

*Present address: Arbeitsstelle Physik, Universität Bielefeld, 48 Bielefeld, W. Germany.

¹A. H. Völkel, Phys. Rev. D 1, 3377 (1970).

²A. H. Völkel, Phys. Rev. D 3, 917 (1971).

³For the sake of fairness we should mention that this is denied by some physicists (H. Kleinert and B. Hamprecht, private communication).

⁴B. Renner, *Current Algebras and their Applications* (Pergamon, New York, 1968).

⁵S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, Amsterdam, 1968).

⁶The distribution and use of indices at the currents throughout this paper are as follows:

(i) Upper Greek indices $\mu, \nu, \lambda \dots = 0, 1, 2, 3$ to the left of the argument(s) indicate tensor properties with respect to the Lorentz group, the corresponding Latin indices $k, l, r \dots = 1, 2, 3$ their restriction to the space parts.

(ii) Lower Latin indices $a, b, c = V, A$ to the right of the argument(s) differentiate between vectors (V) and axial vectors (A). In the commutation relations we have the connection $a \neq b \rightarrow c = A$ and $a = b \rightarrow c = V$.

(iii) Lower Greek indices α, β, γ to the left of the argument(s) refer to the internal broken-symmetry group with structure constants $\epsilon_{[\alpha, \beta] \gamma}$. The usual summation convention for double indices is used. The Minkowski metric is $(+1, -1, -1, -1)$.

⁷Further restrictions on the class of admitted δ sequences like $\varphi(x^0) \geq 0$, etc., would not influence our results.

⁸R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

⁹L. Gårding and A. S. Wightman, Arkiv Fysik 28, 129 (1964).

¹⁰R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, R. I., 1965).

¹¹B. Schroer and P. Stichel, Commun. Math. Phys. 3, 258 (1966).

¹²U. Völkel and A. H. Völkel, Nuovo Cimento 63A, 203 (1969).

¹³L. Schwartz, *Théorie des Distributions* (Hermann, Paris, 1959), Vol. II.

¹⁴From the work of Schroer and Stichel (Ref. 11) it follows that this is very probably not a new and severe assumption beyond the existence of equal-time commutators.

¹⁵Watch that the form factors depend also on the isospin quantum numbers of the nucleon states, which we have not written out explicitly. Therefore, in the following formulas, a summation over some of these quantum numbers is to be understood.

¹⁶D. Schildknecht, DESY Report No. 69/41, 1969 (unpublished).

¹⁷We want to point out that the one-pion pole in the t channel does not contribute to the invariant function $A_{ij}^1(\dots)$.

¹⁸R. E. Marshak, Riazuddin, and Ciaran P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969).

¹⁹K. Dietz and J. Kupsch, Nucl. Phys. B2, 581 (1967).

Analyticity, Covariance, and Unitarity in Indefinite-Metric Quantum Field Theories*

A. M. Gleeson, R. J. Moore,† H. Rechenberg,‡ and E. C. G. Sudarshan
Center for Particle Theory, University of Texas at Austin, Austin, Texas 78712
(Received 15 December 1970)

The general properties of analyticity, covariance, and unitarity are studied in quantum field theories regularized by finite-mass, indefinite-norm states. After reviewing the general status of indefinite-metric theories, a relativistic scalar model is analyzed for covariance and analyticity. This model shows that a commonly accepted prescription for treating the negative-norm states is not covariant, and more sophisticated methods are required. The technique of shadow states developed elsewhere is reviewed and applied to this problem.

I. INTRODUCTION

The problem of constructing finite quantum field theories has received considerable attention in recent years. Although it was invented by Dirac¹ as early as 1941 and has been the subject of careful studies for the past decade, the possibility of using quantized fields which are linear operators in a space whose metric is not positive definite is only gradually coming to be widely appreciated. Some of the confusion was due to the misconception

that such a quantum field theory with indefinite metric is the same as the recipe of regularization introduced by Pauli and Villars in 1949.² The conceptual framework, including the question of probability interpretation and the need to select out a subset of states to be the physical states, has been reviewed by one of the present authors in his report to the 14th Solvay Congress.³

The main conclusions of the above-mentioned investigations are the following:

(i) Every order of perturbation theory yields

finite contributions, provided masses and coupling constants are suitably chosen.

(ii) It is essential to select a subset of all available states and consider these alone as physical states insofar as questions of the physical S -matrix elements and unitarity are considered.

(iii) This selection of physical states entails certain features with regard to the analytic properties of the scattering amplitudes. The threshold for the production of one of the unphysical particles signals the change from one analytic function to another. This means that the transition amplitudes are continuous functions, but are only piecewise analytic.

(iv) Since the higher-mass quanta, which are necessary for the convergence of the theory, could decay to the lower-lying states, one should find kinematic resonances.⁴ This is necessary even for the electron and neutrino, as well as for the nucleons.

(v) The usual assumed analytic properties of scattering amplitudes are no longer expected to be valid. Both complex poles and branch cuts are expected.

Despite these various radical differences it is possible to show that many familiar results would remain unchanged. As a particular example, it has been shown that the finite quantum electrodynamics so constructed agrees with the standard results.⁵ The leptonic resonances that are predicted by the theory have not yet been identified, but neither have there been any decisive experimental tests.

Recently the theme has been taken up by Lee and Wick⁶⁻⁸ and by Lee.^{9,10} These authors have explored the possibility that the negative probabilities are all associated with complex-mass quanta, and have given a set of rules of computation of the S -matrix elements in such a theory. It was their declared conclusion that in such a theory, at least in the lowest-order diagrams, the unitarity troubles did not arise provided their computational rules were followed. We will examine this claim in this paper and show that the Lee-Wick rules do not lead to amplitudes with the properties which they require; in particular, their results are not relativistically invariant. The method must therefore be abandoned.

In an earlier paper two of the present authors investigated the physics of complex-mass particles in an indefinite-metric static-model field theory.¹¹ In this model it was made clear that a field theory of complex-mass particles was not automatically free of unitarity difficulties. It was shown that the primary difficulties are associated with real effective-mass states composed of two particles of complex-conjugate mass. A realistic program for

the formulation of indefinite-metric theories without unitarity troubles was reviewed.

In this paper we elaborate on the proper method of constructing S -matrix elements in an indefinite-metric theory; for the simplest diagrams the result is naturally identical with the result desired (but not obtained) by Lee and Wick. The general rule for arbitrary diagrams is outlined; it is important to recognize that the scattering amplitude so calculated is only piecewise analytic. Hence, the notions of crossing symmetry and analytic properties of the amplitudes have to be viewed in a new light.

These unusual properties are a reflection of the presence of "shadow states" in the theory. These are states which enter the dynamical theory but which do not contribute to the unitarity sum.¹²⁻¹⁴ The indefinite metric is operative only within the shadow states; the physical states are all positive-norm states. Shadow states contribute to dynamical effects like one-particle exchanges and resonance enhancements as if they were real particles, but they are not to be included in counting complete sets of physical states.

The plan of this paper is as follows. In Sec. II a simplified model theory without spin or internal symmetries is defined. The method of making such theories finite by the use of an indefinite-metric space is discussed. The possibility of complex masses is noted. In Sec. III the rules given by Lee and Wick are examined. In Sec. IV we outline a computation, which is carried out according to the Lee-Wick rules, of the simplest possible diagram in perturbation theory. It is shown that the result is *not* relativistically invariant. The details of this computation are given in an Appendix. Section V extends the proper method of calculation to three-body amplitudes. In Sec. VI we argue that piecewise analyticity, complex poles, and shadow states are characteristic of finite relativistic quantum field theories and that one should look for these characteristics in particle physics.

II. INTERACTING FIELDS WITH INDEFINITE METRIC

We consider model field theories¹⁵ which involve two distinct scalar fields ψ and ϕ with an interaction Lagrangian density of the form

$$L_I = \sum_{n, m} f_{nm} \psi^n \phi^m, \quad (2.1)$$

where m, n are integers. The simplest nontrivial interactions are of the form

$$L_I = f \psi \phi^2, \quad (2.2)$$

$$L_I = f \psi^2 \phi^2. \quad (2.3)$$

We shall calculate the lowest-order results for the propagator of the ψ field for the interaction (2.2); this will be one of the two terms contributing to the lowest-order scattering amplitude of the quanta of the ψ field. For the interactions (2.2) and (2.3) the relevant diagrams are illustrated in Figs. 1 and 2.

According to the standard rules for Feynman diagrams,¹⁶ the lowest-order contribution of L_1 to the self-energy is the loop

$$\Sigma(p) = \frac{-if^2}{4\pi^4} \int d^4q D_\phi(\frac{1}{2}p + q) D_\phi(\frac{1}{2}p - q), \quad (2.4)$$

where

$$D_\phi(q) = \frac{-i}{q^2 - m^2 + i\epsilon}, \quad (2.5)$$

and m is the mass of the quanta of the ϕ field. With this choice for $D_\phi(q)$ the expression (2.4) is logarithmically divergent and therefore meaningless. This is the standard divergence of relativistic quantum field theory. In the formal renormalization program this infinity is simply canceled by an infinite renormalization counterterm.

The attempt to make a finite quantum field theory for this model starts with the choice of a generalized free field for ϕ with propagator

$$D_\phi(q) = D_{\phi_1 + \phi_2}(q) = -i \left(\frac{1}{q^2 - m_1^2 + i\epsilon} - \frac{1}{q^2 - m_2^2 + i\epsilon} \right) = \frac{-i(m_1^2 - m_2^2)}{(q^2 - m_1^2 + i\epsilon)(q^2 - m_2^2 + i\epsilon)}. \quad (2.6)$$

The field ϕ_1 with mass m_1 has the standard commutation relations, but the ϕ_2 field with mass m_2 (chosen to be larger than m_1) has the opposite sign of commutation relations. Consequently, any state with an odd number of excitations of the ϕ_2 field has negative norm. The substitution of (2.6) into (2.4) yields a finite and unambiguous result, which explicitly depends upon p^2 , m_1^2 and m_2^2 . The expression so obtained would have branch points at

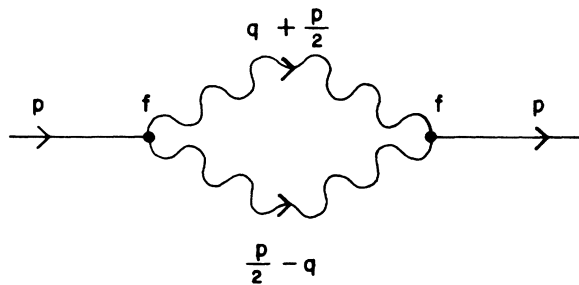


FIG. 1. Self-energy diagram of the field ψ in the theory with $L_I = f \psi \phi^2$.

$4m_1^2$, $(m_1 + m_2)^2$, and $4m_2^2$. The discontinuities across the branch cuts associated with the first and third branch points have usual (positive) sign while the discontinuity across the second branch cut is of the opposite sign. This is consistent with the norm of the expected intermediate states.

Lee explores the possibility of choosing instead of (2.6) the expression

$$D_\phi(q) = -i \left(\frac{1}{q^2 - m^2} - \frac{1/2}{q^2 - M^2} - \frac{1/2}{q^2 - M^{*2}} \right), \quad (2.7)$$

where M is a suitable complex number.¹⁷ This propagator also falls off as $(q^2)^{-2}$ for large values of q^2 , and hence leads to convergence. In the usual case of real masses, there is an infinitesimal imaginary part attached to all real masses to indicate how the integrals are to be performed in the momentum space. In the present context they are omitted, and instead a certain definite rule regarding the path of integration over the momentum variables is specified.

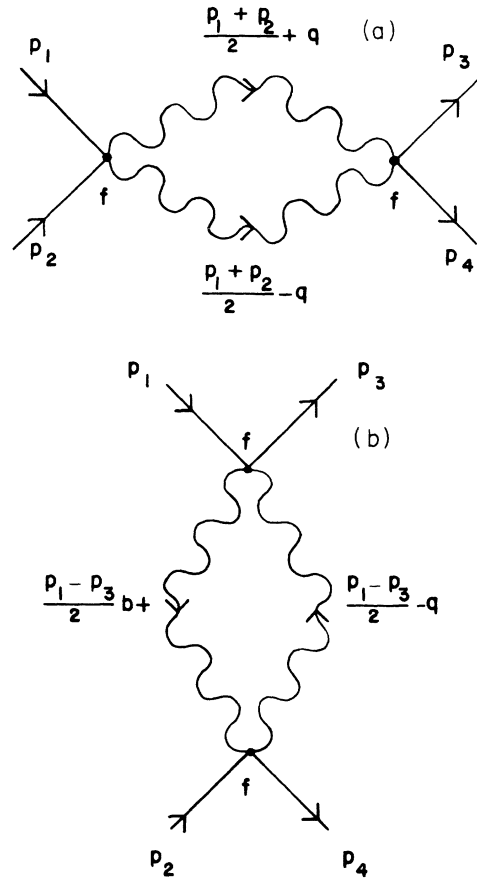


FIG. 2. (a), (b) Lowest-order elastic ψ scattering in the theory with $L_I = f \psi^2 \phi^2$.

We note that instead of (2.7) we could have chosen the expression

$$D_\phi(q) = -i \left(\frac{1}{q^2 - m^2 + i\epsilon} - \frac{1/2}{q^2 - M_1^2 + i\epsilon} - \frac{1/2}{q^2 - M_2^2 + i\epsilon} \right), \quad (2.7')$$

considered the quantity (2.4) computed for real M_1^2, M_2^2 as the boundary value of an analytic function, and then analytically continued the expression for $M_1 \rightarrow M, M_2 \rightarrow M^*$. The result obtained in this way would, however, have a branch point at

$$s = p_0^2 - \vec{p}^2 = (M + M^*)^2 \quad (2.8)$$

and a discontinuity on the real axis which would have violated unitarity. It is the removal of just this problem that prompted the introduction of a computational prescription that differs from standard Feynman rules.

There are apparently many ways out of the prob-

lem of the "unphysical" intermediate states, and these should lead to no branch cut (imaginary part) originating at $(M + M^*)^2$. The method of shadow states, treating the two-complex-mass-particle intermediate state as a shadow state, automatically gives such a result. A prescription which also leads to such a result has been developed by Cutkosky, Landshoff, Olive, and Polkinghorne, who suggest it as an arithmetic rule for certain diagrams.¹⁸

The method advocated by Lee uses the rule that in a frame in which p has a *nonzero* spatial component, p_x , one must use real virtual three-momentum \vec{q} , but q_0 can be complex.¹⁰ The complex path of the q_0 integration is continuously deformed from the Feynman path for (2.7), so that no poles cross the contour. Lee carried out the calculation to the point where he was able to deduce that the result of this integration gave a Lorentz-invariant, finite expression¹⁹:

$$\begin{aligned} \Sigma(p) = & -(32\pi^2)^{-1} f^2 \left\{ \frac{2}{s} (M^2 - m^2) \ln \frac{M^2}{m^2} + \frac{2}{s} (M^{*2} - m^2) \ln \frac{M^{*2}}{m^2} - \frac{1}{s} (M^2 - M^{*2}) \ln \frac{M^2}{M^{*2}} \right. \\ & - (8m^2 - 2s)J(s; m, m) + \left[4(M^2 + m^2) - \frac{2}{s}(M^2 - m^2)^2 - 2s \right] J(s; M, m) \\ & + \left[4(M^{*2} + m^2) - \frac{2}{s}(M^{*2} - m^2)^2 - 2s \right] J(s; M^*, m) - (2M^2 - \frac{1}{2}s)J(s, M, M) \\ & \left. - (2M^{*2} - \frac{1}{2}s)J(s, M^*, M^*) - \left[2(M^2 + M^{*2}) - \frac{1}{s}(M^2 - M^{*2})^2 - s \right] J(s; M, M^*) \right\}, \end{aligned}$$

where

$$s = p^2 = p_0^2 - \vec{p}^2,$$

$$J(s; \mu_1, \mu_2) = F^{-1} \ln P,$$

$$F = [s - (\mu_1 + \mu_2)^2]^{1/2} [s - (\mu_1 - \mu_2)^2]^{1/2},$$

and

$$P = \frac{(\mu_1^2 + \mu_2^2) - s + F}{(\mu_1^2 + \mu_2^2) - s - F}.$$

We shall investigate the computation and show that Lee's deduction is incorrect; his integration prescription gives an amplitude which is *not* the above expression and is not Lorentz-invariant.

III. THE METHOD OF LEE AND WICK

As mentioned above, the scheme proposed by Lee and Wick is as follows: In every Feynman diagram in any order of perturbation theory integrate over real three-dimensional circulating momenta only. The integration over the time components of any circulating momentum is along a complex path defined as follows: Break up the integrands into indi-

vidual terms so that each term has internal lines with well-defined masses; the complex path of the circulating integration variables is obtained by a continuous deformation of the Feynman-Stückelberg path as the complex masses in the problem are realized starting with real masses. While there are some questions about the uniqueness of this definition of the integration path for complicated diagrams, for the simpler diagram that we shall investigate it is unambiguously defined.

It is somewhat curious that the prescription is supposed to be valid in any Lorentz frame except that for the evaluation of the integral (2.4) we are supposed to exclude the center-of-momentum frame. The suggestion was made that for more complicated diagrams such a restriction is not necessary.

It was argued by Lee and Wick that the absence of any imaginary part due to intermediate states containing one or more complex-mass particles was to be expected from physical considerations as follows: If the two intermediate particles have the same sign of imaginary part, their total energy

(for real individual three-momenta) will always be complex and would therefore never satisfy energy-momentum conservation for real external energy and momentum. For the case of two complex-conjugate-mass particles, if their three-momenta have the same magnitude their total energy would also be real. However, the set of real spatial momenta which satisfy this criterion is a set of measure zero when the total three-momentum is non-zero. Hence, according to the argument, the contribution comes only from a set of measure zero and therefore may be ignored.⁷

This argument is incomplete in two respects. First, the states of a complex-mass particle in a relativistic theory have to be defined with some care. States with well-defined spatial momenta do not exist. Rather, one should admit only those states for which the momentum-space wave functions are analytic.¹² Under such a constraint, the complex-momentum states are no different from highly singular linear combinations of real momenta. Also, when we have a pair of complex-conjugate particles, it may be sufficient if their total

momentum is real. Secondly, even in the usual perturbation theory it is quite evident that while the discontinuities of scattering amplitudes are associated with the possibility of intermediate physical states with all the particles on the mass shell, the actual contributions to the discontinuity come not only from the "mass-shell" part of the propagator but also from the "off-mass-shell" part. For the single loop in Fig. 1, with real masses, equal amounts stem from the completely "on-shell" and the completely "off-shell" parts.²⁰ For the case of complex masses we must not therefore simply assume that the contribution is zero.

Since the prescription is frame-dependent, we must investigate whether or not the results are frame-dependent. One cannot rely on the relativistic invariance of the theory or analytic continuation since the integral is defined by the explicit form (2.4) and *not* by analytic continuation. Otherwise, we would get the standard result anyway.

As a general observation we can assert that *no* modification of integration contours can yield amplitudes which would satisfy the proper unitarity condition. This stems from the fact that the unitarity condition, in its basic form, deals with the entire state and should thus be associated with the propagator for the entire state rather than for individual particles.

In view of these considerations and the indirect justifications given by Lee for the expression obtained for the quantity (2.4), we must evaluate it directly and study the resulting expression.

IV. STRUCTURE OF THE VACUUM POLARIZATION LOOP

We calculate the integral

$$\begin{aligned} I(p) &= \frac{2\pi^3}{f^2} \Sigma(p) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} d^3q \int_{-\infty}^{\infty} dq_0 D_{\phi}(\frac{1}{2}p + q) D_{\phi}(\frac{1}{2}p - q), \end{aligned}$$

with

$$D_{\phi}(q) = -i \left(\frac{1}{q^2 - m^2} - \frac{1/2}{q^2 - M^2} - \frac{1/2}{q^2 - M^{*2}} \right) \quad (4.1)$$

for real values of p_0 and \vec{p} . This expression contains six terms which can be classified as follows:

- (a) Both propagators have mass m .
- (b) Both propagators have mass M or both mass M^* .
- (c) One has mass m and the other M or M^* .
- (d) One propagator has mass M and the other mass M^* .

Each of these terms separately is logarithmically divergent, but this divergence cancels when we take their sum.

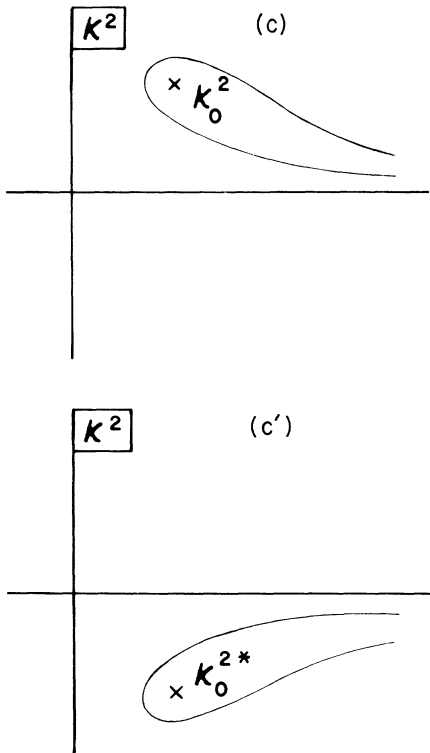


FIG. 3. Areas of discontinuity for the terms of type (c): The contours of the d^4k^2 integrals go within the fish-shaped figures (c) and (c'), whose surface is given by the values of k^2 which satisfy Eq. (4.8). The "physical" thresholds $\kappa_0^2 = (m + M)^2$ and $\kappa_0^{2*} = (m + M^*)^2$, respectively, do not play a specific role as boundary points, but lie in the interior of the figures (c) and (c').

Considered as a function of q_0 the integrand has four poles at the points

$$q_0 = -\frac{1}{2}p_0 - [(\frac{1}{2}\vec{p} + \vec{q})^2 + \mu_1^2]^{1/2}, \quad (4.2)$$

$$q_0 = \frac{1}{2}p_0 - [(\frac{1}{2}\vec{p} - \vec{q})^2 + \mu_2^2]^{1/2}$$

and

$$q_0 = -\frac{1}{2}p_0 + [(\frac{1}{2}\vec{p} + \vec{q})^2 + \mu_1^2]^{1/2}, \quad (4.3)$$

$$q_0 = \frac{1}{2}p_0 + [(\frac{1}{2}\vec{p} - \vec{q})^2 + \mu_2^2]^{1/2},$$

where μ_1, μ_2 have the values

$$\mu_1 = \mu_2 = m \text{ for the terms of type (a),}$$

$$\mu_1 = \mu_2 = M \text{ or } \mu_1 = \mu_2 = M^* \text{ for terms of type (b),}$$

$$\mu_1 = m, \mu_2 = M \text{ or } M^* \text{ for terms of type (c),}$$

$$\mu_1 = M, \mu_2 = M^* \text{ for terms of type (d).}$$

The integration over q_0 is to be carried out over a path that goes from $-\infty$ to $+\infty$ below the poles (4.2) and above the poles (4.3). The integration over \vec{q} goes from $-\infty$ to $+\infty$ along the real axis for each component. In the case where at least one of the masses is complex the open contour of integration becomes detoured off the real axis to complex values to q_0 .

Expanding the $D_\phi(q)$, the six terms in I can be displayed directly as

$$\begin{aligned} I(p) = & I(m, m; p) + \frac{1}{4}I(M, M, p) + \frac{1}{4}I(M^*, M^*, p) \\ & - I(m, M, p) - I(m, M^*, p) + \frac{1}{2}I(M, M^*, p), \end{aligned} \quad (4.4)$$

where

$$I(\mu_1, \mu_2; p) = \pi \int_{-\infty}^{\infty} dl \int_0^{\infty} dk^2 \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{p_0^2 - (E_1 + E_2)^2},$$

with

$$\begin{aligned} E_1 = & [k^2 + (\frac{1}{2}p_z + l)^2 + \mu_1^2]^{1/2}, \\ E_2 = & [k^2 + (\frac{1}{2}p_z - l)^2 + \mu_2^2]^{1/2}. \end{aligned} \quad (4.5)$$

We have carried out the transition to cylindrical coordinates in computing (4.5). Details of this substitution and the subsequent manipulations are contained in the Appendix. By introducing the complex variable κ^2 , which is defined by

$$\begin{aligned} (\kappa^2 + p_z^2)^{1/2} = & [k^2 + (\frac{1}{2}p_z + l)^2 + \mu_1^2]^{1/2} \\ & + [k^2 + (\frac{1}{2}p_z - l)^2 + \mu_2^2]^{1/2}, \end{aligned} \quad (4.6)$$

(4.5) may be written in the form

$$\begin{aligned} I(\mu_1, \mu_2; p) = & \pi \int_{-\infty}^{\infty} dl \\ & \times \int_{\text{fixed } l} d\kappa^2 \frac{1}{(\kappa^2 + p_z^2)^{1/2}} \frac{1}{p_0^2 - p_z^2 - \kappa^2}. \end{aligned} \quad (4.7)$$

The integration over κ^2 is to be carried out over the complex contour defined by (4.6) for each value of l as k^2 varies from 0 to ∞ . For different values of l the variability domain of κ^2 is, in general, along a different complex contour. As $k^2 \rightarrow \infty$, $\kappa^2 \rightarrow \infty$, and the other end point of the κ^2 contour is given by

$$[\kappa^2(l) + p_z^2]^{1/2} = [(\frac{1}{2}p_z + l)^2 + \mu_1^2]^{1/2} + [(\frac{1}{2}p_z - l)^2 + \mu_2^2]^{1/2}. \quad (4.8)$$

In all except the terms of the type (a), the κ^2 integration runs over complex values within a fish-shaped domain which includes $(\mu_1 + \mu_2)^2$ at the inner point.

This fish-shaped domain avoids the real axis altogether for terms of the type (b) and (c), as we may see from Eq. (4.8) or (4.6), since in both cases the imaginary part of $\kappa^2(l)$ is either positive or negative. [See the cases (c) and (c') in Fig. 3.] Since the complex masses enter these terms in a symmetric fashion, it follows that the net contribution from these terms for real p_z is real. For terms of type (a), the κ^2 contour is always real and the resulting function has a branch point at $p^2 = 4m^2$.

For the term of type (d) the fish-shaped domain sits astraddle the real axis. The shape and extent of the fish-shaped region depend on p_z and as long as $p_z \neq 0$, there are contributions off the real axis. The integral clearly is real for real \vec{p} . For different values of l we get a series of gentle curves peeling off the edges of the fish (Fig. 4).

The tip of the fish, κ^2_{\min} , can be calculated and is

$$\begin{aligned} \kappa^2_{\min} = & M^2 + M^{*2} - \frac{1}{2}p_z^2 \\ & + [(2MM^* + \frac{1}{2}p_z^2)^2 + (M - M^*)^2 p_z^2]^{1/2}. \end{aligned} \quad (4.9)$$

Clearly, for $p_z \neq 0$ and $\text{Im}M \neq 0$,

$$\kappa^2_{\min} < \kappa_0^2 \equiv (M + M^*)^2. \quad (4.10)$$

This inequality shows that the result of the integration is *not* Lorentz-invariant since κ^2_{\min} *explicitly* depends on p_z^2 for the complex-mass case. It is therefore not only a function of the invariant quantity p^2 , but also of p_z . Rewriting (4.7) in the form

$$\begin{aligned} I(M, M^*; p) = & \pi \int_0^{\infty} dl \\ & \times \int_{\kappa^2(l)}^{\infty} d\kappa^2 \frac{1}{(\kappa^2 + p_z^2)^{1/2}} \frac{1}{p^2 - \kappa^2} + \text{c.c.}, \end{aligned} \quad (4.11)$$

we now integrate (4.11) by parts with respect to l . Dropping the unimportant surface term (see the Appendix) and changing the name of the integration variable κ^2 to s' , we get (A13):

$$\begin{aligned}
I(M, M^*, p) = & \pi \int_{\kappa_{\min}^2}^{\infty} ds' \frac{1}{2s'} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2} \frac{1}{p^2 - s'} + \text{c.c.} \\
& + \pi \int_{\kappa_{\min}^2}^{\infty} ds' \frac{1}{(s' + p_z^2)^{1/2}} \frac{p_z(M^2 - M^{*2})}{2s'} \frac{1}{p^2 - s'} + \text{c.c.} \quad (4.12)
\end{aligned}$$

The integration goes along the complex path which is the boundary of the fish-shaped region in Fig. 4. To be more explicit, there are two paths: one in the first contribution of each term from κ_{\min}^2 to infinity in the lower s' plane, and the other path which has to be taken from κ_{\min}^2 to infinity in the upper s' plane for the complex-conjugate term. Both paths are described by the parametrization

$$s'(l) = \left\{ \left[\left(\frac{1}{2} p_z + l \right)^2 + M^2 \right]^{1/2} + \left[\left(\frac{1}{2} p_z - l \right)^2 + M^{*2} \right]^{1/2} \right\}^2 - p_z^2, \quad (4.13)$$

with $l=0$ to $+\infty$ and $l=0$ to $-\infty$, respectively.

We are now in a position to study the complete vacuum-polarization integral (4.1). For this purpose we divide the domain of values of the invariant $s = p^2$ into two ranges:

- I. $s < \kappa_{\min}^2$,
- II. $\kappa_{\min}^2 < s < \infty$.

The contribution of the terms from type (a) does distinguish between the two regions, because there exists one covariant threshold which corresponds to κ_0^2 in this case. The fish figures from the types (b) and (c) do not include parts of the s axis, therefore the region I covers the total s axis in both cases. To summarize, for real values of s the contributions of type (a), (b), and (c) are all Lorentz-invariant. The terms of type (a) have the usual branch point at $s = 4m^2$. The only interesting term is the one of type (d), which comes from a pair of particles in the intermediate state with complex masses which are complex conjugates of each other. In what follows we confine our attention to this term only.

In region I, the integrand of (4.12) has no singularities and we may therefore deform the s' integration contour to lie along the real axis from κ_{\min}^2 to ∞ . It is then verified that the second term of (4.12) does not contribute at all and we get the standard Lorentz-invariant contribution from the first term. This result is therefore the boundary value of a real analytic function; the analytic continuation of this function has a branch point at $s = (M + M^*)^2$. However, this branch point does not lie in the region I, where this is the proper expression.

In region II the first term in (4.12) can be split into two pieces by integrating from κ_{\min}^2 to κ_0^2 and from κ_0^2 to $+\infty$. The second contribution is the analytic continuation of the function from region I and

is invariant. The first piece gives an invariant pole contribution. However, the second term in (4.12) does not vanish in this region, and is not invariant. Therefore, the function in this region is *not* relativistically invariant. The amplitude here is not the analytic continuation of the region-I amplitude, because its derivative is discontinuous at the pseudothreshold κ_{\min}^2 .

Consequently, explicit calculation reveals that, contrary to the hopes and declarations of Lee, the rule for evaluating amplitudes does not give relativistically invariant results. A theory based on such rules should therefore be abandoned.

V. THE TRANSITION AMPLITUDES IN A THEORY WITH INDEFINITE METRIC AND SHADOW STATES

We are led back to the question of unitarity of indefinite-metric theories. The aim of any such theory is to produce finite results, but the states with complex mass and negative norm should not contribute to probability and hence should not enter the unitarity sum. The principle of these theories is to deal with the propagator for the complex state: We chose it to be a standing-wave propagator for the shadow states and a forward propagator for physical states. The physical states consist exclusively of physical (real mass, positive metric) quanta; all other states are shadow states. The general theory of shadow states has been developed by one of us and is discussed in several other pa-

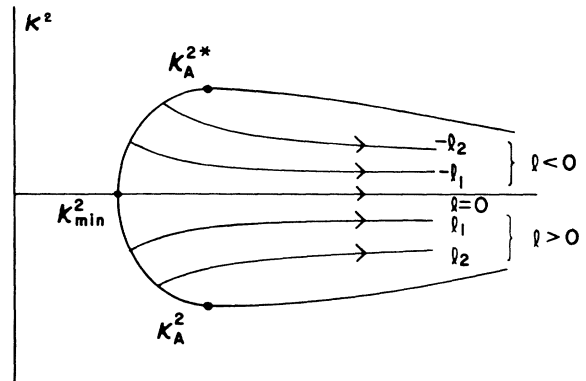


FIG. 4. Area of discontinuity for the terms of type (d): The tip of the fish is called κ_{\min}^2 , Eq. (4.9). The surface points with maximal imaginary part are $\kappa_A^2 = (M + M^*)^2 + 2M^*[(M^2 + p_z^2)^{1/2} - M]$ and κ_A^{2*} .

pers.^{11, 12} Here we shall consider them only within the context of the two diagrams that we have analyzed.²¹

For the single-loop diagram (Fig. 1 or 2) the forward propagating Green's function for the two-particle state is given by the product of the two causal propagators:

$$G^s(q, \mu_1; p-q, \mu_2) = \frac{-i}{q^2 - \mu_1^2 + i\epsilon} \frac{-i}{(p-q)^2 - \mu_2^2 + i\epsilon}. \quad (5.1)$$

But the standing-wave propagator is obtained by taking the average of the forward and the backward propagators²²:

$$G^s(q, \mu_1; p-q, \mu_2) = \frac{1}{2} \frac{-i}{q^2 - \mu_1^2 + i\epsilon} \frac{-i}{(p-q)^2 - \mu_2^2 + i\epsilon} - \frac{1}{2} \frac{-i}{q^2 - \mu_1^2 - i\epsilon} \frac{-i}{(p-q)^2 - \mu_2^2 - i\epsilon}. \quad (5.2)$$

$$J(p) = \frac{1}{2} \int d^4q \int d^4k \left[\frac{-i}{q^2 - m^2 + i\epsilon} \frac{-i}{k^2 - M^2 + i\epsilon} \frac{-i}{(p-q-k)^2 - M^{*2} + i\epsilon} + \frac{-i}{q^2 - m^2 - i\epsilon} \frac{-i}{k^2 - M^2 - i\epsilon} \frac{-i}{(p-q-k)^2 - M^{*2} - i\epsilon} \right]. \quad (5.3)$$

This contribution to the vacuum polarization is a real function for all real values of p^2 given by $J(p) + J^*(p)$. The function is continuous across the point $p^2 = (M + M^* + m)^2$, but on either side of it we have different analytic functions.

For more complicated graphs, like the box diagram, these considerations have to be elaborated somewhat. We should arrange the calculation so that the intermediate states are explicitly displayed. Among these only the physical states have forward propagators associated with them, and they alone lead to imaginary parts for real values of the momenta. The shadow states have standing-wave propagators. While the recipe sounds tedious, for the simpler diagrams for which explicit computation is feasible the shadow-state prescription is straightforward.²¹

The net result of the introduction of shadow states is as follows: We calculate the contributions from various Feynman diagrams in the usual manner. From these diagrams we subtract out the discontinuity coming from shadow states in a systematic fashion. We subtract the contribution from the second-order diagrams, and this result is used to compute the next order, the shadow-state contribution to the discontinuity is then again subtracted out, and so on. Of course the shadow states affect the scattering amplitude through their continuous part, and hence contribute to the total amplitude indirectly. But all imaginary parts are directly

Of the four types of contributions considered in Sec. IV, only those of type (a) are physical states and should use propagators of the type (5.1). The integrations over the circulating momenta for shadow states are to be carried out covariantly according to the prescriptions (5.2). With this prescription the remaining contributions to (4.1) and (4.4) are explicitly real and covariant. Since they have no contributions to the imaginary part of the diagram there is no contribution to unitarity by the shadow states. This of course occurs at the cost of the analyticity of the loop diagram. It follows that *unitarity is satisfied to the lowest order by the physical states alone.*

These considerations can be extended to the calculation of the double loop. (See Fig. 5.) The intermediate state containing the particles mMM^* is by definition a shadow state and the integral becomes

associated with physical intermediate states, and these alone satisfy the unitarity condition.²³

VI. DISCUSSION

We have shown that the method of shadow states leads to finite quantum field theories with explicitly Lorentz-invariant transition amplitudes. Unitarity is satisfied by physical states alone. The amplitudes so obtained have local analyticity and continuity properties, but they are never given by a single analytic function. Rather they are represented by piecewise analytic functions which are continuous at their joining points along the real axis of the invariants. This is a fundamental property of any theory with shadow states ("hidden" states) and we do expect that scattering matrix elements should not be represented by global analytic functions in

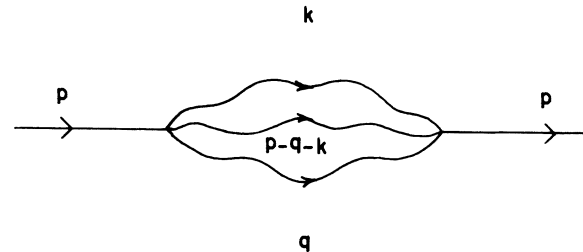


FIG. 5. Self-energy diagram of the field ψ in the theory with $L_I = f \psi \phi^3$.

any consistent quantum theory.

It is curious to note that all the rigorous arguments for analyticity based on basic quantum mechanical considerations only require local analyticity. It is only quantum field theory with its twin assumptions of the completeness of scattering states and the symmetry between emission and absorption that suggests global analytic functions. In any theory with shadow states (and these appear to be the only consistent theories at the present time) the scattering states satisfying unitarity by themselves are the subset comprising the physical states and are, therefore, *not complete* in the sense of quantum field theory. It is therefore not surprising that the use of standing-wave propagators naturally leads to violations of the global analyticity.

Such violations of analyticity would exhibit themselves in the asymptotic behavior of wave functions. A discussion of this question is beyond the scope of the present paper. We are content here to point out that the departure from the assumption of global analyticity as well as the possibility of complex poles would make it imperative that we reexamine many of the conclusions that we have arrived at in particle physics.

We have shown above that avoiding shadow states, and yet dealing with indefinite-metric theory using certain rules of computation, fails to produce a relativistic theory. Quantum field theory with indefinite metric, however, has a satisfactory theoretical framework and should be systematically tested against experiment.

Added note. After we had completed this work, Professor H. P. Stapp brought to our attention a report of lectures, delivered by T. D. Lee at the "Ettore Majorana Summer School, 1970" at Erice, in which he seems to have completely retreated from the previous Lee-Wick prescriptions. We understand from his writing that he agrees with the fact that the old prescriptions are not covariant and that he proposed a new set of rules. Professor S. Okubo has kindly informed us about a paper by N. Nakanishi, who also has reached the conclusion that the prescriptions given by Lee and Wick are not covariant.²⁴ We would like to thank Dr. N. Nakanishi for several critical remarks.

New Variables in the Three-Momentum Integration

Introducing cylindrical coordinates

$$q_x = k \cos \phi, \quad q_y = k \sin \phi, \quad q_z = l, \quad (\text{A2})$$

and defining the direction of the incoming three-momentum \vec{p} as the z axis,²⁵ the angular integration becomes trivial, and the integral (A1) reduces to

$$I(\mu_1, \mu_2; p) = \pi \int_{-\infty}^{\infty} dl \int_0^{\infty} dk^2 \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{p_0^2 - (E_1 + E_2)^2}. \quad (\text{A3})$$

APPENDIX: QUADRATURE OF THE VACUUM-POLARIZATION DIAGRAM IN THE LEE-WICK PRESCRIPTION

In calculating perturbation diagrams of quantum field theory, several methods have been developed. In our case, we perform a direct integration of the vacuum-polarization diagram and avoid the use of the more sophisticated methods. There are several advantages to this direct approach. The direct integration does not involve any implicit analytic continuation and, since analyticity is not preserved in these amplitudes, direct integration does not introduce any misleading simplifications. The form of the propagation functions is not manifestly covariant and direct integration of the diagrams will reveal any frame-dependent terms. The obvious disadvantage is the difficulty of studying more complicated diagrams.

The Energy Integration

Starting from the integral (4.1) we carry out the dq_0 integration. According to the Lee-Wick prescription, the contour should pass below the poles

$$q_0 = -\frac{1}{2}p_0 - \left[\left(\frac{1}{2}\vec{p} + \vec{q} \right)^2 + \mu_1^2 \right]^{1/2}$$

and

$$q_0 = \frac{1}{2}p_0 - \left[\left(\frac{1}{2}\vec{p} - \vec{q} \right)^2 + \mu_2^2 \right]^{1/2}$$

and above the poles

$$q_0 = -\frac{1}{2}p_0 + \left[\left(\frac{1}{2}\vec{p} + \vec{q} \right)^2 + \mu_1^2 \right]^{1/2}$$

and

$$q_0 = \frac{1}{2}p_0 + \left[\left(\frac{1}{2}\vec{p} - \vec{q} \right)^2 + \mu_2^2 \right]^{1/2}.$$

Since the integrand (4.1) is eventually regularized, I is supposed to behave well at $q_0 = \infty$ and we may close the contour at infinity. Therefore the q_0 integral is $2\pi i$ times the residue of either the two poles above or the two poles below the contour. The integral is

$$I(\mu_1, \mu_2; p) = \frac{1}{2} \int d^3q \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{p_0^2 - (E_1 + E_2)^2},$$

with

$$\begin{aligned} E_1 &= \left[\left(\frac{1}{2}\vec{p} + \vec{q} \right)^2 + \mu_1^2 \right]^{1/2}, \\ E_2 &= \left[\left(\frac{1}{2}\vec{p} - \vec{q} \right)^2 + \mu_2^2 \right]^{1/2}. \end{aligned} \quad (\text{A1})$$

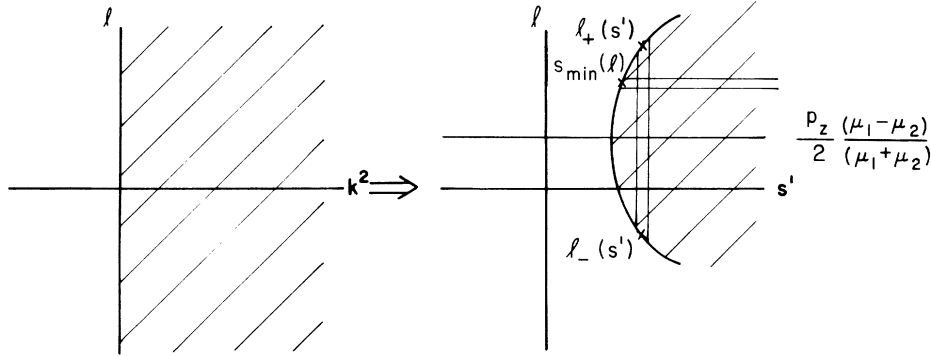


FIG. 6. Mapping of the k^2-l half plane to s', l variables.

In the case of the usual Feynman prescriptions with real masses, the further evaluation of (A3) is straightforward and we review it so that the required changes for complex masses will become apparent. The integration variable k^2 is replaced by a new variable s' , defined by

$$(s' + p_z^2)^{1/2} = [k^2 + (\frac{1}{2}p_z + l)^2 + \mu_1^2]^{1/2} + [k^2 + (\frac{1}{2}p_z - l)^2 + \mu_2^2]^{1/2}, \tag{A4}$$

and

$$I(\mu_1, \mu_2; p) = \pi \int_{-\infty}^{\infty} dl \int_{s'_{\min}(l)}^{\infty} ds' \frac{1}{(s' + p_z^2)^{1/2}} \frac{1}{s - s'}. \tag{A5}$$

The path of integration goes along the real s' axis from a minimum value s'_{\min} , depending on l and p_z , to infinity. The mapping between the k^2-l plane and the $s'-l$ plane is shown in Fig. 6. After interchanging the dl and the ds' integrations, $I(\mu_1, \mu_2; p)$ becomes

$$I(\mu_1, \mu_2; p) = \pi \int_{(\mu_1 + \mu_2)^2}^{\infty} ds' \int_{l_-(s')}^{l_+(s')} dl \frac{1}{(s' + p_z^2)^{1/2}} \frac{1}{s - s'}, \tag{A6}$$

where

$$l_{\pm} = \frac{p_z}{2s'} (\mu_1^2 - \mu_2^2) \pm \frac{1}{2s'} (s' + p_z^2)^{1/2} [s' - (\mu_1 - \mu_2)^2]^{1/2} [s' - (\mu_1 + \mu_2)^2]^{1/2}. \tag{A7}$$

The trivial l integration can now be done directly, and the result is

$$I(\mu_1, \mu_2; p) = I(\mu_1^2, \mu_2^2; s = p^2) = \pi \int_{(\mu_1 + \mu_2)^2}^{\infty} \frac{ds'}{s'} [s' - (\mu_1 - \mu_2)^2]^{1/2} [s' - (\mu_1 + \mu_2)^2]^{1/2} \frac{1}{s - s' + i\epsilon}, \tag{A8}$$

where we have now introduced the $-i\epsilon$ prescription.

The steps leading from Eq. (A5) to Eq. (A8) are not legitimate in the case of complex masses with the prescriptions of Lee and Wick. Equation (A4) does not define a real quantity s' for real values of the three-momentum \vec{q} . This in turn implies, after change in the order of integrations, that the dl integration must be over a complex path. We remark, however, that none of these complications would arise in the special case of a complex pair in a frame with $p_z = 0$. This case Lee and Wick exclude as pathological.

The Complex-Pair Loop Integral

To be specific, let us now turn to the case of $\mu_1 = M$ and $\mu_2 = M^*$, and to a real s . In this case the integral (A5) should be real because to each contribution from a path $s'(l)$ we find the complex-conjugate contribution from a path $s'(-l)$. Therefore, we can replace Eq. (A5) by

$$I(M, M^*; p) = \pi \left(\int_0^{\infty} dl \int_{s'_{\min}}^{\infty} ds' \frac{1}{(s' + p_z^2)^{1/2}} \frac{1}{s - s'} + \text{c.c.} \right). \tag{A9}$$

The paths in the complex s' plane are shown in Fig. 4.

It is very tempting to pass from the area integral (A9) to a surface or line integral by integration by

parts. The derivation of the ds' integral with respect to the variable l yields two terms. One is coming from the boundary according to Eq. (4.13). The other results from the fact that the paths in the ds' integration are also l dependent. These latter contributions cancel against the "surface" term of

$$I(M, M^*; p) = 2\pi \int_0^\infty dl l \left[\frac{\frac{1}{2}p_z + l}{[(\frac{1}{2}p_z + l)^2 + M^2]^{1/2}} - \frac{\frac{1}{2}p_z - l}{[(\frac{1}{2}p_z - l)^2 + M^{*2}]^{1/2}} \right] \frac{1}{[s'(l) + p_z^2]^{1/2}} \frac{1}{s - s'(l)} + \text{c.c.} + \text{surface term.} \quad (\text{A10})$$

This surface term is finite at $l = \infty$.

Now we replace the dl integration by a ds' integration via the s' -defining relation (A4), and obtain

$$I(M, M^*; p) = \pi \int_{\kappa_{\min}^2}^\infty ds' l(s') \frac{1}{(s' + p_z^2)^{1/2}} \frac{1}{s - s'} + \text{c.c.} \quad (\text{A11})$$

The remaining l in Eq. (A11) can be expressed in terms of s' , according to

$$l(s') = \frac{p_z}{2s'} (M^2 - M^{*2}) + \frac{1}{2s'} (s' + p_z^2)^{1/2} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2}. \quad (\text{A12})$$

In Eq. (A12) we recognize an old Eq. (A7); however, in the present case only the positive square root has to be taken. Therefore, different terms from those expressed in Eq. (A8) will appear. We insert the result (A12) into the integral (A11) and obtain

$$I(M, M^*; p) = I_1 + I_2,$$

where

$$I_1 = \pi \int_{\kappa_{\min}^2}^\infty ds' \frac{p_z (M^2 - M^{*2})}{2s' (s' + p_z^2)^{1/2}} \frac{1}{s - s'} + \text{c.c.}$$

and

$$I_2 = \pi \int_{\kappa_{\min}^2}^\infty ds' \frac{1}{2s'} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2} \frac{1}{s - s'} + \text{c.c.} \quad (\text{A13})$$

The paths in the s' integration go from κ_{\min}^2 to infinity in the lower half plane for the first contribution to each integral, and in the upper half plane for the complex-conjugate contributions.

Evaluation of the Integral (A13) in the Different Regions of s

Let us assume a definite three-momentum p_z of the incoming particle. Then we can talk about different regions in the variable $s = p_0^2 - p_z^2$: (I) In the first region, s is below the pseudothreshold κ_{\min}^2 . (II) In the second region, s lies between κ_{\min}^2 and the "physical" threshold $(M + M^*)^2$. (III) The third region extends above $(M + M^*)^2$ to infinity.

In the first region, I, the integrands of both integrals I_1 and I_2 have no singularity, because s does not fall in the fish-shaped region. Then we may detour the s' path to the real axis, and the same paths may be taken for both contributions to each integral. Now, in the first integral, all factors are real except for $(M^2 - M^{*2})$. But then the conjugate-complex term cancels the other one. The second integral, I_2 , we may split into two parts, according to

$$\int_{\kappa_{\min}^2}^\infty ds' \dots = \int_{\kappa_{\min}^2}^{(M + M^*)^2} ds' \dots + \int_{(M + M^*)^2}^\infty ds' \dots \quad (\text{A14})$$

Then the first factor, $[s' - (M + M^*)^2]^{1/2}$, is purely imaginary since s' is below the threshold, and the conjugate-complex term cancels the other. The final contribution is

$$I(M, M^*; p) = \pi \int_{(M + M^*)^2}^\infty \frac{ds'}{s'} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2} \frac{1}{s - s'}, \quad \text{for } s < \kappa_{\min}^2. \quad (\text{A15})$$

In region II, that is, $\kappa_{\min}^2 < s < (M + M^*)^2$, the integrands of both integrals 1 and 2 show singularities, and the s' integral cannot be detoured to the real s' axis at the point $s = s'$, but can only be brought to slightly above and slightly below the real axis.

Thus we have to replace

$$\frac{1}{s-s'} \rightarrow P \frac{1}{s-s'} + i\delta(s-s'), \quad (\text{A16})$$

where P refers to principal value, and the (+) sign to the complex-conjugate term.

With the formula (A16) the integral in region II becomes

$$I(M, M^*; p) = \pi^2 i \left(-\frac{p_z(M^2 - M^{*2})}{s(s+p_z^2)^{1/2}} + \frac{[s - (M - M^*)^2]^{1/2} (+) [s - (M + M^*)^2]^{1/2}}{s} \right) \\ + \pi \int_{(M+M^*)^2}^{\infty} \frac{ds'}{s'} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2} \frac{1}{s-s'}, \quad \text{for } \kappa^2_{\min} < s < (M + M^*)^2. \quad (\text{A17})$$

Here the explicitly indicated (+) sign before the imaginary square root reminds us that we have to take the (+i) root. The result (A17) is obviously noncovariant.

Finally we evaluate in region III above the physical threshold and obtain

$$I(M, M^*; p) = -\pi^2 i \frac{p_z(M^2 - M^{*2})}{s(s+p_z^2)^{1/2}} + \pi \int_{(M+M^*)^2}^{\infty} \frac{ds'}{s'} [s' - (M - M^*)^2]^{1/2} [s' - (M + M^*)^2]^{1/2} P \frac{1}{s-s'}, \quad (\text{A18})$$

for

$$s \geq (M + M^*)^2.$$

In the expression (A18) the second term of the right-hand side of Eq. (A17) is missing. However, the transition from region II to region III is analytic.²⁶ Thus we may consider the regions II and III as a single one. We note that in the limit $p_z \rightarrow 0$ the fish-shaped region shrinks to the real axis, region II disappears, and all the noncovariant contributions are removed.

*Partly supported by the Atomic Energy Commission under Contract No. AEC(40-1)3992.

†Permanent address: University of Wisconsin at Parkside, Wis.

‡On leave of absence from the Max-Planck Institut für Physik and Astrophysik, München.

¹P. A. M. Dirac, Proc. Roy. Soc. (London) **A180**, 1 (1942).

²W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

³E. C. G. Sudarshan, in *Fundamental Problems in Elementary Particle Physics: Proceedings of the Fourteenth Solvay Institute of Physics Conference* (Wiley, New York, 1968), p. 97.

⁴These resonances are called "kinematic" because there exists no necessary connection between the width and the coupling constant. In theories with indefinite metric, there is no need for the diagonal elements of the noninteracting Hamiltonian to be real and, therefore, the bare states may have masses with independent real and imaginary parts.

⁵M. Arons, M. Y. Han, and E. C. G. Sudarshan, Phys. Rev. **137**, B1085 (1965).

⁶T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969); **B10**, 1 (1969).

⁷T. D. Lee and G. C. Wick, Columbia report, 1969 (unpublished).

⁸T. D. Lee and G. C. Wick, Phys. Rev. **D 2**, 1033 (1970).

⁹T. D. Lee, in *Proceedings of the Topical Conference on Weak Interactions, CERN, 1969* (CERN, Geneva, 1969).

¹⁰T. D. Lee, in *Quanta: Essays in Theoretical Physics*, Dedicated to Gregor Wentzel, edited by P. G. O. Freund, C. J. Goebel, and Y. Nambu (Univ. of Chicago Press,

Chicago, 1970).

¹¹A. M. Gleeson and E. C. G. Sudarshan, Phys. Rev. **D 1**, 474 (1970).

¹²E. C. G. Sudarshan, Austin Report No. AEC 5-CPT 41 (unpublished). For the use of the wave function, which must not only be square-integrable but also analytic, see, for example, J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, J. Math. Phys. **9**, 2100 (1968).

¹³E. C. G. Sudarshan, *Coral Gables Conference on Fundamental Interactions at High Energy II*, edited by A. Perlmutter, G. J. Iversen, and R. M. Williams (Gordon and Breach, New York, 1970). This paper discusses the relevant issue of unitarity in a quantum field theory with shadow states. The associated problem of cluster decomposition will be discussed in a separate paper.

¹⁴E. C. G. Sudarshan, Fields and Quanta **2**, 175 (1971).

¹⁵This is the same model as the one treated by T. D. Lee (see Ref. 10). We study it for two reasons: first, because we wish to compare with the results given by Lee, and second, because as far as general unitarity considerations are concerned, the complications brought in by spin, etc., do not change the results obtained from these simple models.

¹⁶The translation of the d^4q integral, associated with symmetrization of $\Sigma(p)$, does not change $\Sigma(p)$ since at worst $\Sigma(p)$ diverges logarithmically.

¹⁷We emphasize that complex masses arise naturally in a theory with indefinite metric.

¹⁸R. E. Cutkosky, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, Nucl. Phys. **B12**, 281 (1969).

¹⁹See Ref. 10, pp. 283-284, and replace δ in this formula by m .

²⁰If the individual propagators are broken up into a principal-value part and a δ -function part, then the term

in the product of the two propagators involving a "product" of principal values (completely "off-shell" part) has a singular piece as well as the term involving the "product" of the δ functions (completely "on-shell" part). Both of these pieces are included in the *physical* discontinuity of the amplitude. In the case of the box diagram with scalar and equal-mass fields this fact has been explicitly demonstrated by M. E. Ebel and R. J. Moore, Phys. Rev. **177**, 2470 (1969). See also J. C. Taylor, Phys. Rev. **150**, 1321 (1966), footnote 23.

²¹Added in the final draft: Since we have submitted this paper, we have made an extended study concerning the unitarity, the cluster properties, and the physical interpretation of the scattering amplitudes in a quantum theory with shadow states. The results do not modify the statements made in this section. For details we refer to the forthcoming paper by H. Rechenberg and E. C. G. Sudarshan.

²²According to our prescription we actually add half the amplitude T and half the amplitude T^* . In the case of

one loop we then have to multiply the antipropagator with a minus sign. In general $(-)^{k-1}$ appears, where k refers to the number of s -channel internal lines.

²³Added in the final draft: See the forthcoming paper by H. Rechenberg and E. C. G. Sudarshan. The discussion of singularities which are not associated with "on-shell" intermediate states will be given elsewhere.

²⁴Added in the galley proof: See N. Nakanishi, Phys. Rev. D **3**, 811 (1971); **3**, 3235 (1971). Already in the paper of Cutkosky *et al.* (Ref. 18) the essential noncovariance of the original Lee-Wick prescription was suspected.

²⁵The integral $I(\mu_1, \mu_2; p)$ may not be explicitly covariant, but it is explicitly rotationally symmetric.

²⁶It would seem as if we had a branch point at $s = (M + M^*)^2$. However, due to the fact that we can take in the integral I_2 , Eq.(A13), the sum of the upper and the lower paths, the apparent nonanalyticity is dissolved. Therefore, the amplitude is not only continuous at the normal threshold $s = (M + M^*)^2$ but also analytic.

PHYSICAL REVIEW D

VOLUME 4, NUMBER 8

15 OCTOBER 1971

Anomalies of the Axial-Vector Currents in a Thirring Model with Internal Symmetry*

Howard Georgi†

Physics Department, Yale University, New Haven, Connecticut 06520

(Received 26 July 1971)

We consider a two-dimensional field theory which is a generalization of the Thirring model to include $SU(2)$ symmetry, and investigate the properties of the isospin currents in this theory. The anomalous divergences of the axial-vector currents are calculated using functional methods. The results agree with those obtained by Hagen using nonperturbative techniques. The anomalies are found to have the form suggested by a dynamical theory of currents in two dimensions.

I. INTRODUCTION

Field theories describing massless fermions in two-dimensional space-time have some peculiar properties which can sometimes be used to simplify calculations. In the Thirring model, for instance, these properties can be used to write down an explicit solution.¹ The trouble with the Thirring model is that it is *too* simple. While it may give some insights into purely field-theoretical problems, it is not an interacting theory in the usual sense. Thus it might be interesting to look at nontrivial generalizations of this model. One possibility is to give the fermions a mass.² This leads to a nontrivial theory, but at the expense of giving up the simplifying features associated with massless fermions. In this paper we take a small step in a different direction by investigating the properties of the currents in a Thirring model with internal symmetry.

The theory we have in mind is described (for-

mally) by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2} \sigma j_a^\mu j_{a\mu}, \quad (1.1)$$

where

$$\mathcal{L}_0 = \frac{1}{2} i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} i (\partial_\mu \bar{\psi}) \gamma^\mu \psi. \quad (1.2)$$

In (1.1), j_a^μ are the vector currents $\bar{\psi} \gamma^\mu \frac{1}{2} \tau^a \psi$, where the τ_a for $a=1, 2,$ or 3 are Pauli matrices acting on the internal-symmetry indices [we work with $SU(2)$ for simplicity, but everything we do works for $SU(3)$]. We use the notation $-g^{00} = g^{11} = 1$. The Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}. \quad (1.3)$$

We also introduce the matrix

$$\gamma^5 = \gamma^0 \gamma^1 \quad (1.4)$$

and the invariant pseudotensor density

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}, \quad \epsilon^{01} = 1. \quad (1.5)$$

For a vector A^μ , we will introduce the notation