discussed are the generalization of the Omnès variables[7]; for a nonrelativistic spinless three-particle state in the over-all c.m. frame, Omnès used the energy of the three particles and three angles, the transformation coefficients of which were the total angular momentum, and spin projections along a body-fixed and space-fixed axis.

The generalization of such a three-body state to a N-body relativistic state (in the over-all c.m. frame), with the individual particles having arbitrary mass and spin is

$$|[M_1J_1]\vec{p}_1\sigma_1;\cdots[M_NJ_N]\vec{p}_N\sigma_N\rangle=\sum_{r_1,\ldots,r_N}\sum_J\sum_{r\sigma}(2J+1)^{1/2}D^J_{\sigma r}(R)\prod_{i=1}^N D^{J_i}_{r_i\sigma_i}(p_i,R^{-1})\,|[\sqrt{s}\,J]\vec{p}=\vec{0}\sigma;r,r_1\cdots r_N\{s_q\}\{\text{sgn}i\}\rangle\,. \tag{2.1}$$

The derivation of Eq. (2.1) is given in Appendix A, Eq. (A12). Here all the quantities appearing in (2.1) will be discussed with a view towards their physical significance. It is to be noted that the multiparticle state is written in such a way as to indicate that the first six labels of the state correspond to the usual single-particle labels; only the remaining labels indicate the multiparticle nature. $\sqrt{s}=[(p_1+\cdots+p_N)^2]^{1/2}$ is the invariant "mass" (energy) of the multiparticle state, while J is the total angular momentum (invariant spin of the multiparticle). Since the multiparticle state is in its rest frame (over-all c.m. frame), its total momentum $\vec{p}$ is zero; finally, there appears a spin component σ which is the projection of the total angular momentum along an axis defined relative to the rotation R.

As shown in Appendix A, R is shorthand for $R(\text{obs}\to\text{bf})$; that is, a rotation between a space-fixed or observer's frame to a body-fixed frame, a frame fixed by the momentum vectors. Then the coefficients appearing in the transform of R are the total angular momentum J and spin projections σ relative to the space-fixed axis and r relative to the body-fixed axis. How the body-fixed axis is chosen is arbitrary; a convenient choice is to write

$$\hat{z}\ (\text{body-fixed axis})=\hat{p}_1\,,$$
$$\hat{x}\ (\text{body-fixed axis})=(\hat{p}_2-\hat{p}_1\cos\theta_{12})/\sin\theta_{12}\,, \tag{2.2}$$
$$\cos\theta_{12}=\hat{p}_1\cdot\hat{p}_2,\quad 0\leqslant\theta_{12}\leqslant\pi\,,$$

where $\hat{p}_i$ is the unit vector direction of the ith particle. It is to be noted that J and J_i will always denote the total angular momentum and the intrinsic spin angular momentum of the individual particles, respectively.

The multiparticle state, Eq. (2.1), is labeled by a number of parameters $\{s_q\}$ and $\{\text{sgn}i\}$. Simple counting shows that $3N-3$ independent momentum labels can equivalently be given in terms of three Euler angles appearing in the rotation R and an energy variable $\sqrt{s}$, with $3N-7$ variables remaining; these $3N-7$ variables are collectively denoted by $\{s_q\}$ (s will be used for subenergy) and are formed out of invariants of four-momenta. For example, in the three-particle case, the possibilities are $(p_1+p_2)^2$, $(p_1+p_3)^2$, and $(p_2+p_3)^2$.

But the three subenergies possible for the three-particle state are not linearly independent. There are many ways of choosing a linearly independent set of $3N-7$ subenergies from the $\frac{1}{2}N(N-1)$ scalar products $(p_i+p_j)^2$. Asribekov[8] has shown that the relations between the various scalar products can be given in terms of Gram determinants. The notation $\{s_q\}$, $q=1,\ldots,3N-7$, is meant to denote any linearly independent set. When more than three-particle states are considered, other combinations of subenergies such as $(p_1+p_3+p_4)^2$ also are possible; Asribekov has shown how to include any of these more general invariants in the linearly independent set $\{s_q\}$. It should be noted that as far as the analytic properties of generalized multiparticle PWA are concerned, the subenergies $\{s_q\}$ may not be the most convenient choice of variables; ratios like s_q/s or linear combinations of such ratios seem to have more convenient analytic properties.[9] But since we will not be concerned with analytic properties of multiparticle PWA, we will continue to use the subenergies $\{s_q\}$.

It would seem that, having exhausted the $3N-7$ variables by the subenergies $\{s_q\}$, no other variables would appear in the multiparticle state (when particle $1,\ldots,N$ are all spinless). What then is the significance of the labels sgn$i$? If one knows the $3N-3$ independent momentum labels, it is easy to compute the angles between any two particles. However, when the momentum labels are replaced by the subenergies, it is possible to compute only the cosine of the angle between two particles:

$$(p_i+p_j)^2 = M_i^{\,2}+M_j^{\,2}+2E_iE_j-2|\vec{p}_i||\vec{p}_j|\cos\theta_{ij}\,. \tag{2.3}$$

Now the energies E_i and E_j of particles i and j can always be computed from the subenergies $\{s_q\}$, so that $\cos\theta_{ij}$ also is fixed by the subenergies. However, it is not possible to ascertain whether θ_{ij} or $2\pi-\theta_{ij}$ is the angle between i and j. Hence the need to find another set of relativistic invariants which distinquish between θ_{ij} and $2\pi-\theta_{ij}$. Such a set can be formed from the over-all momentum, along with the four-momenta of the particles defining the body-fixed axis. If the body-fixed axes are defined, for example, by Eq. (2.2), then a suitable set of invariants is $\epsilon_{\alpha\beta\gamma\delta}p^\alpha p_1^\beta p_2^\gamma p_i^\delta$, the sign of which resolves the ambiguity of θ_{ij} and $2\pi-\theta_{ij}$. Hence it is necessary to have an additional set of $N-3$ invariants,

$$\text{sgn}i \equiv \text{sgn}\,\epsilon_{\alpha\beta\gamma\delta}p^\alpha p_1^\beta p_2^\gamma p_i^\delta\,, \quad i=4,\ldots,N\,. \tag{2.4}$$

Finally, we come to the spin complications of the multiparticle state. There are in general N Wigner D functions relating the spin projections of the N particles in the body-fixed and space-fixed frames. Though the multiparticle state of Eq. (2.1) is written for arbitrary boosts, as discussed in Appendix A, it is instructive to show that if the canonical or spin-component basis[10] is used (rather than helicity,[5] for example), that the Wigner rotations (p_i, R^{-1}) all become R^{-1}, indicating that for the spin-component basis, in which the spin projection of every particle is measured in the same direction, the rotations relating body-fixed and space-fixed axes for the individual particles are all the same.

To see this, note that the relevant boosts for the canonical basis consist of pure Lorentz transformation which in SL(2, C) are Hermitian matrices that can be written [see Eqs. (A3) and (A5)]

$$\begin{aligned} &B_c(p) = R(\hat{p})\Lambda_z(|\vec{p}|)R^{-1}(\hat{p})\,,\\ &R(\hat{p}), \Lambda_z(|\vec{p}|) \in \text{SL}(2,C)\,. \end{aligned} \tag{2.5}$$

Then the Wigner rotation (p_i, R^{-1}), by definition, is

$$\begin{aligned}(p_i, R^{-1}) = {}&[R(R^{-1}\hat{p}_i)\Lambda_z^{-1}(|\vec{p}_i|)R^{-1}(R^{-1}\hat{p}_i)]\\ &\times R^{-1}(R(\hat{p}_i)\Lambda_z(|\vec{p}_i|)R^{-1}(\hat{p}_i))\,.\end{aligned} \tag{2.6}$$

As shown in Appendix A, $R^{-1}(R^{-1}\hat{p}_i)R^{-1}R(\hat{p}_i)$ is a rotation about the z axis, so that the pure Lorentz transformation along the z axis, $\Lambda_z(|\vec{p}_i|)$ and $\Lambda_z^{-1}(|\vec{p}_i|)$, cancels. Then

$$\begin{aligned}(p_i, R^{-1}) &= R(R^{-1}\hat{p}_i)R_zR^{-1}(\hat{p}_i)\\ &= R(R^{-1}\hat{p}_i)R^{-1}(R^{-1}\hat{p}_i)R^{-1}R(\hat{p}_i)R^{-1}(\hat{p}_i)\\ &= R^{-1}\end{aligned} \tag{2.7}$$

and the D functions become

$$\begin{aligned}D^{J_i}_{r_i\sigma_i}(p_i, R^{-1}) &= D^{J_i}_{r_i\sigma_i}(R^{-1})\\ &= D^{J_i\,*}_{\sigma_i r_i}(R)\,.\end{aligned} \tag{2.8}$$

The significance then, of the quantities r_i, $i=1,\ldots,N$ in the multiparticle state, Eq. (2.1), is that of spin projections of the N particles in the body-fixed frame of reference.

Finally, it is necessary to specify the normalization of the multiparticle state. If the single-particle states are normalized so that

$$\langle p_i'\sigma_i' | p_i\sigma_i\rangle = 2(2\pi)^3 E_i \delta^3(\vec{p}_i - \vec{p}_i)\delta_{\sigma_i\sigma_i'}, \tag{2.9}$$

then the multiparticle-state normalization can be computed. The Jacobian arising from the transformation of the $3N$ momentum components to the set $\{s_q\}$ of $3N-7$ invariants, along with s, $R(\text{obs}\to\text{bf})$, and $\vec{p}=\sum\vec{p}_i$ is

$$\frac{d^3p_1}{2(2\pi)^3E_1}\cdots\frac{d^3p_N}{2(2\pi)^3E_N} = (2\pi)^{-3N}\frac{d^3p_1}{2E_1}\cdots\frac{d^3p_{N-1}}{2E_{N-1}}d^4p_N\delta(p_N{}^2 - M_N{}^2). \tag{2.10}$$

Assuming that the momenta $\vec{p}_1$ and $\vec{p}_2$ fix the body-fixed coordinate system in some way, such as for example Eq. (2.2), gives

$$\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} = \frac{p_1{}^2}{2E_1}dp_1 d\Omega_{\hat{p}_1}\frac{d^3p_2}{2E_2} = \frac{p_1{}^2}{2E_1}\frac{p_2{}^2}{2E_2}8\pi^2 dR dp_1 dp_2 d(\hat{p}_1\cdot\hat{p}_2), \tag{2.11}$$

where dR is normalized so that $\int dR = 1$. Now $\hat{p}_1\cdot\hat{p}_2$, $p_1{}^2/2E_1$, and $p_2{}^2/2E_2$ can be expressed in terms of relativistic invariants, so that Eq. (2.10) becomes

$$\begin{aligned}\frac{d^3p_1}{2E_1}\cdots\frac{d^3p_N}{2E_N} &= (2\pi)^{-3N}d^4p\,dR\,8\pi^2\frac{p_1{}^2}{2E_1}\frac{p_2{}^2}{2E_2} \\ &\quad\times dp_1dp_2d\hat{p}_1\cdot\hat{p}_2\frac{d^3p_3}{2E_3}\cdots\frac{d^3p_{N-1}}{2E_{N-1}} \\ &\quad\times\delta[(p-(p_1+\cdots+p_{N-1}))^2 - M_N{}^2] \\ &= d^4p\,dR\,J(s,s_q)\prod_{q=1}^{3N-1}ds_q\sum_{\text{sgn}i},\end{aligned} \tag{2.12}$$

where $J(s,s_q)$ is the relativistically invariant Jacobian of the transformation and includes the factors $8\pi^2$, $p_1{}^2/2E_1$, etc. The actual functional form of $J(s,s_q)$ depends, of course, on the choice made of the invariants $\{s_q\}$. But the significance of Eq. (2.12) is that it separates the Euler angles specifying the body-fixed frame relative to the observer frame and the over-all four-momentum from the rest of the invariants.

It is now possible to carry out a partial-wave expansion for arbitrary reactions, and in so doing define the natural generalization of a partial-wave amplitude for multiparticle reactions. Consider a reaction $1'+\cdots+N'\to 1+\cdots+N$, where we are anticipating notation to be used in the following sections. Then

$$\begin{aligned}\langle 1\cdots N|T|1'\cdots N'\rangle &= \sum_{r_1,\ldots,r_N;r_1',\ldots,r_N'}\sum_{J;J'}\sum_{r\sigma;r'\sigma'}(2J+1)^{1/2}(2J'+1)^{1/2}D^{J*}_{\sigma r}(R)D^{J'}_{\sigma' r'}(R') \\ &\quad\times\left(\prod_{i=1}^{N'}D^{J_i'}_{r_i'\sigma_i'}(p_i',R'^{-1})\right)\left(\prod_{i=1}^{N}D^{J_i *}_{r_i\sigma_i}(p_i,R^{-1})\right)\delta_{JJ'}\delta_{\sigma\sigma'}\delta^4[(p_1+\cdots+p_N)-(p_1'+\cdots+p_N')] \\ &\quad\times\langle[\sqrt{s}\,J]r,r_1\cdots r_N\{s_q\}\{\text{sgn}i\}|\,|T|\,|[\sqrt{s'}\,J']r',r_1'\cdots r_N'\{s_q'\}\{\text{sgn}i'\}\rangle \\ &= \delta^4[(p_1+\cdots+p_N)-(p_1'+\cdots+p_N')]\sum_{r_1,\ldots,r_N;r_1',\ldots,r_N'}\sum_J\sum_{r;r'}(2J+1)D^J_{rr'}(R^{-1}R') \\ &\quad\times\left(\prod_{i=1}^{N'}D^{J_i'}_{r_i'\sigma_i'}(p_i',R'^{-1})\right)\left(\prod_{i=1}^{N}D^{J_i *}_{r_i\sigma_i}(p_i,R^{-1})\right)A_{Jrr'r_1\cdots r_Nr_1'\cdots r_N'}(s,\{s_q\}\{\text{sgn}i\},\{s_q'\}\{\text{sgn}i'\}).\end{aligned} \tag{2.13}$$

Only the relativistic invariance of the T operator has been used in Eq. (2.13);

$$A_{Jrr'r_1\cdots r_Nr_1'\cdots r_N'}(s,\{s_q\}\{\text{sgn}i\},\{s_q'\}\{\text{sgn}i'\})$$

is the generalization of a partial-wave amplitude and has all of its variables written out. The rotation R' can be written more explicitly as $R(\text{obs}\to\text{bf}')$; that is, the rotation from the observer's frame to the "primed" body-fixed or, in the notation Sec. III, the initial frame. Similarly, the rotation R is $R(\text{obs}\to\text{bf})$ and is the rotation from the observer's frame to the body-fixed frame.

It might seem strange that the multiparticle-to-multiparticle scattering amplitude contain any reference to an observers frame, since by relativistic invariance, the variables appearing in the scattering amplitude should refer only to quantities related to the reacting particles. But the point is that, as discussed earlier in this section, the body-fixed frames are specified only by momentum vectors. It is, however, also possible to (partially) specify frames by polarization directions of particles. The fact that an observer's frame still appears in the final expression allows for these polarization directions to be used in specifying frames.

For example, for two incoming particles ($N'=2$), with one particle polarized perpendicular to the beam direction, one could choose the observer's frame such that the z axis were given by the polarization direction, while the $x-z$ plane were given by the beam direction. On the other hand, one might choose the observer's frame to be the body-fixed frame for a multiparticle state, in which case any polarization would be measured relative to the body-fixed frame; then $R(\text{obs} \rightarrow \text{bf})$ would be the identity rotation.

Finally, it is to be noted that a multiparticle-to-multiparticle scattering amplitude may not be square-integrable because of rescattering singularities which can occur in multiparticle scattering amplitudes.[11] Such difficulties will be dealt with in succeeding papers; here we will always assume that it is meaningful to carry out the expansion of a multiparticle amplitude.

III. ANALYSIS OF PARTIAL-WAVE AMPLITUDES OCCURRING IN THE UNITARITY EQUATIONS

Because the variables occurring in multiparticle partial-wave amplitudes are in general quite complicated, including as they do spin labels of the reacting particles, subenergies and the "sgn" invariants, this section will assume all particles are spinless, and ignore the functional dependence on subenergies and "sgn" invariants. By so doing, the basically simple considerations necessary to derive the unitarity equations can be emphasized while the complications of spin, subenergy, and "sgn-" invariant dependence are relegated to Sec. IV.

The starting point is the unitarity equation $S^\dagger S = I$; we will take matrix elements of the operator equation for the multiparticle connected T matrix, expand the resulting scattering amplitudes in a partial-wave series of the type defined in Sec. II, and then eliminate the kinematic factors so that the remaining equations contain only partial-wave amplitudes or integrals over partial-wave amplitudes weighted by phase-space factors.

The T operator is related to the scattering operator by $iT = S - I$, so that $i(T^\dagger - T) = T^\dagger T$. Matrix elements of an arbitrary reaction $1' + \cdots + N' \rightarrow 1'' + \cdots + N''$ then gives

$$i\langle 1'' \cdots N'' | T^\dagger - T | 1' \cdots N' \rangle = \langle 1'' \cdots N'' | T^\dagger T | 1' \cdots N' \rangle . \quad (3.1)$$

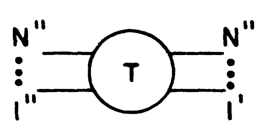

FIG. 1. Multiparticle bubble diagram.

We wish to insert between $T^\dagger$ and T on the right-hand side of Eq. (3.1) a complete set of states; now clearly for an arbitrary multiparticle reaction there will in general be an enormous number of such intermediate states. However, from the point of view of the analysis being carried out in this paper only four types of intermediate states need be analyzed. The left-hand side of Eq. (3.1) and the four types of intermediate states are lettered from a to e so that the unitarity Eq. (3.1) becomes

$$i\langle 1'' \cdots N'' | T^\dagger - T | 1' \cdots N' \rangle \quad (3.2a)$$

$$= \sum_{(1\cdots N)_b} \langle 1'' \cdots N'' | T^\dagger | 1 \cdots N \rangle \times \langle 1 \cdots N | T | 1' \cdots N' \rangle \quad (3.2b)$$

$$+ \cdots$$

$$+ \sum_{(1\cdots N)_e} \langle 1'' \cdots N'' | T^\dagger | 1 \cdots N \rangle \times \langle 1 \cdots N | T | 1' \cdots N' \rangle . \quad (3.2e)$$

The four types of intermediate states can best be understood in terms of so-called "bubble" diagrams.

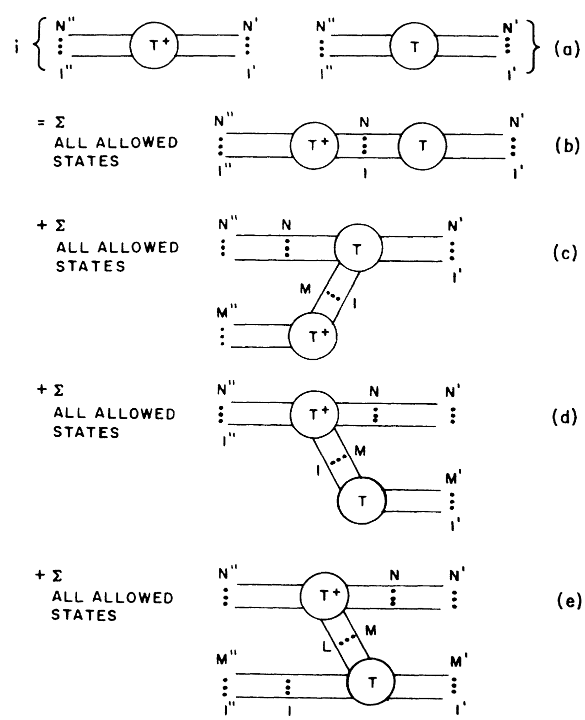

FIG. 2. Unitarity equations in terms of multiparticle bubble diagrams.

grams.[12] The bubble diagram for a multiparticle scattering amplitude is given in Fig. 1, in which particles $1',\ldots,N'$ react to produce particles $1'',\ldots,N''$. The unitarity equations, (3.2), in terms of bubble diagrams are displayed in Fig. 2, and from this figure it should be clear in what sense only four types of diagrams appear. For any one of the four types of diagrams there will, of course, be a host of actual reactions involving in general many different intermediate particles. Notice that each diagram is lettered; this lettering will be used later when each bubble diagram is separately analyzed.

The diagrams in Fig. 2 can be used to see why Racah coefficients of the Poincaré group arise. In Figs. 2(b) all particles in the intermediate state are coupled together symmetrically as discussed in Sec. II. However in Fig. 2(c) only the first M intermediate-state particles are coupled symmetrically in the T-matrix element, while all N should be coupled symmetrically for the $T^\dagger$ matrix element. In order to have only subenergies and "sgn" variables as arguments of the partial-wave amplitudes, it is necessary to compute Racah coefficients so as to eliminate the labels that would occur in the $T^\dagger$ matrix elements as a result of coupling the first M particles symmetrically, coupling particles $M+1,\ldots,N$ symmetrically, and then coupling the resulting two states together to form an N-particle state. Actually, in this section, Racah coefficients will not be explicitly used; rather a technique equivalent to using Racah coefficients will be used to derive the unitarity equations. It is to be noted that Fig. 2 contains no disconnected diagram, for example, of the type shown in Fig. 3, since such diagrams are spurious and can always be eliminated.[12]

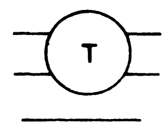

FIG. 3. A disconnected diagram.

We wish then to analyze successively all of the diagrams of Fig. 2, carrying out partial-wave expansions and eliminating kinematic factors whenever possible. The diagram of (3.2a) [Fig. 2(a)] determines which kinematic factors are common to all diagrams since it contains no intermediate states. Thus

$$\begin{aligned}(3.2\mathrm{a}) &\equiv i\{\langle 1''\cdots N''|T^\dagger - T|1'\cdots N'\rangle\}\\ &= i\{\langle 1'\cdots N'|T|1''\cdots N''\rangle^* - \langle 1''\cdots N''|T|1'\cdots N'\rangle\}\end{aligned} \tag{3.3}$$

and, according to Eq. (2.13), the matrix element $\langle 1''\cdots N''|T|1'\cdots N'\rangle$ can be expanded in multiparticle partial waves, so that

$$\langle 1''\cdots N''|T|1'\cdots N'\rangle = \delta^4(p_f - p_{\rm in})\sum_{Jr''r'}(2J+1)D^J_{r''r'}(R(f\to \mathrm{in}))A^{\mathrm{in}\to f}_{Jr''r'}\,; \tag{3.4}$$

similarly the matrix element for $T^\dagger$ can be expanded so that

$$\langle 1'\cdots N'|T|1''\cdots N''\rangle^* = \delta^4(p_f - p_{\rm in})\sum_{Jr''r'}(2J+1)D^{J*}_{r'r''}(R(\mathrm{in}\to f))\,A^{f\to\mathrm{in}*}_{Jr'r''}\,. \tag{3.5}$$

Here $R(f\to \mathrm{in})$ means the rotation from the coordinate system fixed by the final (outgoing) particles to the coordinate system fixed by the initial (incoming) particles. It is to be noted that no arguments other than J, r', and r'' are attached to the partial-wave amplitudes since these are the only labels which are being considered in this section. By combining Eqs. (3.4) and (3.5), Eq. (3.3) becomes

$$i\{\langle 1''\cdots N''|T^\dagger - T|1'\cdots N'\rangle\} = \delta^4(p_f - p_{\rm in})\sum_{Jr''r'}(2J+1)D^J_{r''r'}(R(f\to \mathrm{in}))i[A^{f\to\mathrm{in}*}_{Jr'r''} - A^{\mathrm{in}\to f}_{Jr''r'}]\,; \tag{3.6}$$

the term in square brackets can be collapsed even further using time-reversal invariance, but we shall leave it as is in this paper.

The goal now is to analyze diagrams (b) through (e) of Fig. 2 in such a way as to factor out a

$$\delta^4(p_f - p_{\rm in})\sum_{Jr''r'}(2J+1)D^J_{r''r'}(R(f\to \mathrm{in}))$$

term, with the remaining expression containing only partial-wave amplitudes or integrals over partial-wave amplitudes weighted by phase-space factors. Consider first Fig. 2(b), the diagram of (3.2b),

$$(3.2\text{b}) \equiv \int \frac{d^3p_1}{(2\pi)^3 2E_1} \cdots \frac{d^3p_N}{(2\pi)^3 2E_N} \langle 1'' \cdots N'' | T^\dagger | 1 \cdots N \rangle \langle 1 \cdots N | T | 1' \cdots N' \rangle$$
$$= \int d^4p_{\text{int}} dR(\text{obs} \to \text{int}) J(s, s_q) \prod_{q=1}^{3N-7} ds_q \delta^4(p_f - p_{\text{int}}) \delta^4(p_{\text{int}} - p_{\text{in}})$$
$$\times \sum_{Jr''rJ'r'\bar{r}} (2J+1)(2J'+1) D^{J*}_{rr''}(R^{-1}(\text{obs}\to\text{int})R(\text{obs}\to f)) A^{f\to\text{int}*}_{Jrr''} D^{J'}_{\bar{r}r'}(R^{-1}(\text{obs}\to\text{int})R(\text{obs}\to\text{in})) A^{\text{in}\to\text{int}}_{J'\bar{r}r'} . \tag{3.7}$$

The change of variables from the $3N$ momentum variables to the Jacobian $J(s, s_q)$ and the $3N-7$ subenergies s_q was discussed in Sec. II, Eq. (2.10) *ff.* By rewriting both the D functions appearing in Eq. (3.7), it is possible to eliminate all dependence on $R(\text{obs}\to\text{int})$ (int stands for the intermediate particles $1, \ldots, N$):

$$D^{J'}_{\bar{r}r'}(R^{-1}(\text{obs}\to\text{int})R(\text{obs}\to\text{in})) = \sum_{\sigma'=-J'}^{+J'} D^{J'*}_{\sigma'\bar{r}}(R(\text{obs}\to\text{int})) D^{J'}_{\sigma'r'}(R(\text{obs}\to\text{in})) ,$$
$$D^{J*}_{rr''}(R^{-1}(\text{obs}\to\text{int})R(\text{obs}\to f)) = \sum_{\sigma=-J}^{+J} D^{J}_{\sigma r}(R(\text{obs}\to\text{int})) D^{J*}_{\sigma r''}(R(\text{obs}\to f)) , \tag{3.8}$$
$$\int dR(\text{obs}\to\text{int}) D^{J'*}_{\sigma'\bar{r}}(R(\text{obs}\to\text{int})) D^{J}_{\sigma r}(R(\text{obs}\to\text{int})) = \frac{\delta_{JJ'}\delta_{\sigma\sigma'}\delta_{\bar{r}r}}{2J+1} .$$

Then Eq. (3.7) becomes

$$(3.2\text{b}) = \delta^4(p_f - p_{\text{in}}) \int J(s, s_q) \prod_{q=1}^{3N-7} ds_q \sum_{J\sigma r'r''r} (2J+1) D^J_{\sigma r'}(R(\text{obs}\to\text{in})) D^{J*}_{\sigma r''}(R(\text{obs}\to f)) A^{\text{in}\to\text{int}}_{Jrr'} A^{f\to\text{int}*}_{Jrr''}$$
$$= \delta^4(p_f - p_{\text{in}}) \sum_{Jr''r'} (2J+1) D^J_{r''r'}(R(f\to\text{in})) \sum_{r=-J}^{+J} \int J(s, s_q) \prod_{q=1}^{3N-7} ds_q A^{f\to\text{int}*}_{Jrr''} A^{\text{in}\to\text{int}}_{Jrr'} \tag{3.9}$$

which is of the desired form in that the correct common term has been factored out while what remains involves only integrals over multiparticle partial-wave amplitudes weighted by phase-space factors.

The analysis of the next diagram becomes somewhat more complicated. For the diagram of (3.2c) [Fig. 2(c)] we have

$$(3.2\text{c}) \equiv \int \frac{d^3p_1}{(2\pi)^3 2E_1} \cdots \frac{d^3p_N}{(2\pi)^3 2E_N} \langle 1'' \cdots N'' | T^\dagger | 1 \cdots N \rangle \langle 1 \cdots M | T | 1' \cdots M' \rangle (2\pi)^{3(N-M)} 2^{N-M} \prod_{i=M+1}^{N} E_i \delta^3(\vec{p}_i - \vec{p}_i{}') . \tag{3.10}$$

It is to be noted that (3.2c) could also have been written so as to have integrations over only the first M intermediate particles, with the state $|1\cdots N\rangle$ written as $|1\cdots M, M'+1, \ldots, N'\rangle$ and the $3(N-M)$ δ functions eliminated. An N-particle state would be formed from $|1\cdots M, M'+1, \ldots, N'\rangle$ by first coupling particles $1\cdots M$ and particles $M'+1, \ldots, N'$ symmetrically and then coupling the two resulting multiparticles. While such a procedure is straightforward, it results in partial-wave amplitudes with arguments different than the arguments given in Sec. II. Such partial-wave amplitudes can be transformed to partial-wave amplitudes with the correct arguments by using the generalization of the Racah coefficients derived in Appendix B.

Rather than using these coefficients, we will carry out an equivalent procedure not involving the Racah coefficients directly. To that end carry out a change of variables and partial-wave expansion of Eq. (3.10) as was done for (3.7):

$$(3.2\text{c}) = \int d^4p_{\text{int}} dR(\text{obs}\to\text{int}) J(s, s_q) \prod_{q=1}^{3N-7} ds_q$$
$$\times \sum_{Jrr''jmm'} (2J+1)(2j+1) \delta^4(p_f - p_{\text{int}}) \delta^4((p_1 + \cdots + p_M) - (p_1' + \cdots + p_M'))$$
$$\times D^{J*}_{rr''}(R^{-1}(\text{obs}\to\text{int})R(\text{obs}\to f)) A^{f\to\text{int}*}_{Jrr''} D^{j}_{mm'}(R^{-1}(\text{obs}\to\text{int}_{1,M}[1\cdots M\text{ c.m.}])R(\text{obs}\to\text{in}_{1',M'}[1'\cdots M'\text{c.m.}]))$$
$$\times A^{1'\cdots M'\to 1\cdots M}_{JMM'} (2\pi)^{3(N-M)} 2^{N-M} \prod_{i=M+1}^{N} E_i \delta^3(\vec{p}_i - \vec{p}_i{}') . \tag{3.11}$$

In order to bring Eq. (3.11) into the desired form, the $3(N-M)$ δ functions must be rewritten so that three variables specify a rotation from the observer to a coordinate frame fixed by the $M+1, \ldots, N$ momentum vectors (in the over-all c.m. frame) written $R(\text{obs}\to\text{int}_{M+1,N})$[13] while the remaining $3(N-M)-3$ variables are subenergies. The change of variables involves something like an inverse of the kind of Jacobian used in Eq. (3.7):

$$(2\pi)^{3(N-M)}2^{N-M}\prod_{i=M+1}^{N}\delta^3(\vec{p}_i-\vec{p}_i')=\tilde{J}(s_q)\prod_{q=3M-4}^{3N-7}\delta(s_q-s_q')\delta^3(R(\text{obs}\to\text{int}_{M+1,N})-R(\text{obs}\to\text{in}_{M'+1,N'})). \quad (3.12)$$

Now $\delta^3(R(\text{obs}\to\text{int}_{M+1,N})-R(\text{obs}\to\text{in}_{M'+1,N'}))$ can be represented as an infinite series of Wigner D functions:

$$\delta^3(R(\text{obs}\to\text{int}_{M+1,N})-R(\text{obs}\to\text{in}_{M'+1,N'}))=\sum_{Lkk'}(2L+1)D^{L*}_{kk'}(R(\text{obs}\to\text{int}_{M+1,N}))D^{L}_{kk'}(R(\text{obs}\to\text{in}_{M'+1,N'}))$$
$$=\sum_{Lkk'\tilde{k}}D^{L*}_{k\tilde{k}}(R(\text{obs}\to\text{int}))D^{L*}_{\tilde{k}k'}(R(\text{int}\to\text{int}_{M+1,N}))D^{L*}_{kk'}(R(\text{obs}\to\text{in}_{M'+1,N'})). \quad (3.13)$$

Further, the D^j function arising from the T-matrix element can be written as

$$D^{j}_{mm'}(R^{-1}(\text{obs}\to\text{int}[1\cdots M\text{ c.m.}])R(\text{obs}\to\text{in}[1'\cdots M'\text{c.m.}]))$$
$$=\sum_{\tilde{m}\tilde{m}'}D^{j*}_{\tilde{m}\tilde{m}'}(R(\text{obs}\to\text{int}))D^{j*}_{\tilde{m}'m}(R(\text{int}\to\text{int}[1\cdots M\text{ c.m.}]))D^{j}_{\tilde{m}m'}(R(\text{obs}\to\text{in}[1'\cdots M'\text{c.m.}])) \quad (3.14)$$

and the $D^{j*}(R(\text{obs}\to\text{int}))$ and $D^{L*}(R(\text{obs}\to\text{int}))$ combined to give

$$D^{j*}_{\tilde{m}\tilde{m}'}(R(\text{obs}\to\text{int}))D^{L}_{k\tilde{k}}(R(\text{obs}\to\text{int}))=\sum_{J'\tilde{r}\tilde{r}'}\langle J'\tilde{r}\,|\,j\tilde{m},Lk\rangle\langle J'\tilde{r}'\,|\,j\tilde{m}',L\tilde{k}\rangle D^{J'*}_{\tilde{r}\tilde{r}'}(R(\text{obs}\to\text{int})). \quad (3.15)$$

Then making use of the orthogonality relations of the D functions, with

$$\int dR\,D^{J}_{r'r}(R(\text{obs}\to\text{int}))D^{J'*}_{\tilde{r}\tilde{r}'}(R(\text{obs}\to\text{int}))=\frac{\delta_{JJ'}\delta_{r'\tilde{r}}\delta_{r\tilde{r}'}}{2J+1},$$

results in the following matrix element:

$$(3.2\text{c})=\delta^4(p_f-p_{\text{in}})\sum_{Jr'r''rjmm'\tilde{m}\tilde{m}'Lkk'\tilde{k}}(2j+1)(2L+1)D^{J*}_{r'r''}(R(\text{obs}\to f))\int J\tilde{J}\prod_{q=1}^{3M-5}ds_q\,\langle Jr'\,|\,j\tilde{m},Lk\rangle\langle Jr\,|\,j\tilde{m}',L\tilde{k}\rangle$$
$$\times D^{j}_{\tilde{m}'m}(R(\text{int}\to\text{int}[1\cdots M\text{c.m.}]))D^{j}_{\tilde{m}m'}(R(\text{obs}\to\text{in}[1'\cdots M'\text{c.m.}]))$$
$$\times D^{L*}_{\tilde{k}k'}(R(\text{int}\to\text{int}_{M+1,N}))D^{L}_{kk'}(R(\text{obs}\to\text{in}_{M'+1,N'}))A^{f\to\text{int}\,*}_{Jrr''}A^{1'\cdots M'\to1\cdots M}_{jmm'}. \quad (3.16)$$

There are several ways of eliminating the observer's frame from Eq. (3.16). One could carry out analogous procedures to that used in eliminating $R(\text{obs}\to\text{int})$; but an equivalent and simpler procedure is to choose the observer's frame as the initial frame and note that all rotations are either functions of appropriate subenergies or of the form $R(f\to\text{in})$ as desired. The final expression becomes

$$(3.2\text{c})=\delta^4(p_f-p_{\text{in}})\sum_{Jrr'r''jmm'}(2J+1)D^{J}_{r''r'}(R(f\to\text{in}))\int J\tilde{J}\prod_{q=1}^{3M-5}ds_q F_{Jrr'r''jmm'}A^{f\to\text{int}\,*}_{Jrr''}A^{1'\cdots M'\to1\cdots M}_{jmm'}, \quad (3.17)$$

where $F_{Jrr'r''jmm'}$ is a function of angular momentum labels and subenergies only:

$$F_{Jrr'r''jmm'}=\left(\frac{2j+1}{2J+1}\right)\sum_{Lkk'\tilde{k}\tilde{m}\tilde{m}'}(2L+1)\langle Jr'\,|\,j\tilde{m},Lk\rangle\langle Jr\,|\,j\tilde{m}',L\tilde{k}\rangle D^{j*}_{\tilde{m}'m}(R(\text{int}\to\text{int}[1\cdots M\text{ c.m.}]))$$
$$\times D^{j}_{\tilde{m}m'}(R(\text{in}\to\text{in}[1'\cdots M'\text{c.m.}]))D^{L}_{\tilde{k}k'}(R(\text{int}\to\text{int}_{M+1,N}))D^{L}_{kk'}(R(\text{in}\to\text{in}_{M'+1,N'})). \quad (3.18)$$

Note that the arguments of the D functions of Eq. (3.18) are of two kinds. First, an argument such as $R(\text{int}\to\text{int}_{M+1,N})$ means the rotation from the coordinate system fixed by the N intermediate particles in the over-all c.m. frame [int means $\text{int}_{1,N}(1\cdots N\text{c.m.})$][13] to the coordinate system fixed by the $M+1,\ldots,N$ intermediate particles, also in the over-all c.m. frame; since specification of a coordinate system requires only two nonparallel momentum vectors, it is possible to choose both "int" and "$\text{int}_{M+1,N}$" by, for example, the N and $N-1$ intermediate particles, in which case the rotation is the identity. But it is better at this point to leave all coordinate systems unspecified, so that, depending on the physical problem under consideration, the most convenient coordinate system can be chosen.

The second type of rotation has as a typical argument $R(\text{int}\to\text{int}(1\cdots M\text{ c.m.}))$, which means the rotation from the "int" coordinate frame to a coordinate frame fixed by the $1\cdots M$ intermediate particles in the

$1\cdots M$ c.m. frame. Thus, this rotation requires a boost from the over-all c.m. frame to the c.m. of the first M intermediate particles. The boost transforms the momenta of these particles from their over-all c.m. configurations to new configurations in the $1\cdots M$ c.m. frame, and the rotation is from "int" to this new configuration. Although this rotation is in general more complicated than the first rotation, it is still a function of subenergies only. Notice that if $R(\text{int}\to\text{int}_{M+1,N})$ is a simple rotation, $R(\text{int}\to\text{int}(1\cdots M\text{ c.m.}))$ is in general complicated and vice versa. But generally two of the rotations in Eq. (3.18) can be made simple.

The next diagram that should be analyzed is Fig. 2(d). But this diagram of (3.2d) is something like the time-reversed diagram of (3.2c) [Fig. 2(c)] which was just analyzed. Since the techniques for bringing (3.2d) into a standard form are exactly the same as those used for (3.2c), we could proceed directly to the diagram of (3.2e) [Fig. 2(e)]. However, this diagram is the most complicated of all the diagrams in terms of the algebraic manipulation involved, and since the techniques used are a straightforward generalization of those used in diagram of (3.2c), the analysis is relegated to Appendix C, which has been deposited with the National Auxiliary Publication Service.[14] It should be noted that the simplest example of this diagram, a so-called rescattering diagram, has already been analyzed by Dashen and Ma.[15]

What remains to be done in terms of being able to write down unitarity equations is to include spin properly, and then worry about subenergy and "sgn"- invariant counting. These problems will be taken up in Sec. IV.

IV. UNITARITY EQUATIONS FOR MULTIPARTICLE PARTIAL-WAVE AMPLITUDES

Section III showed how expressions could be derived for the four basic types of diagrams that occur in the unitarity equation. These expressions neglected spin complications and subenergy counting. In this section the complications due to spin will be dealt with by considering the three simplest diagrams and showing that certain spin D functions are common factors to all diagrams, while those that are not have the property that they are functions of subenergies only.

Once these spin functions are known it is possible to write down the basic unitarity equation which relates the various partial-wave amplitudes to one another. In so doing the dependence of the partial-wave amplitudes on the various subenergies will be exhibited, for because of the subenergy δ functions, a given partial-wave amplitude may have as variables initial, intermediate, and final subenergies.

As in Sec. III the simplest diagram is of (3.2a) [Fig. 2(a)], which is $i\{\langle 1''\cdots N''|T^\dagger - T|1'\cdots N'\rangle\}$. The spin D functions arise in the partial-wave expansion Eq. (2.13), and we consider them only since all other terms in (3.2a) are given in Eq. (3.6):

$$[\text{spin terms for }(3.2\text{a})]=\left(\prod_{i=1}^{N'} D^{J_i'}_{r_i'\sigma_i'}(p_i', R^{-1}(\text{obs}\to\text{in}))\right)\left(\prod_{i=1}^{N''} D^{J_i''*}_{r_i''\sigma_i''}(p_i'', R^{-1}(\text{obs}\to f))\right). \tag{4.1}$$

Since there are no intermediate states for (3.2a), the spin terms (4.1) can be factored out of all the diagrams; thus, these spin D functions are combined with the other terms already given in Eq. (3.6) to give the over-all common factor

$$\delta^4(p_f - p_{\text{in}}) \sum_{r_1',\ldots,r_{N'}'}\ \sum_{r_1'',\ldots,r_{N''}''}\ \sum_{J,r',r''} (2J+1) D^J_{r''r'}(R(f\to\text{in}))$$
$$\times\left(\prod_{i=1}^{N'} D^{J_i'}_{r_i'\sigma_i'}(p_i', R^{-1}(\text{obs}\to\text{in}))\right)\left(\prod_{i=1}^{N''} D^{J_i''*}_{r_i''\sigma_i''}(p_i'', R^{-1}(\text{obs}\to f))\right).$$

The next diagram, that of (3.2b) [Fig. 2(b)], is expanded in two partial-wave series, so that intermediate-particle spin functions arise; the spin functions are of the form

$$[\text{spin terms for }(3.2\text{b})]=\sum_{\sigma_i}\left(\prod_{i=1}^{N'} D^{J_i'}_{r_i'\sigma_i'}(p_i', R^{-1}(\text{obs}\to\text{in}))\right)\left(\prod_{i=1}^{N} D^{J_i*}_{r_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int}))\right)$$
$$\times\left(\prod_{i=1}^{N''} D^{J_i''*}_{r_i''\sigma_i''}(p_i'', R^{-1}(\text{obs}\to f))\right)\left(\prod_{i=1}^{N} D^{J_i}_{\bar{r}_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int}))\right). \tag{4.2}$$

Now two of these products, those involving the initial-particle and final-particle spin functions, are already of the form of Eq. (4.1); the remaining spin functions coming from the intermediate particles cancel out, since

$$\sum_{\sigma_i} D^{J_i}_{\tilde{r}_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int}))D^{J_i *}_{r_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int})) = D^{J_i}_{\tilde{r}_i r_i}((p_i, R^{-1})(p_i, R^{-1})^{-1})$$
$$= \delta_{\tilde{r}_i r_i}\,. \tag{4.3}$$

As before, the first real complications come from (3.2c) [Fig. 2(c)], and again have to do with Racah coefficients. While the spin D functions of the final particles are of the form given in Eq. (4.1), the Wigner rotations of the initial spin D functions are not written in the over-all c.m. frame, but in the $1\cdots M$ c.m. frame. Also, only the first M D functions arising from the intermediate particles will cancel as in Eq. (4.3). That is,

$$[\text{spin terms for (3.2c)}] = \sum_{\sigma_i}\left(\prod_{i=1}^{M'} D^{J_i'}_{r_i'\sigma_i'}(p_i', \Lambda')\right)\left(\prod_{i=1}^{M} D^{J_i *}_{\tilde{r}_i\sigma_i}(p_i, \Lambda)\right)$$
$$\times\left(\prod_{i=1}^{N''} D^{J_i'' *}_{r_i''\sigma_i''}(p_i'', R^{-1}(\text{obs}\to f))\right)\left(\prod_{i=1}^{N} D^{J_i}_{r_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int}))\right). \tag{4.4}$$

Now the first M (intermediate-particle) D functions with Wigner rotations $(p_i, R^{-1}(\text{obs}\to\text{int}))$ can be combined with the M D functions with Wigner rotations (p_i, Λ) by writing, for each term in the product

$$\sum_{\sigma_i} D^{J_i *}_{\tilde{r}_i\sigma_i}(p_i, \Lambda)D^{J_i}_{r_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int})) = \sum_{\sigma_i} D^{J_i *}_{\tilde{r}_i\sigma_i}(p_i, \Lambda)D^{J_i *}_{\sigma_i r_i}((p_i, R^{-1}(\text{obs}\to\text{int}))^{-1})$$
$$= D^{J_i *}_{\tilde{r}_i r_i}((p_i, \Lambda)(p_i, R^{-1}(\text{obs}\to\text{int}))^{-1})\,. \tag{4.5}$$

Writing out the two Wigner rotations gives

$$(p_i, \Lambda)(p_i, R^{-1}(\text{obs}\to\text{int}))^{-1} = [B^{-1}(\Lambda p_i)\Lambda B(p_i)][B^{-1}(R^{-1}p_i)R^{-1}(\text{obs}\to\text{int})B(p_i)]^{-1}$$
$$= B^{-1}(\Lambda p_i)\Lambda R(\text{obs}\to\text{int})B(R^{-1}p_i)$$
$$= B^{-1}(\Lambda p_i)\Lambda R(\text{obs}\to\text{int})B(p_i(\text{int}))\,, \tag{4.6}$$

where use has been made of the fact that $R^{-1}(\text{obs}\to\text{int})p_i(\text{obs}) = p_i(\text{int})$; that is, the rotation $R^{-1}(\text{obs}\to\text{int})$ acting on p_i (defined relative to the observer's frame) is equivalent to p_i relative to the "int" frame. From Appendix A, Eq. (A9), it is seen that $\Lambda = R^{-1}(\text{obs}\to\text{int}(1\cdots M\text{ c.m.}))B^{-1}(p_1+\cdots+p_M)$; further,[16] one has $\Lambda p_i \equiv \Lambda p_i(\text{obs}) = \Lambda R(\text{obs}\to\text{int})R^{-1}(\text{obs}\to\text{int})p_i(\text{obs}) = \Lambda R(\text{obs}\to\text{int})p_i(\text{int})$, and the Wigner rotation becomes $(p_i(\text{int}), \Lambda R(\text{obs}\to\text{int}))$. But $\Lambda R(\text{obs}\to\text{int})$ can be written as

$$\Lambda R(\text{obs}\to\text{int}) = R^{-1}(\text{obs}\to\text{int}(1\cdots M\text{ c.m.}))B^{-1}(p_1+\cdots+p_M)R(\text{obs}\to\text{int})$$
$$= R^{-1}(\text{int}\to\text{int}(1\cdots M\text{ c.m.}))R^{-1}(\text{obs}\to\text{int})B^{-1}(p_1+\cdots+p_M)R(\text{obs}\to\text{int})\,. \tag{4.7}$$

Now a boost by definition satisfies $p = B(p)p^+$, $p^+ = (M, \vec{0})$. Then

$$R^{-1}(\text{obs}\to\text{int})B(p)p^+ = R^{-1}(\text{obs}\to\text{int})B(p)R(\text{obs}\to\text{int})p^+$$
$$= R^{-1}(\text{obs}\to\text{int})p = p(\text{int}) = B(p(\text{int}))p^+ \tag{4.8}$$

so that $R^{-1}(\text{obs}\to\text{int})B(p)R(\text{obs}\to\text{int}) = B(p(\text{int}))$, and Eq. (4.7) becomes

$$\Lambda R(\text{obs}\to\text{int}) = R^{-1}(\text{int}\to\text{int}(1\cdots M\text{ c.m.}))B^{-1}(p_1+\cdots+p_M(\text{int}))\,. \tag{4.9}$$

Thus, the Wigner rotations of Eq. (4.6) can be written as $(p_i(\text{int}), R^{-1}(\text{int}\to\text{int}(1\cdots M\text{ c.m.}))B^{-1}(p_1+\cdots+p_M(\text{int})))$. The point of writing the Wigner rotation in this way is to show that it is completely independent of the observer's frame and depends only on subenergies of the intermediate particles.

The remaining spin functions of the intermediate particles, $D^{J_i}_{r_i\sigma_i}(p_i, R^{-1}(\text{obs}\to\text{int}))$, $i = M+1, \ldots, N$, can be written in such a way as to give the correct form for the initial spin functions. Since the $M+1, \ldots, N$ intermediate particles are the same as the $M'+1, \ldots, N'$ initial particles, the Wigner rotations can be written

$$(p_i, R^{-1}(\text{obs}\to\text{int})) = (p_i', (R(\text{obs}\to\text{in})R(\text{in}\to\text{int}))^{-1})$$
$$= (R^{-1}(\text{obs}\to\text{in})p_i', R^{-1}(\text{in}\to\text{int}))(p_i', R^{-1}(\text{obs}\to\text{in}))$$
$$= (p_i'(\text{in}), R^{-1}(\text{in}\to\text{int}))(p_i', R^{-1}(\text{obs}\to\text{in}))\,, \qquad i = M'+1, \ldots, N'\,. \tag{4.10}$$

Depending on how the intermediate and initial frames are fixed, both $p_i'(\text{in})$ and $R^{-1}(\text{in}\to\text{int})$ can be seen to depend on subenergies of the intermediate and/or initial systems only. Thus, the Wigner rotation, Eq. (4.10), has been split into a properly invariant Wigner rotation and a Wigner rotation of the correct form for the initial spin functions appearing in the common factor, Eq. (4.1).

There remains then the M' spin functions of the initial system, $D^{J_i'}_{r_i'\sigma_i'}(p_i', \Lambda')$, which are not of the correct form to be used in the common factor. By rewriting the Wigner rotation it is again possible to split off a term which depends only on initial subenergies times a Wigner rotation of the desired form:

$$\begin{aligned}(p_i', \Lambda') &= (p_i', \Lambda' R(\text{obs}\to\text{in})R^{-1}(\text{obs}\to\text{in}))\\ &= (R^{-1}(\text{obs}\to\text{in})p_i', \Lambda' R(\text{obs}\to\text{in}))(p_i', R^{-1}(\text{obs}\to\text{in}))\\ &= (p_i'(\text{in}), \Lambda' R(\text{obs}\to\text{in}))(p_i', R^{-1}(\text{obs}\to\text{in}))\,.\end{aligned}$$

Now $\Lambda' = R^{-1}(\text{obs}\to\text{in}(1'\cdots M'\text{c.m.}))B^{-1}(p_1' + \cdots + p_M')$, and, as shown in Eq. (4.9), $\Lambda' R(\text{obs}\to\text{in})$ can be equivalently written as $R^{-1}(\text{in}\to\text{in}(1'\cdots M'\text{c.m.}))B^{-1}(p_1' + \cdots + p_M'(\text{in}))$, so that Eq. (4.11) becomes

$$(p_i', \Lambda') = (p_i'(\text{in}), R^{-1}(\text{in}\to\text{in}(1'\cdots M'\text{c.m.}))B^{-1}(p_1' + \cdots p_M'(\text{in})))(p_i', R^{-1}(\text{obs}\to\text{in}))\,. \tag{4.12}$$

Collecting all these results, Eq. (4.4) can be written

[spin terms for (3.2c)]

$$\begin{aligned}&= \left(\prod_{i=1''}^{N''} D^{J_i''*}_{r_i''\sigma_i''}(p_i'', R^{-1}(\text{obs}\to f))\right)\sum_{\tilde r_i'\tilde r_i}\left(\prod_{i=1}^{N'} D^{J_i'}_{\tilde r_i'\sigma_i'}(p_i', R^{-1}(\text{obs}\to\text{in}))\right)\\ &\quad\times\left(\prod_{i=1}^{M'} D^{J_i'}_{r_i'\tilde r_i'}(p_i'(\text{in}), R^{-1}(\text{in}\to\text{in}(1'\cdots M'\text{c.m.}))B^{-1}(p_1' + \cdots + p_M'(\text{in})))\right)\\ &\quad\times\left(\prod_{i=M'+1}^{N'} D^{J_i'}_{\tilde r_i'\tilde r_i'}(p_i'(\text{in}), R^{-1}(\text{in}\to\text{int}))\right)\\ &\quad\times\left(\prod_{i=1}^{M} D^{J_i*}_{\tilde r_i r_i}(p_i(\text{int}), R^{-1}(\text{int}\to\text{int}(1\cdots M\,\text{c.m.})B^{-1}(p_1 + \cdots + p_M(\text{int})))\right).\end{aligned} \tag{4.13}$$

The spin functions for the remaining diagrams, those of (3.2d) [Fig. 2(d)] and (3.2e) [Fig. 2(e)], can be computed along similar lines to those given for the diagram of (3.2c) [Fig. 2(c)]. Again there are the common factors which still retain a dependence on the observer's frame, while the remaining spin functions are all dependent on appropriate subenergies only.

The natural way to conclude this section would be to write out the complete unitarity equation, involving sums over all allowed intermediate states of the four diagrams of Eq. (3.2). But, for arbitrary multiparticle processes, with arbitrary coordinate frames, the equation becomes so long and involved that it is not clear what is gained. Also, to write down a unitarity equation means choosing normalization factors with respect to the various disconnected lines that have been ignored in the bubble diagrams.

Rather, we will write down the most general form for the three terms for which the spin factors were explicitly computed in this section, leaving to the interested reader the task of computing the form of the terms (3.2d) and (3.2e). In so doing we will also show how the subenergies $\{s_q\}$ and the invariants $\{\text{sgn}i\}$ come in.

Consider first the diagram corresponding to Eq. (3.2a) [Fig. 2(a)]. We have

$$(3.2\text{a}) = i[A^{f\to\text{in}*}_{Jr'r''r_1'\cdots r_N'r_1''\cdots r_N''}(s, \{s_q'\}\{\text{sgn}i'\}, \{s_q''\}\{\text{sgn}i''\}) - A^{\text{in}\to f}_{Jr''r'r_1''\cdots r_N''r_1'\cdots r_N'}(s, \{s_q''\}\{\text{sgn}i''\}, \{s_q'\}\{\text{sgn}i'\})]. \tag{4.14}$$

All the common terms have been factored out, including the spin D functions of Eq. (4.1), leaving only the partial-wave amplitudes. There are $3N' - 7$ subenergies $\{s_q'\}$ and $N' - 3$ sgn invariants for the initial system, and $3N'' - 7$ subenergies $\{s_q''\}$ and $N'' - 3$ sgn invariants for the final system.

The diagram of (3.2b) [Fig. 2(b)] becomes that of

$$\begin{aligned}(3.2\text{b}) = \sum_{\text{sgn}i}\sum_{r,r_1,\ldots,r_N}\int J(s, s_1, \ldots, s_{3N-7})\Bigg(\prod_{q=1}^{3N-7} ds_q\, A^{f\to\text{int}*}_{Jrr''r_1\cdots r_Nr_1''\cdots r_N''}(s, s_1, \ldots, s_{3N-7}, \text{sgn}4, \ldots, \text{sgn}N; \{s_q''\}\{\text{sgn}i''\})\\ \times A^{\text{in}\to\text{int}}_{Jrr'r_1\cdots r_Nr_1'\cdots r_N'}(s, s_1, \ldots, s_{3N-7}, \text{sgn}4, \ldots, \text{sgn}N; \{s_q'\}\{\text{sgn}i'\})\Bigg)\,.\end{aligned}$$

Note that there is a summation over the two values that the invariant sgn$i$ can take, for $i=4,\ldots,N$. Those arguments of the multiparticle partial-wave amplitudes arising from the intermediate particles are written out explicitly, while the arguments pertaining to the initial and final systems are written in the same way as in Eq. (4.14).

Finally there is the diagram of (3.2c) [Fig. 2(c)]:

$$
\begin{aligned}
(3.2\text{c}) = &\sum_{\text{sgn}\, i} \; \sum_{r;r_1,\ldots,r_N;\tilde r_1,\ldots,\tilde r_N;\tilde r_1',\ldots,\tilde r_N'} \int J(s,s_1,\ldots,s_{3N-7}) \prod_{P=1}^{3N-7} ds_p \tilde J(s_q) \prod_{q=3M-4}^{3N-7} \delta(s_q - s_q') \\
&\times \sum_{jmm'} F_{Jrr'r''jmm'} \left(\prod_{i=1}^{M} D^{J_i *}_{r_i \tilde r_i}(p_i(\text{int}), R^{-1}(\text{int} \to \text{int}(1\cdots M\ \text{c.m.}))B^{-1}(p_1+\cdots+p_M(\text{int}))) \right) \\
&\times \left(\prod_{i=1}^{M'} D^{J_i'}_{\tilde r_i' r_i'}(p_i'(\text{in}), R^{-1}[\text{in}\to \text{in}(1'\cdots M'\text{c.m.})]B^{-1}[p_1'+\cdots+p_{M'}'(\text{in})]) \right) \left(\prod_{i=M'+1}^{N'} D^{J_i'}_{\tilde r_i r_i}(p_i'(\text{in}), R^{-1}(\text{in}\to \text{int})) \right) \\
&\times A^{f\to \text{int}\, *}_{Jrr''r_1\cdots r_N r_1''\cdots r_N''}(s,s_1,\ldots,s_{3N-7},\text{sgn}4,\ldots,\text{sgn}M,\text{sgn}M'+1,\ldots,\text{sgn}N';\{s_q''\}\{\text{sgn}i''\}) \\
&\times A^{1'\cdots M'\to 1\cdots M}_{jmm'\tilde r_1\cdots \tilde r_M \tilde r_1'\cdots \tilde r_M'}(s_{1'M'},s_1,\ldots,s_{3M-7},\text{sgn}4,\ldots,\text{sgn}M;s_1',\ldots,s_{3M'-7}',\text{sgn}4',\ldots,\text{sgn}M')\,.
\end{aligned}
$$

The sum over the two values of sgn$k$ extends from $k=4$ to $k=M$; the remaining sgn invariants of the intermediate state are equal to the corresponding sgn invariants of the initial state and are written out explicitly in the $f\to$ int multiparticle partial-wave amplitude. This equality of sgn invariants comes about because particles $M'+1,\ldots,N'$ are the same as particles $M+1,\ldots,N$; since, in particular, they have the same momentum direction, a change of variables from momentum labels to subenergy labels brings in not only subenergy δ functions $\delta(s_q - s_q')$ and the Jacobian $\tilde J(s_q)$, but also δ functions over sgn invariants.

V. IDENTICAL - PARTICLE PROBLEMS

Thus far the distinguishability of all particles has been implicitly assumed. In this section we wish to see how multiparticle states and partial-wave amplitudes are modified by the presence of identical particles. It will be shown that, as compared with the conventional stepwise coupling schemes, in which permuted particles must be coupled and uncoupled to generate suitably symmetrized or antisymmetrized multiparticle states, the symmetric coupling scheme offers distinct advantages in the ease with which identical-particle problems can be handled. Since all particle labels are treated on the same footing, with no labels arising from intermediate coupling schemes, it is quite easy to carry out the interchange of particle labels resulting in the correct permutation symmetry.

To see the effects of particle interchange consider Eq. (2.1), where N particles are symmetrically coupled together in the over-all c.m. frame:

$$
\begin{aligned}
|[M_1J_1]\vec p_1\sigma_1;\ldots[M_NJ_N]\vec p_N\sigma_N\rangle = &\sum_{r_1,\ldots,r_N} \sum_{Jr\sigma} (2J+1)^{1/2} D^J_{\sigma r}(R(\text{obs}\to\text{bf})) \\
&\times \left(\prod_{i=1}^{N} D^{J_i}_{r_i \sigma_i}(p_i, R^{-1}) \right) |[\sqrt{s}\ J]\vec p = \vec 0 \sigma; r, r_1\cdots r_N\{s_q\}\{\text{sgn}i\}\rangle\,. \qquad (5.1)
\end{aligned}
$$

It is clear that, under a permutation of particles, only the degeneracy labels $\{s_q\}$ and $\{\text{sgn}i\}$ will be permuted in the multiparticle state. The rotation $R(\text{obs}\to\text{bf})$ will in general also change depending on which particles specify the body-fixed (bf) frame and how they are permuted. Let bf$^{\text{per}}$ designate the body-fixed frame as specified by the permuted particles. Then

$$
R(\text{obs}\to \text{bf}^{\text{per}}) = R(\text{obs}\to\text{bf})R(\text{bf}\to\text{bf}^{\text{per}})\,. \qquad (5.2)
$$

The significance of Eq. (5.2) is that it expresses the rotation of the permuted particles in terms of the original rotation occurring in Eq. (5.1) and a rotation which depends on the permutation and subenergies of the particles only. Further the "permuted" spin rotations can be written

$$
\begin{aligned}
(p_i, R^{-1})^{\text{per}} &\equiv (p_i, R(\text{bf}^{\text{per}}\to\text{obs})) \\
&= (p_i, R(\text{bf}^{\text{per}}\to\text{bf})R(\text{bf}\to\text{obs})) \\
&= (R(\text{bf}\to\text{obs})p_i, R(\text{bf}^{\text{per}}\to\text{bf}))(p_i, R(\text{bf}\to\text{obs})) \\
&= (p_i(\text{bf}), R(\text{bf}^{\text{per}}\to\text{bf}))(p_i, R^{-1}(\text{obs}\to\text{bf}))\,, \qquad (5.3)
\end{aligned}
$$

where $p_i(\mathrm{bf})$ is the momentum of the ith particle relative to the body-fixed frame. That is, in writing p_i, one usually means the momentum of the ith particle relative to the observer's frame, so that a more accurate notation for p_i would be $p_i(\mathrm{obs})$. To get the momentum of the ith particle relative to the body-fixed frame requires the rotation $R^{-1}(\mathrm{obs}\to\mathrm{bf})$ on $p_i(\mathrm{obs})$, so

$$p_i(\mathrm{bf}) = R^{-1}(\mathrm{obs}\to\mathrm{bf})p_i(\mathrm{obs}) = R(\mathrm{bf}\to\mathrm{obs})p_i(\mathrm{obs})\,.$$

The significance of Eq. (5.3) is that the permuted Wigner rotation is written as a Wigner rotation which is a function of the permutation and subenergies only times the Wigner rotation occurring in the multiparticle expression, Eq. (5.1).

Collecting these results on rotations then allows the permuted multiparticle state to be written as

$$\begin{aligned}
|[M_1J_1]\vec{p}_1\sigma_1;\ldots[M_NJ_N]\vec{p}_N\sigma_N\rangle^{\mathrm{per}} &= \sum_{r_1,\ldots,r_N}\sum_{Jr\sigma}(2J+1)^{1/2}D^J_{\sigma r}(R(\mathrm{obs}\to\mathrm{bf}^{\mathrm{per}}))\\
&\quad\times\left(\prod_{i=1}^N D^{J_i}_{r_i\sigma_i}(p_i,R^{-1}(\mathrm{obs}\to\mathrm{bf}^{\mathrm{per}}))\right)|[\sqrt{s}\,J]\,\vec{p}=\vec{0}\,\sigma;r,r_1\cdots r_N\{s_q\}^{\mathrm{per}}\{\mathrm{sgn}i\}^{\mathrm{per}}\rangle\\
&=\sum_{r_1,\ldots,r_N;r_1',\ldots,r_N'}\sum_{Jr\sigma}(2J+1)^{1/2}D^J_{\sigma r'}(R(\mathrm{obs}\to\mathrm{bf}))D^J_{r'r}(R(\mathrm{bf}\to\mathrm{bf}^{\mathrm{per}}))\\
&\quad\times\left(\prod_{i=1}^N D^{J_i}_{r_ir_i'}(p_i(\mathrm{bf}),R(\mathrm{bf}^{\mathrm{per}}\to\mathrm{bf}))D^{J_i}_{r_i'\sigma_i}(p_i,R^{-1}(\mathrm{obs}\to\mathrm{bf}))\right)\\
&\quad\times|[\sqrt{s}\,J]\,\vec{p}=\vec{0}\,\sigma;r,r_1\cdots r_N\{s_q\}^{\mathrm{per}}\{\mathrm{sgn}i\}^{\mathrm{per}}\rangle\\
&=\sum_{r_1',\ldots,r_N'}\sum_{Jr'\sigma}(2J+1)^{1/2}D^J_{\sigma r'}(R(\mathrm{obs}\to\mathrm{bf}))\left(\prod_{i=1}^N D^{J_i}_{r_i'\sigma_i}(p_i,R^{-1}(\mathrm{obs}\to\mathrm{bf}))\right)\\
&\quad\times\sum_{r,r_1,\ldots,r_N}[D^J_{r'r}(R(\mathrm{bf}\to\mathrm{bf}^{\mathrm{per}}))D^{J_i}_{r_ir_i'}(p_i(\mathrm{bf}),R(\mathrm{bf}^{\mathrm{per}}\to\mathrm{bf}))]\\
&\quad\times|[\sqrt{s}\,J]\,\vec{p}=\vec{0}\,\sigma;r,r_1\cdots r_N\{s_q\}^{\mathrm{per}}\{\mathrm{sgn}i\}^{\mathrm{per}}\rangle\,.
\end{aligned}\tag{5.4}$$

Several things are to be noted from Eq. (5.4). First, the proper symmetrization of a multiparticle state involving arbitrary clusters of identical particles is carried out from knowledge of the symmetric and antisymmetric representation of the permutation group applied to $\{s_q\}^{\mathrm{per}}$ and $\{\mathrm{sgn}i\}^{\mathrm{per}}$, for the main complication in constructing properly symmetrized states is to express $\{s_q\}^{\mathrm{per}}$ and $\{\mathrm{sgn}i\}^{\mathrm{per}}$ in terms of $\{s_q\}$ and $\{\mathrm{sgn}i\}$, respectively. Second, the rotations involving bf $\to\mathrm{bf}^{\mathrm{per}}$ (and their inverses) can be eliminated by choosing, if possible, the body-fixed coordinate frame to be specified by distinguishable particles, so that no interchange of the body-fixed frame under particle permutation takes place. Third, the quantities in square brackets in Eq. (5.4) all are functions of invariant subenergies only, so that the permuted multiparticle state is a known (though generally complicated) linear combination of the unpermuted multiparticle state.

Most important, when a partial-wave analysis is carried out using multiparticle states containing identical particles, the result can be written in such a way as to have the same form as for distinguishable particles. Since the quantity in square brackets of Eq. (5.4) is a function of subenergies only, it can be absorbed into the multiparticle partial-wave amplitude, or conversely, a multiparticle partial-wave amplitude containing identical particles can be written as a known linear combination of "distinguishable" multiparticle partial-wave amplitudes.

As a simple example of these general conclusions, consider a three-particle final state, in which two particles – say 1 and 2 – are identical,[17] and the z axis of the final (body-fixed) frame is particle 1, while the final x-z plane is given by particle 2 [see Eq. (2.2)]. Then under the interchange of particles 1 and 2, the rotation $R(\mathrm{bf}\to\mathrm{bf}^{\mathrm{per}})$ is – since particle 2 becomes the body-fixed z axis – $R(\pi,\theta_{12},0)$, where $\cos\theta_{12}=\hat{p}_1\cdot\hat{p}_2$, $0\le\theta_{12}\le\pi$, and is a function of subenergies only. The Wigner rotations $(p_i(\mathrm{bf}),R^{-1}(\mathrm{bf}\to\mathrm{bf}^{\mathrm{per}}))$ will, of course, depend on the boost being used; for example, as shown in Sec. II, for a spin-component boost, the Wigner rotations are all just $R^{-1}(\mathrm{bf}\to\mathrm{bf}^{\mathrm{per}})$. It remains to specify $\{s_q\}^{\mathrm{per}}$. (There are no $\{\mathrm{sgn}i\}$ for three-particle states.) The most convenient choice of $\{s_q\}$ is $s_{13}=(p_1+p_3)^2$ and $s_{23}=(p_2+p_3)^2$. If these two variables are chosen as Dalitz variables, then $\{s_{13}, s_{23}\}^{\text{per}} = \{s_{23}, s_{13}\}$ means interchanging the abscissa and ordinate of the Dalitz plot.

VI. CONCLUSION

Unitarity equations have been derived for arbitrary multiparticle (including two-particle) processes. In succeeding papers these very general unitarity equations will be used in a host of problems, including analyzing pion-nucleon scattering data (where phase-shift analyses have been performed, and hence the partial-wave amplitudes are known) in terms of pion production amplitudes, and putting unitarity bounds on production partial-wave amplitudes analogous to the unitarity bound for two-particle partial-wave amplitudes, in which the inelasticity parameter η must be less than one. Thus, the point of this paper has been to present a formalism general enough to be applied to various kinds of problems which involve unitarity; it is to be hoped that it can also be applied to current high-energy physics models, where unitarity always seems to be a difficult constraint to satisfy.

The philosophy of this series of papers will be to exploit model-independent features of high-energy reactions as much as possible and to see how various models and experimental data agree with these model-independent features. The term "model independent" is being used here basically as meaning relativistic invariance and unitarity, with the added proviso that partial-wave amplitudes have as their arguments quantities which are related as directly as possible to quantities of experimental interest.

The unitarity equations for partial-wave amplitudes were not explicitly derived because then a normalization convention would have had to have been chosen. It is not clear what the most convenient normalization convention is for multiparticle processes; probably this question can best be answered in terms of specific problems and reactions.

Finally, the Racah coefficients, derived in Appendix B, can be used in various ways in relativistic reactions, and it remains to work out some of their specific properties.

Note added in proof. Because of the length of time needed to prepare this work for publication, another paper by Wang[18] has appeared which overlaps with this work to some extent.

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APPENDIX A: MATHEMATICS OF MULTIPARTICLE STATES

It is not at all obvious how one gets from the Clebsch-Gordan coefficients given in Ref. 19 to the multiparticle state of Eq. (2.1); the purpose of this Appendix is the sketch the connection and then perform some mathematical manipulations on multiparticle helicity states.

The Clebsch-Gordan coefficients resulting from the reduction of an N-fold tensor product of irreducible representations of the Poincaré group are given in Ref. 19 as

$$\langle[\sqrt{s}\,J]\vec{\mathrm{p}}\sigma; r, r_1\cdots r_N, D_i\,|\,[M_1J_1]\vec{\mathrm{p}}_1\sigma_1;\cdots[M_NJ_N]\vec{\mathrm{p}}_N\sigma_N\rangle = \int_{\mathrm{SL}(2,C)} d\Lambda\, D^{\sqrt{s}J\,*}_{p_D r p\sigma}(\Lambda)\prod_{i=1}^{N} D^{M_iJ_i}_{p_i^+ r_i p_i\sigma_i}(\Lambda_{D_i}\Lambda), \tag{A1}$$

where $D^{M_iJ_i}_{p_i' m_i p_i \sigma_i}(\Lambda)$ are Poincaré group matrix elements,

$$\langle[M_iJ_i]\,p_i'm_i\,|\,U(\Lambda)\,|\,[M_iJ_i]\,p_i\sigma_i\rangle .$$

To evaluate such a matrix element, the transformation properties of a plane-wave state of momentum $\vec{\mathrm{p}}$ and spin projection σ must be given:

$$U(\Lambda)\,|\,p\sigma\rangle = \sum_{\sigma'=-J}^{+J} D^J_{\sigma'\sigma}(p,\Lambda)\,|\,\Lambda p, \sigma'\rangle , \tag{A2}$$

where (p,Λ) is a Wigner rotation defined as

$$(p,\Lambda) = B^{-1}(\Lambda p)\Lambda B(p) \tag{A3}$$

and $B(p)$ is a boost satisfying, by definition, $p = B(p)p^+$; p^+ is the rest-frame four-momentum vector $(M, \vec{0})$ while σ is a spin projection. The actual type of boost is not relevant at this point, but later on in the Appendix a helicity boost, defined as

$$B_H(p) = R(\hat{p})\Lambda_z(|\vec{\mathrm{p}}|) , \tag{A4}$$

will be considered in some detail. In order to emphasize that there are many different types of boosts, a spin-component or canonical-basis boost is also written out[10]; such boosts consist of rotationless Lorentz transformations, which in the SL$(2, C)$ notation are Hermitian matrices and can be written

$$B_c(p) = R(\hat{p})\Lambda_z(|\vec{\mathrm{p}}|)R^{-1}(\hat{p}) . \tag{A5}$$

Here Λ_z is a pure Lorentz transformation in the $+z$ direction, while $R(\hat{p})$ is a rotation specified by the polar and azimuthal angles of the unit vector $\hat{p}$.

Once the transformation properties of plane-wave states are known [Eq. (A2)], it is clear that the matrix elements Eq. (A1) can be computed. Since the plane-wave states are non-normalizable,

δ functions will occur in the matrix elements and these δ functions can be used to eliminate not only the integration over SL(2, C) of Eq. (A1), but also the unphysical double coset labels, appearing as subscripts on Lorentz transformations, Λ_{D_i}.

In all of the matrix elements appearing in Eq. (A1), the ket states are chosen for their physical significance, whereas the bra states are chosen for their mathematical convenience. Mathematical convenience means that the unphysical double coset factors should all be eliminated; this can be done by defining special boosts for the bra states called Rideau boosts[20]:

$$B_R(p) = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}(2, C), \quad \lambda \text{ real},$$
$$p \equiv \sigma_\mu p^\mu = M B_R{}^{-1}(p) B_R{}^{-1\dagger}(p) . \tag{A6}$$

Eq. (A6) defines the relationship, worked out in Ref. 19, between the four-momentum vector and the boost parameters λ and z. The importance of the boosts $B_R(p)$, as pointed out by Rideau, is that they form a group. Hence any sequence of Rideau boosts produces no Wigner rotation. As shown in Ref. 2, the Lorentz transformations Λ_{D_i} are all of the form (A6) and are equivalent to relativistically invariant subenergies $\{s_q\}$ and invariants of the form $\epsilon_{\alpha\beta\gamma\delta} p^\alpha p_1^\beta p_2^\gamma p_i^\delta$, both of which are discussed in Sec. II. p_D of Eq. (A1) is defined to be

$$p_D = \sum_{i=1}^{N} \Lambda_{D_i}{}^{-1} p_i^+ .$$

The δ functions appearing in the matrix elements of Eq. (A1) can be written $\delta^3(\Lambda_{D_i}{}^{-1} p_i^+ - \Lambda p_i)$ so that Eq. (A1) becomes

$$\begin{aligned} &\langle [\sqrt{s}\, J]\vec{\mathrm{p}}\sigma ; r_1 \cdots r_N, D_i \,|\, [M_1 J_1]\vec{\mathrm{p}}_1\sigma_1 \cdots [M_N J_N]\vec{\mathrm{p}}_N \sigma_N \rangle \\ &\qquad = \int_{\mathrm{SL}(2,C)} d\Lambda\, D^{\sqrt{s}\,J\,*}_{p_D r p \sigma}(\Lambda) \left(\prod_{i=1}^{N} D^{M_i J_i}_{\Lambda_{D_i}{}^{-1} p_i^+ r_i p_i \sigma_i}(\Lambda) \right) \\ &\qquad = \delta^3\left(\vec{\mathrm{p}} - \sum_{i=1}^{N} \vec{\mathrm{p}}_i \right) \int_{\mathrm{SL}(2,C)} d\Lambda\, D^{J\,*}_{r\sigma}(p,\Lambda) \left(\prod_{i=1}^{N} D^{J_i}_{r_i \sigma_i}(p_i,\Lambda)\, \delta^3(\Lambda_{D_i}{}^{-1} p_i^+ - \Lambda p_i) \right) . \end{aligned} \tag{A7}$$

It follows then that a multiparticle state can be written

$$| p_1\sigma_1 \cdots p_N \sigma_N \rangle = \sum_{J r \sigma r_i} (2J+1)^{1/2} D^{J*}_{r\sigma}(p,\Lambda) \left(\prod_{i=1}^{N} D^{J_i}_{r_i \sigma_i}(p_i, \Lambda) \right) | [\sqrt{s}, J]\vec{\mathrm{p}}\sigma ; r, r_1 \cdots r_N \{s_q\}, \{\mathrm{sgn}\,i\} \rangle \tag{A8}$$

Here, $s = (p_1 + \cdots + p_N)^2$ is the invariant energy squared and the double coset labels D_i have been replaced by the $3N-7$ subenergies $\{s_q\}$ and the invariants $\epsilon_{\alpha\beta\gamma\delta} p^\alpha p_1^\beta p_2^\gamma p_i^\delta$, denoted by $\{\mathrm{sgn}\,i\}$; how an independent set of subenergies is to be chosen, and the physical significance of $\{\mathrm{sgn}\,i\}$ is discussed in Sec. II.

It remains to determine the Lorentz transformation Λ; it will be shown that

$$\Lambda = R B^{-1}(p) , \tag{A9}$$

where the physical significance of R will be discussed. Note that if the multiparticle state appearing in Eq. (A8) is in its over-all rest frame, where $p = p^+ = (\sqrt{s}, \vec{0})$, then $(p, \Lambda) = (p^+, \Lambda) = (p^+, R) = R$, since $B(p^+)$ is the identity. Further

$$\begin{aligned} (p, \Lambda) &= B^{-1}(\Lambda p)\Lambda B(p) \\ &= B^{-1}(R B^{-1}(p) p) R B^{-1}(p) B(p) \\ &= B^{-1}(R p^+) R \\ &= B^{-1}(p^+) R \\ &= R . \end{aligned} \tag{A10}$$

The consistency of the solution (A9) with (A8) can also be checked by starting with the multiparticle states in its c.m. frame, where $p = p^+$, and then boosting it and all the single-particle states to an arbitrary frame. The Wigner rotations which appear as a result of this boosting can then be combined in such a way as to again give Eq. (A8).

What then is the rotation R? It is a rigid-body rotation in the c.m. frame satisfying

$$\begin{aligned} \Lambda_{D_i}{}^{-1} p_i^+ &= \Lambda p_i \\ &= R B^{-1}(p^+) p_i \\ &= R p_i , \end{aligned} \tag{A11}$$

where p_i is the momentum of the ith particle in the over-all c.m. frame. The momentum $\Lambda_{D_i}{}^{-1} p_i^+$ represents a standard set of vectors so that R is a rotation to the frame fixed by the set of momentum vectors of the i particles from a standard or observer's frame.[2] The frame fixed by the i particles can be considered as a body-fixed frame, so that the rotation R can be written $R(\mathrm{bf} \to \mathrm{obs})$.

In the over-all c.m. frame, Eq. (A8) then becomes

$$|p_1\sigma_1\cdots p_N\sigma_N\rangle = \sum_{Jr\sigma r_i}(2J+1)^{1/2}D^{J\,*}_{r\sigma}(R(\text{bf}\to\text{obs}))\left(\prod_{i=1}^{N}D^{J_i}_{r_i\sigma_i}(p_i, R(\text{bf}\to\text{obs}))\right)|[\sqrt{s}J]\vec{p}=\vec{0},\,\sigma;r,r_1\cdots r_N\{s_q\}\{\text{sgn}i\}\rangle\,. \tag{A12}$$

Since it is standard convention to specify rotations from the observer and not to the observer's frame, Eq. (A12) becomes

$$|p_1\sigma_1\cdots p_N\sigma_N\rangle = \sum_{r_i=-J_i}^{+J_i}\sum_{J}\sum_{r,\sigma=-J}^{+J}(2J+1)^{1/2}D^{J}_{\sigma r}(R)\left(\prod_{i=1}^{N}D^{J_i}_{r_i\sigma_i}(p_i,R^{-1})\right)|[\sqrt{s}J]\vec{p}=\vec{0},\,\sigma;r,r_1\cdots r_N\{s_q\}\{\text{sgn}i\}\rangle \tag{A13}$$

which is the starting point of Sec. I. Here $R\equiv R(\text{obs}\to\text{bf})$.

The multiparticle states considered thus far consist of more than two particles. To see what happens for a two-particle state consider the inverse of Eq. (A13) in which the multiparticle state is written in terms of the N single-particle states:

$$|[\sqrt{s}J]\vec{p}=\vec{0},\,\sigma;r,r_1\cdots r_N\{s_q\}\{\text{sgn}i\}\rangle = \sum_{\sigma_i=-J_i}^{+J_i}\int dR\,D^{J\,*}_{\sigma r}(R)\prod_{i=1}^{N}D^{J_i\,*}_{r_i\sigma_i}(p_i,R^{-1})|p_1\sigma_1\cdots p_N\sigma_N\rangle\,. \tag{A14}$$

If there are only two single-particle states, then an integration can be carried out over one of the azimuthal Euler angles in the integration dR (note that $\int dR=1$). This integration will be carried out explicitly only for the case when the single-particle states are helicity states, but general considerations show that R becomes $R(\text{obs}\to p_1)=R(\hat{p}_1)$, where $\hat{p}_1$ is the unit vector direction of particle one in the two-particle c.m. frame.

Consider then a state $|p,\lambda\rangle$, where the general spin index σ is replaced by the conventional helicity index λ. The Wigner rotations $(p_i, R^{-1}(\text{obs}\to\text{bf}))$ appearing in Eq. (A13) and (A14) can be computed explicitly. To see this consider a general Wigner rotation (p,R) where p and R are unspecified for the moment:

$$\begin{aligned}(p,R) &= B_H^{-1}(Rp)RB_H(p)\\ &= \Lambda_z^{-1}(|\vec{p}|)R^{-1}(R\hat{p})RR(\hat{p})\Lambda_z(|\vec{p}|)\,.\end{aligned} \tag{A15}$$

Here $\Lambda_z(|\vec{p}|)$ is a pure Lorentz transformation along the positive z axis satisfying $(E,0,0,|\vec{p}|)=\Lambda_z(|\vec{p}|)(M,\vec{0})$ and $R(\hat{p})\equiv R(\varphi,\theta,-\varphi)$, where (θ,φ) are the polar and azimuthal angles of the unit vector $\hat{p}$. It will be shown that $R^{-1}(R\hat{p})RR(\hat{p})$ is a rotation about the z axis, $R_z(\delta)$; if that is the case, then $(p,R)=R_z(\delta)$, for R_z commutes with Λ_z and hence the Wigner rotations corresponding to helicity states generate phases only, for $D^J_{\lambda\lambda'}(p,R)=D^J_{\lambda\lambda'}(R_z(\delta))=e^{-i\lambda\delta}\delta_{\lambda\lambda'}$. To see that $R^{-1}(R\hat{p})RR(\hat{p})=R_z(\delta)$ note that any rotation $R\in\text{SO}(3,R)$ can be decomposed into left cosets with respect to the subgroup $\text{SO}(2,R)$, so that

$$\begin{aligned}R(\alpha,\beta,\gamma) &= R(\alpha,\beta,0)R_z(\gamma)\\ &= R(\alpha,\beta,-\alpha)R_z(\gamma+\alpha)\\ &= R(\hat{p}(\beta,\alpha))R_z(\gamma+\alpha)\,.\end{aligned} \tag{A16}$$

The last line of Eq. (A16) indicates that any rotation can be decomposed into a rotation about the z axis and a rotation specified by a unit vector with polar and azimuthal angles (β,α). Now $RR(\hat{p})$ is a rotation, so that it can be written

$$RR(\hat{p}) = R(\hat{p}')R_z\,, \tag{A17}$$

where the unit vector $\hat{p}'$ is to be determined. Apply the unit vector $\hat{z}=(0,0,1)$ to both sides of Eq. (A17). Then

$$\begin{aligned}RR(\hat{p})\hat{z} &= R(\hat{p}')R_z\hat{z}\,,\\ R\hat{p} &= R(\hat{p}')\hat{z}\\ &= \hat{p}'\,.\end{aligned} \tag{A18}$$

Hence $RR(\hat{p})=R(R\hat{p})R_z(\delta)$ or $R^{-1}(R\hat{p})RR(\hat{p})=R_z(\delta)$, which was to be shown. It remains to compute δ in order to completely specify the phases in a multiparticle helicity state.

To evaluate δ, notice that, in analogy with Wigner rotations, it is possible to define $(\hat{p},R)\equiv R^{-1}(R\hat{p})RR(\hat{p})=R_z(\delta)$. In particular $(\hat{p},R_1R_2R_3)=(R_2R_3\hat{p},R_1)(R_3\hat{p},R_2)(\hat{p},R_3)$, so that, if a general rotation is specified by its Euler angles, $R(\alpha,\beta,\gamma)=R_z(\alpha)R_y(\beta)R_z(\gamma)$, then

$$\begin{aligned}R_z(\delta) &= (\hat{p},R(\alpha\beta\gamma))\\ &= (\hat{p},R_z(\alpha)R_y(\beta)R_z(\gamma))\\ &= (R_y(\beta)R_z(\gamma)\hat{p},R_z(\alpha))(R_z(\gamma)\hat{p},R_y(\beta))(\hat{p},R_z(\gamma))\\ &= R_z(\alpha+\gamma)(R_z(\gamma)\hat{p},R_y(\beta))\,,\end{aligned} \tag{A19}$$

since $(\hat{p},R_z(\alpha))=R_z(\alpha)$ ($\hat{p}$ arbitrary, but $\theta\neq 0$).

We must thus evaluate $(\tilde{p},R_y(\beta))$, where $\tilde{p}=R_z(\gamma)\hat{p}(\theta,\varphi)$; write

$$\begin{aligned}R_z(\tilde{\delta}) &= (\tilde{p},R_y(\beta))\\ &= R^{-1}(R_y(\beta)\tilde{p})R_y(\beta)R(\tilde{p})\\ &= R^{-1}(\tilde{p}')R_y(\beta)R(\tilde{p})\end{aligned} \tag{A20}$$

or

$$R_y(\beta)R(\tilde{p}) = R(\tilde{p}')R_z(\tilde{\delta}),$$
$$R_y(\beta)R_z(\tilde{\varphi})R_y(\tilde{\theta})R_z^{-1}(\tilde{\varphi}) = R_z(\tilde{\varphi}')\,R_y(\tilde{\theta}')R_z^{-1}(\tilde{\varphi}')R_z(\tilde{\delta}),$$
$$R_y(\beta)R_z(\tilde{\varphi})R_y(\tilde{\theta}) = R_z(\tilde{\varphi}')R_y(\tilde{\theta}')R_z(\tilde{\delta}+\tilde{\varphi}-\tilde{\varphi}'). \tag{A21}$$

Carrying out the matrix multiplication and taking the trace of both sides of Eq. (A21) then gives

$$\cos\beta\cos\tilde{\theta}\cos\tilde{\varphi} - \sin\beta\sin\tilde{\theta} + \cos\tilde{\varphi} - \sin\beta\cos\tilde{\varphi}\sin\tilde{\theta} + \cos\beta\cos\tilde{\theta} = \cos\tilde{\theta}'\cos(\tilde{\delta}+\tilde{\varphi})\cos(\tilde{\delta}+\tilde{\varphi}) + \cos\tilde{\theta}'. \tag{A22}$$

Now $\tilde{p} \equiv R_z(\gamma)\hat{p}(\theta, \varphi)$ has polar angle $\tilde{\theta} = \theta$ and azimuthal angle $\tilde{\varphi} = \gamma + \varphi$; further, $\tilde{p}' \equiv R_y(\beta)\tilde{p}$ has polar angle

$$\cos\tilde{\theta}' = \cos\beta\cos\theta - \sin\beta\sin\theta - \sin\beta\sin\theta\cos(\gamma+\varphi).$$

Therefore, solving for $\cos(\tilde{\delta}+\varphi+\gamma)$ in Eq. (A22) gives

$$\cos(\tilde{\delta}+\varphi+\gamma) = \frac{\cos\beta\cos\theta\cos(\varphi+\gamma) - \sin\beta\sin\theta + \cos(\varphi+\gamma)}{1+\cos\tilde{\theta}'} = \frac{\cos\beta\cos\theta\cos(\gamma+\varphi) - \sin\beta\sin\theta + \cos(\gamma+\varphi)}{1+\cos\beta\cos\theta - \sin\beta\sin\theta\cos(\gamma+\varphi)}. \tag{A23}$$

Then from Eq. (A19), $\delta = \alpha + \gamma + \tilde{\delta}$. Somewhat more work is needed to resolve the angle ambiguity in going from $\cos(\tilde{\delta}+\varphi+\gamma)$ to the solution for $\tilde{\delta}+\varphi+\gamma$; the ambiguity can be resolved by geometric arguments which will not be given here.

With the angle δ of Eq. (A19) known from Eq. (A23) it is possible to evaluate the phases occurring in a two-particle state and compare them with the phases chosen by Jacob and Wick. From Eq. (A14) we see that

$$|[\sqrt{s}\,J]\,\vec{p} = \vec{0}M; \lambda, r_1 r_2\rangle = \sum_{\lambda_i=-J_i}^{+J_i} \int \frac{d\alpha\, d\cos\beta\, d\gamma}{8\pi^2} D^{J\,*}_{M\lambda}(\alpha\beta\gamma)\left(\prod_{i=1}^{2} D^{J_i\,*}_{r_i\lambda_i}(p_i, R^{-1})\right)|p_1\lambda_1, p_2\lambda_2\rangle,$$
$$|[\sqrt{s}\,J]\,\vec{p} = \vec{0}M; \lambda\lambda_1\lambda_2\rangle = \int \frac{d\Omega_{\hat{p}_1}}{4\pi}\frac{d\gamma}{2\pi} D^{J\,*}_{M\lambda}(\varphi, \theta, -\varphi)e^{+i\lambda(\varphi+\gamma)}e^{i\lambda_1\delta_1}e^{i\lambda_2\delta_2}|\vec{p}_1\lambda_1, -\vec{p}_1\lambda_2\rangle, \tag{A24}$$

where the Euler angles β and α have been chosen as the angles θ, φ of $\hat{p}_1(\theta, \varphi)$ while the labels M, λ are those used by Jacob and Wick (their Eq. (16), Ref. 5). To carry out the integration over γ, it is necessary to evaluate the phase factors δ_1 and δ_2 defined as

$$\begin{aligned} R_z(\delta_i) &\equiv (p_i, R^{-1}) \\ &= (p_i, [R(\alpha, \beta, -\alpha)R_z(\alpha+\gamma)]^{-1}) \\ &= (p_i, R_z^{-1}(\varphi+\gamma)R^{-1}(\hat{p}_1)) \\ &= (R^{-1}(\hat{p}_1)p_i, R_z^{-1}(\varphi+\gamma))(p_i, R^{-1}(\hat{p}_1)). \end{aligned} \tag{A25}$$

By direct evaluation

$$(R^{-1}(\hat{p}_1)p_1, R_z^{-1}(\varphi+\gamma)) = R_z^{-1}(\varphi+\gamma),$$
$$(R^{-1}(\hat{p}_1)p_2, R_z^{-1}(\varphi+\gamma)) = R_z(\varphi+\gamma),$$

and

$$(p_1, R^{-1}(\hat{p}_1)) = R_z(0);$$

$(p_2, R^{-1}(\hat{p}_1))$ is evaluated by either using Eq. (A23) or noting that

$$(p_2, R^{-1}(\hat{p}_1)) = (p_2, R_z(\pi+\varphi)R_y(\theta)R_z^{-1}(\pi+\varphi)),$$

which by Eq. (A19) is

$$R_z^{-1}(\pi+\varphi)R_z^{-1}(\pi+\varphi) = R_z(-2\varphi).$$

Hence

$$\delta_1 = -(\varphi+\gamma), \quad \delta_2 = (\varphi+\gamma) - 2\varphi \tag{A26}$$

and upon changing variables of integration from $\gamma \to \gamma + \varphi$, for φ fixed, Eq. (A24) becomes

$$|[\sqrt{s}\,J]\,\vec{p} = \vec{0}M; \lambda_1\lambda_2\rangle = \int \frac{d\Omega_{\hat{p}_1}}{4\pi} D^{J\,*}_{M\,\lambda_1-\lambda_2}(\hat{p}_1(\theta, \varphi))e^{-2i\lambda_2\varphi} \times |\vec{p}_1\lambda_1, -\vec{p}_1\lambda_2\rangle. \tag{A27}$$

This phase convention differs from that of Jacob and Wick by the factor $e^{-2i\lambda_2\varphi}$. The difference hinges on the definition of a two-particle state. Whereas our definition of a two-particle state is fixed by the general definition of a multiparticle state given by Eqs. (A8) and (A13), Jacob and Wick define their two-particle state so that

$$R(\varphi, \theta, -\varphi)\,|\vec{p}_1\lambda_1, -\vec{p}_1\lambda_2\rangle$$

by definition produces no $e^{-2i\lambda_2\varphi}$ $[\vec{p}_1=(0,0,|\vec{p}|)]$ [their Eq. (15)]. It is to be noted that Eq. (A24) can be written so that it has exactly the same form as Eq. (16) of Jacob and Wick; as a consequence our analysis and that of Jacob and Wick will lead to the same partial-wave expansion, for the extra phase appearing in our convention can be lumped into the over-all phase and hence cannot be measured. Where a real difference will occur is in multiparticle scattering processes, where our phase cannot be factored out.

Thus, it is not clear whether at this stage it would be better to get agreement with Jacob and Wick by inserting a phase in Eq. (A14) which would have the effect of canceling the extra factor in Eq. (A27) or just leaving it as is. An argument for leaving it is that then the multiparticle states have no arbitrary phases inserted other than those given from the Wigner rotation. Because this seems aesthetically more satisfying, we will tentatively adopt the latter possibility and use Eq. (A27) for a two-particle state.

APPENDIX B: RACAH COEFFICIENTS FOR THE POINCARÉ GROUP

Racah coefficients for the rotation group[6] are defined as the coefficients carrying one stepwise-coupled basis to another. More precisely, if states of the rotation group are labeled $|[J]M\rangle$, where the Casimir invariant J is the angular momentum and M a spin projection, then the problem arises of taking tensor products of irreducible representations of the rotation group and reducing out the product. Consider, for example, the tensor product of three representations, $J_1\otimes J_2\otimes J_3$. This can be written

$$|[J_1]m_1;\ [J_2]m_2;\ [J_3]m_3\rangle = \sum_{JM\eta}\begin{bmatrix} J & J_1 & J_2 & J_3 \\ M & m_1 & m_2 & m_3 \\ \eta & & & \end{bmatrix}|[J]M;\ \eta J_1J_2J_3\rangle\,. \tag{B1}$$

Now one conventionally couples the three representations in a stepwise fashion, one possibility being

$$\begin{aligned}|[J_1]m_1;\ [J_2]m_2\rangle &= \sum_{j_{12}m}\langle j_{12}m|J_1m_1J_2m_2\rangle|[j_{12}]m;\ J_1J_2\rangle\,,\\ |[J_1]m_1;\ [J_2]m_2;\ [J_3]m_3\rangle &= \sum_{j_{12}m}\langle j_{12}m|J_1m_1J_2m_2\rangle|[j_{12}]m;\ J_1J_2\rangle|[J_3]m_3\rangle \\ &= \sum_{j_{12}m}\sum_{JM}\langle j_{12}m|J_1m_1J_2m_2\rangle\langle JM|j_{12}mJ_3m_3\rangle|[J]M;\ j_{12}J_1J_2J_3\rangle\,.\end{aligned} \tag{B2}$$

Comparing Eqs. (B1) and (B2) we see that the degeneracy parameter η is j_{12} and

$$\begin{bmatrix} J & J_1 & J_2 & J_3 \\ M & m_1 & m_2 & m_3 \\ j_{12} & & & \end{bmatrix}=\sum_{m}\langle j_{12}m|J_1m_1J_2m_2\rangle\langle JM|j_{12}mJ_3m_3\rangle\,. \tag{B3}$$

But there is no reason that, for example, 2 and 3 could not have been coupled first, and then 1; Eq. (B2) would then become

$$\begin{bmatrix} J & J_1 & J_2 & J_3 \\ M & m_1 & m_2 & m_3 \\ j_{23} & & & \end{bmatrix}=\sum_{m}\langle j_{23}m|J_2m_2J_3m_3\rangle\langle JM|j_{23}mJ_1m_1\rangle\,. \tag{B4}$$

Since in the tensor-product space one basis is equivalent to another, there must be operators which transform one coupling scheme to another. That is,

$$|[J]M;\ j_{12}J_1J_2J_3\rangle=\sum_{j_{23}}\begin{Bmatrix} J_1 & J_2 & j_{12} \\ J_3 & J & j_{23}\end{Bmatrix}|[J]M;\ j_{23}J_1J_2J_3\rangle\,, \tag{B5}$$

where $\{\ \}$ are the Racah coefficients.

For the Poincaré group a new possibility arises. Because of the induced representation structure, it is possible to decompose tensor products not only in a stepwise fashion, as was just done for the rotation group, but also decompose tensor products in a symmetrical fashion, in which all particles are treated on an equal footing. Such a decomposition was discussed in some detail in Appendix A. Now given this latter possibility for coupling states of the Poincaré group, since all particles are treated symmetrically, it is possible to derive recoupling coefficients using the symmetrical scheme as a standard relative to all the possible stepwise schemes. Such a possibility exists, however, only for groups such as the Poincaré group whose irreducible unitary representations can be written as induced representations.

The goal of this Appendix is to couple three relativistic particles in a stepwise fashion and then derive the coefficients which transform the stepwise scheme to the symmetric scheme. Once these coefficients and their inverses are known, any stepwise scheme, or combination stepwise and symmetrical scheme, as was needed in deriving the unitarity equations, can be readily computed.

Consider then, the coupling of three particles 1, 2, and 3, such that particles 1 and 2 are coupled first:

$$\begin{aligned}
|[M_1J_1]\vec{p}_2\sigma_1\cdots[M_3J_3]\vec{p}_3\sigma_3\rangle &= \sum_{j_{12}\sigma_{12}r_1r_2}(2j_{12}+1)^{1/2}D^{j_{12}*}_{r_1+r_2\,\sigma_{12}}(p_{12},\Lambda)\\
&\qquad\times D^{J_1}_{r_1\sigma_1}(p_1,\Lambda)D^{J_2}_{r_2\sigma_2}(p_2,\Lambda)\,|[\sqrt{s_{12}}\,j_{12}]\vec{p}_{12}=\vec{p}_1+\vec{p}_2,\ \sigma_{12};\,r_1r_2\rangle\,|[M_3J_3]\vec{p}_3\sigma_3\rangle\\
&=\sum_{j_{12}\sigma_{12}r_1r_2}\ \sum_{J\sigma r_{12}r_3}[(2j_{12}+1)(2J+1)]^{1/2}D^{j_{12}*}_{r_1+r_2\,\sigma_{12}}(p_{12},\Lambda)D^{J_1}_{r_1\sigma_1}(p_1,\Lambda)\\
&\qquad\times D^{J_2}_{r_2\sigma_2}(p_2,\Lambda)D^{J}_{\sigma\, r_3+r_{12}}(R(\hat{p}_3))\,D^{j_{12}}_{r_{12}\sigma_{12}}(p_{12},R^{-1}(\hat{p}_3))\\
&\qquad\times D^{J_3}_{r_3\sigma_3}(p_3,R^{-1}(\hat{p}_3))\,|[\sqrt{s}\,J]\vec{p}=\vec{0}\,\sigma;\,r_1r_2r_3r_{12}s_{12}j_{12}\rangle\\
&=\sum_{r_1r_2r_3}\ \sum_{j_{12}r_{12}}\ \sum_{J\sigma}[(2J+1)(2j_{12}+1)]^{1/2}D^{J}_{\sigma r_3+r_{12}}(R(\hat{p}_3))\,D^{J_1}_{r_1\sigma_1}(p_1,\Lambda)\\
&\qquad\times D^{J_2}_{r_2\sigma_2}(p_2,\Lambda)D^{J_3}_{r_3\sigma_3}(p_3,R^{-1}(\hat{p}_3))\,D^{j_{12}}_{r_{12}\,r_1+r_2}((p_{12},R^{-1}(\hat{p}_3)),(p_{12},\Lambda)^{-1})\\
&\qquad\times|[\sqrt{s}\,J]\vec{p}=\vec{0}\,\sigma;\,r_1r_2r_3r_{12}s_{12}\,j_{12}\rangle\,,
\end{aligned}\tag{B6}$$

where $\Lambda=R^{-1}(\text{obs}\to\hat{p}_1(12\text{ c.m.}))B^{-1}(p_{12})$ as discussed in Appendix A Eq. (A9) and $\sqrt{s_{12}}$ and j_{12} are the invariant mass and total angular momentum of the 1-2 system, respectively.

Now the symmetrically coupled three-particle state can be written in terms of the three one-particle states Eq. (A14) and hence, from Eq. (B6) in terms of the stepwise-coupled three-particle state:

$$\begin{aligned}
|[\sqrt{s}\,J']\vec{p}=\vec{0}\,\sigma';\,r'r_1'r_2'r_3's_1s_2\rangle &= \sum_{\sigma_i=-J_i}^{+J_i}\int dR\,D^{J'*}_{\sigma'r'}(R)\Big(\prod_{i=1}^{3}D^{J_i}_{r_i'\sigma_i}(p_i,R^{-1})\Big)|p_1\sigma_1,\,p_2\sigma_2,\,p_3\sigma_3\rangle\\
&=\sum_{\sigma_i}\int dR\,D^{J'*}_{\sigma'r'}(R)\Big(\prod_{i=1}^{3}D^{J_i}_{r_i'\sigma_i}(p_i,R^{-1})\Big)\sum_{r_1r_2r_3}\ \sum_{j_{12}r_{12}}\ \sum_{J\sigma}[(2J+1)(2j_{12}+1)]^{1/2}\\
&\qquad\times D^{J}_{\sigma\, r_{12}+r_3}(R(\hat{p}_3))\,D^{J_1}_{r_1\sigma_1}(p_1,\Lambda)D^{J_2}_{r_2\sigma_2}(p_2,\Lambda)D^{J_3}_{r_3\sigma_3}(p_3,R^{-1}(\hat{p}_3))\\
&\qquad\times D^{j_{12}}_{r_{12}\,r_1-r_2}((p_{12},R^{-1}(\hat{p}_3))(p_{12},\Lambda)^{-1})\,|[\sqrt{s}\,J]\vec{p}=\vec{0}\,\sigma;\,r_1r_2r_3r_{12}s_{12}\,j_{12}\rangle\\
&=\sum_{r_1r_2r_3}\ \sum_{j_{12}r_{12}}[(2J+1)(2j_{12}+1)]^{1/2}D^{J'}_{r'\,r_{12}+r_3}(R(123\to\hat{p}_3))\,D^{J_1}_{r_1r_1'}((p_1,\Lambda)(p_1,R^{-1})^{-1})\\
&\qquad\times D^{J_2}_{r_2r_2'}((p_2,\Lambda)(p_2,R^{-1})^{-1})\,D^{J_3}_{r_3r_3'}((p_3,R^{-1}(\hat{p}_3))(p_3,R^{-1})^{-1})\\
&\qquad\times D^{j_{12}}_{r_{12}\,r_1+r_2}((p_{12},R^{-1}(\hat{p}_3))(p_{12},\Lambda)^{-1})\,|[\sqrt{s}\,J']\vec{p}=\vec{0},\,\sigma';\,r_1r_2r_3r_{12}s_{12}\,j_{12}\rangle\,.
\end{aligned}\tag{B7}$$

Here R means $R(\text{obs}\to 123)$ and is the rotation from the observer to the coordinate frame of the three particles. [123 is shorthand for $\text{bf}_{1,3}(123)$.][13]

It would seem as though some incorrect manipulations were carried out in arriving at the last line of Eq. (B7), for the rotation $R(\hat{p}_3)$ in $D^{J}_{\sigma\, r_{12}+r_3}(R(\hat{p}_3))$ was written as $R(\hat{p}_3)\equiv R(\text{obs}\to\hat{p}_3)=R(\text{obs}\to 123)R(123\to\hat{p}_3)$ $=RR(123\to\hat{p}_3)$ and then the orthogonality relations of D functions used to eliminate the integration over R; now the rotation R also appears in the other D functions, so that it would seem that the orthogonality relations were improperly used. But it will be shown that all the rotations appearing in the spin functions labeled by J_1, J_2, J_3, and j_{12} are functions of s, and squared subenergies s_1 and s_2 only (and masses of the three particles), so that the appearance of the rotation R in the arguments of the D functions is spurious.

To compute the Racah coefficients explicitly, it is necessary to compute the Wigner rotations of Eq. (B7) and show that they are functions of energy, subenergies, and masses only. The simplest rotation to deal with is $R(123\to\hat{p}_3)$ which is the rotation from the coordinate frame fixed by the momenta $\vec{p}_1, \vec{p}_2$, and $\vec{p}_3$ (in the over-all c.m. frame) to the unit vector $\hat{p}_3$. Clearly $R(123\to\hat{p}_3)$ is a function of the subenergies for the rotation depends only on angles between the various momentum vectors; for example, if the coordinate frame defined by Eq. (2.2) is chosen, $R(123\to\hat{p}_3)$ becomes $R(\pi, \theta_{13}, \pi)$, with $\cos\theta_{13}=\hat{p}_1\cdot\hat{p}_3$, $0\leq\theta_{13}\leq\pi$.

A somewhat more complicated rotation is $(p_3, R^{-1}(\hat{p}_3))(p_3, R^{-1})^{-1}$. From Sec. IV, Eq. (5.5) ff, where properties of Wigner rotations were derived, it can be seen that

$$\begin{aligned}(p_3, R^{-1}(\hat{p}_3))(p_3, R^{-1})^{-1} &= [B^{-1}(R^{-1}(\hat{p}_3)p_3)R^{-1}(\hat{p}_3)B(p_3)][B^{-1}(R^{-1}p_3)R^{-1}B(p_3)]^{-1}\\ &= B^{-1}(R^{-1}(\hat{p}_3)p_3)R^{-1}(\hat{p}_3)RB(R^{-1}p_3)\\ &= B^{-1}(R^{-1}(\hat{p}_3)p_3)R(\hat{p}_3\to 123)B(p_3(123)),\end{aligned} \tag{B8}$$

where $p_3(123)$ is the four-momentum of the third particle relative to the 123 (body-fixed) frame. The rotation $R^{-1}(\hat{p}_3)$ can be rewritten as $[R(\text{obs}\to 123)R(123\to\hat{p}_3)]^{-1}$, so that $B^{-1}(R^{-1}(\hat{p}_3)p_3)=B^{-1}(R^{-1}(123\to\hat{p}_3)p_3(123))$, meaning that the Wigner rotation of Eq. (B8) can be written $(p_3(123), R(\hat{p}_3\to 123))$ which shows that it is a function of subenergies only, since it depends only on the magnitude and direction of particle 3 relative to the "123" frame.

The rotation appearing as the argument of $D^{j_{12}}$ is also readily computed although it is somewhat more complicated:

$$\begin{aligned}(p_{12}, R^{-1}(\hat{p}_3))(p_{12}, \Lambda)^{-1} &= [B^{-1}(R^{-1}(\hat{p}_3)p_{12})R^{-1}(\hat{p}_3)B(p_{12})][B^{-1}(\Lambda p_{12})\Lambda B(p_{12})]^{-1}\\ &= B^{-1}(R^{-1}(\hat{p}_3)p_{12})R^{-1}(\hat{p}_3)\Lambda^{-1}B(\Lambda p_{12}).\end{aligned} \tag{B9}$$

Now $\Lambda = R^{-1}(\text{obs}\to\hat{p}_1(12\,\text{c.m.}))B^{-1}(p_{12})$, so that $B(\Lambda p_{12})$ is the identity boost. Further

$$\begin{aligned}R^{-1}(\hat{p}_3)p_{12} &= [R(\text{obs}\to 123)R(123\to\hat{p}_3)]^{-1}p_{12}\\ &= R^{-1}(123\to\hat{p}_3)p_{12}(123),\end{aligned} \tag{B10}$$

and

$$\begin{aligned}R^{-1}(\hat{p}_3)\Lambda^{-1} &= R^{-1}(123\to\hat{p}_3)R^{-1}(\text{obs}\to 123)B(p_{12})R(\text{obs}\to\hat{p}_1(12\,\text{c.m.}))\\ &= R^{-1}(123\to\hat{p}_3)R^{-1}(\text{obs}\to 123)B(p_{12})R(\text{obs}\to 123)R(123\to\hat{p}_1(12\,\text{c.m.}))\\ &= R^{-1}(123\to\hat{p}_3)B(p_{12}(123))R(123\to\hat{p}_1(12\,\text{c.m.}))\end{aligned} \tag{B11}$$

so that the Wigner rotation of Eq. (B9) can be written as

$$\begin{aligned}&B^{-1}(R^{-1}(123\to\hat{p}_3)p_{12}(123))R^{-1}(123\to\hat{p}_3)B(p_{12}(123))R(123\to\hat{p}_1(12\,\text{c.m.}))\\ &\qquad = (p_{12}(123), R^{-1}(123\to\hat{p}_3))R(123\to\hat{p}_1(12\,\text{c.m.})).\end{aligned}$$

Again, all of the arguments of these rotations depend only on subenergies, and all reference to the observer's frame has been eliminated.

The two remaining rotations are of the same complication as the Wigner rotation which was just discussed. They also depend only on subenergies and can be written as

$$(p_i, \Lambda)(p_i, R^{-1})^{-1} = (p_i(123), R^{-1}(123\to\hat{p}_1(12\,\text{c.m.}))B^{-1}(p_{12}(123))), \quad i=1,2. \tag{B12}$$

Collecting all the terms, we have

$$\begin{aligned}|[\sqrt{s}\,J]\vec{p}=\vec{0}\,\sigma; r'r_1'r_2'r_3's_1s_2\rangle &= \sum_{r_1r_2r_3}\sum_{j_{12}r_{12}}[(2J+1)(2j_{12}+1)]^{1/2}D^J_{r'r_{12}+r_3}(R(123\to\hat{p}_3))\\ &\quad\times\prod_{i=1}^{2}D^{J_i}_{r_i r_i'}(p_i(123), R^{-1}(123\to\hat{p}_1(12\,\text{c.m.}))B^{-1}(p_{12}(123)))\\ &\quad\times D^{J_3}_{r_3r_3'}(p_3(123), R^{-1}(123\to\hat{p}_3))\\ &\quad\times D^{j_{12}}_{r_{12}r_1+r_2}(p_{12}(123), R^{-1}(123\to\hat{p}_3)\,R(123\to\hat{p}_1(12\text{ c.m.})))\\ &\quad\times|[\sqrt{s}\,J]\vec{p}=\vec{0}, \sigma; r_1r_2r_3r_{12}s_{12}j_{12}\rangle\end{aligned} \tag{B13}$$

with inverse

$$
\begin{aligned}
|[\sqrt{s}\,J]\vec{\mathrm{p}}=0,\sigma;r_1r_2r_3r_{12}s_{12}\,j_{12}\rangle = \sum_{r_1'r_2'r_3'r'} (2J+1)^{-1/2}\int dR\, D^{J*}_{r'r_{12}+r_3}(R(123\to\hat{p}_3)) \\
\times\left(\prod_{i=1}^{2} D^{J_i*}_{r_ir_i'}(p_i(123),R^{-1}(123\to\hat{p}_1(12\,\mathrm{c.m.}))B^{-1}(p_{12}(123)))\right) \\
\times D^{J_3*}_{r_3r_3'}(p_3(123),R^{-1}(123\to\hat{p}_3)) \\
\times D^{j_{12}}_{r_{12}r_1+r_2}(R)\,|[\sqrt{s}\,J]\vec{\mathrm{p}}=\vec{0}\,\sigma;r'r_1'r_2'r_3's_1s_2\rangle\,. \qquad \text{(B14)}
\end{aligned}
$$

The coupling scheme that has been used in Eqs. (B13) and (B14) first couples particles 1 and 2 together, then 1-2 to 3. But the notation has been so chosen as to indicate, by permuting particle labels, how any other stepwise-coupled scheme is related to the symmetrical scheme. Then using Eqs. (B13) and (B14), the coefficients transforming a stepwise coupling scheme to any other stepwise scheme can be computed. Hence, Eqs. (B12) and (B13) provide the means for computing any Racah coefficient for the Poincaré group.

As with the unitarity equations, these Racah coefficients seem quite complicated; but after specific choices are made for the coordinate frames, many of the D functions will collapse, and the coefficients will look much simpler.

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[1]See any book on high-energy physics, such as R. J. Eden, *High Energy Collision of Elementary Particles* (Cambridge Univ. Press, Cambridge, England, 1967) or A. O. Barut, *The Theory of the Scattering Matrix* (MacMillan, New York, 1967).

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[13]It is at this point that specifying rotations becomes complicated. A frame is specified by both the relevant center of mass and the number of particles in that center-of-mass frame. Hence, $bf_{i,j}(k\cdots l\,\mathrm{c.m.})$ means the body-fixed frame specified by particles i through j in the $p_k+\cdots+p_l$ center-of-mass frame. If no c.m. is specified, then the over-all center of mass is understood. If no particle subscripts are specified, then the particles are understood to be the same as those specifying the center of mass. Thus, "int" really means $\mathrm{int}_{1,N}$ $(1\cdots N\,\mathrm{c.m.})$.

[14]National Auxiliary Publications Service: For a copy of Appendix C, order document NAPS 01605, from ASIS-National Auxiliary Publications Service, c/o CCM Information Corporation, 909 Third Avenue, New York, N. Y. 10022.

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[16]It is usually understood that a four-vector p is specified relative to the observer's coordinate frame; but when p is specified relative to other body-fixed frames, it is necessary to be more explicit and write p as p(obs) or p(bf), etc. This point is discussed in more detail in Sec. V, Eq. (5.3) ff.

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