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$$\vec{Q}(t_0)=\vec{q},\quad \vec{Q}(t_N)=\vec{q}';$$

$$d\vec{Q}(t_j)=dQ_1(t_j)\cdots dQ_f(t_j),\quad \text{similarly for } d\vec{P}(t_j);$$

$$\Delta t_j=t_j-t_{j-1},\quad \dot{\vec{Q}}(t_j)=[\vec{Q}(t_j)-\vec{Q}(t_{j-1})]/\Delta t_j.$$

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Behavior of Commutator Matrix Elements at Small Distances. III. Sum Rules for Form Factors and Their Derivatives at $t=0$

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From general principles of quantum field theory (especially locality and Poincaré invariance), the small-distance behavior of commutator matrix elements, and partial conservation of axial-vector current (smoothness of the matrix elements in the meson masses), sum rules between the four weak isovector form factors and their first derivatives at zero momentum transfer are derived.

I. INTRODUCTION

In two recent publications[1,2] we started a systematic investigation of the behavior of commutator matrix elements at small distances on the basis of general principles of quantum field theory. These investigations gain some physical significance[3] in connection with Gell-Mann's program of current-generated commutator algebras.[4,5]

In the present article we give an application of these ideas and derive from the current-density algebras, especially from the nonoccurrence of q-number Goto-Imamura-Schwinger terms and partial conservation of axial-vector currents (PCAC), a sum rule between the four weak isovector form factors.

We assume the equal-time commutation relations for the current densities,[6]

$$\lim_{\delta\to 0}\langle\Psi|[j^0_\alpha(\varphi_\delta;h)_a,j^\nu_\beta(0)_b]\Phi\rangle^T=\epsilon_{[\alpha,\beta]}{}^\gamma\langle\Psi|j^\nu_\gamma(0)_c|\Phi\rangle h(\vec{0}),\tag{1}$$

to hold for all C^∞ functions $h(\vec{x})$, all state vectors Ψ,Φ from a certain domain D of the Hilbert space of physical states $\mathfrak{H}$ which is stable under the Poincaré group, and all symmetric δ-sequences $\varphi_\delta(x^0)$ $=:(1/\delta)\varphi(x^0/\delta)$. Here $\varphi(x^0)$ is a C^∞ function with compact support contained in $[-a,a]$ and normalized according to[7]

$$\int dx^0\varphi(x^0)=1.\tag{2}$$

(The equality sign with the colon means "is by definition.") Of course, we assume the currents to be members of a complete local Wightman field theory.[8-10]

T means subtraction of the vacuum expectation value before taking the limit.

$$\langle\Psi|[\cdots]|\Phi\rangle^T=\langle\Psi|[\cdots]|\Phi\rangle-\langle 0|[\cdots]|0\rangle\langle\Psi||\Phi\rangle.\tag{3}$$

A consequence of (1) and Poincaré symmetry of the theory are the following two relations[2]:

$$\lim_{\delta\to 0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[\partial_\mu j^\mu_\alpha(x)_a,j^\nu_\beta(0)_b]|\Phi\rangle^T=0\tag{4}$$

and

$$\lim_{\delta\to 0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[j^0_\alpha(x)_a,\partial_\nu j^\nu_\beta(0)_b]|\Phi\rangle^T=0\quad(k=1,2,3).\tag{5}$$

In other words Eqs. (4) and (5) are necessary (but not sufficient) conditions for the absence of q-number

Goto-Imamura-Schwinger (GIS) terms in the Gell-Mann relations (1).

In the following we will use the relations (1), (4), and (5) only for the special test function $h(\vec{x})\equiv 1$ (charge-density relations). However, the conditions (4) and (5), even in this special case, are in general valid only if no GIS terms occur in (1). Suppose we add on the right-hand side of (1) such a term:

$$\langle\Psi|S^{0\nu[t]}_{\alpha\beta}(0)_{ab}|\Phi\rangle\frac{\partial}{\partial z^t}h(\vec{z})\Big|_{\vec{z}=0},$$

with

$$S^{0\nu[t]}_{\alpha\beta}(x)_{ab}=[\delta_{ab}(\delta^{cV}-\delta^{cA})+\delta^{cA}]d_{\alpha\rho}{}^{\gamma}g^{\nu t}\partial_\kappa J^\kappa_\gamma(x)_c$$
$$+[\delta_{ab}(\delta^{cA}-\delta^{cV})+\delta^{cV}]\{f_{\alpha\beta}{}^{\gamma}\epsilon^{0\nu t}{}_{\kappa}\bar{J}^\kappa_\gamma(x)_c+\tfrac{1}{2}e_{\alpha\beta}{}^{\gamma}(1+g^{\nu0})\epsilon^{\nu t}{}_{\mu\kappa}[\partial^\mu A^\kappa_\gamma(x)_c-\partial^\kappa A^\mu_\gamma(x)_c]\},\qquad(6)$$

where $\bar{J}^\mu_\gamma(x)_c$ and $A^\mu_\gamma(x)_c$ are local operators, $\epsilon^{\mu\nu\kappa\lambda}$ is the complete antisymmetric tensor, and the constants e, f, g have the symmetry properties

$$e_{\alpha\beta}{}^{\gamma}=e_{\beta\alpha}{}^{\gamma},\quad f_{\alpha\beta}{}^{\gamma}=-f_{\beta\alpha}{}^{\gamma},\quad d_{\alpha\beta}{}^{\gamma}=-d_{\beta\alpha}{}^{\gamma}.\qquad(7)$$

Then we get, in the same way as in Ref. 2,

$$\lim_{\delta\to0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[\partial_\mu j^\mu_\alpha(x)_a\,;j^0_\beta(0)_b]|\Phi\rangle^T=2[\delta_{ab}(\delta^{cV}-\delta^{cA})+\delta^{cA}]d_{\alpha\beta}{}^{\gamma}\langle\Psi|\partial_\kappa J^\kappa_\gamma(0)_c|\Phi\rangle\frac{\partial}{\partial z_k}h(\vec{z})\Big|_{\vec{z}=0}\qquad(8)$$

and

$$\lim_{\delta\to0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[j^0_\alpha(x)_a\,;\partial_\nu j^\nu_\beta(0)_b]|\Phi\rangle^T$$
$$=-2[\delta_{ab}(\delta^{cV}-\delta^{cA})+\delta^{cA}]\,d_{\alpha\beta}{}^{\gamma}\left(\langle\Psi|\partial^k\partial_\kappa J^\kappa_\gamma(0)_c|\Phi\rangle h(\vec{0})+\langle\Psi|\partial_\kappa J^\kappa_\gamma(0)_c|\Phi\rangle\frac{\partial}{\partial z_k}h(\vec{z})\Big|_{\vec{z}=0}\right).\qquad(9)$$

Obviously, for the special case $h(\vec{x})\equiv1$, the condition (4) holds, but (5) does not if GIS terms of the type occur in (1).

Furthermore, this example shows that the relations (4) and (5) are not sufficient conditions for the nonoccurrence of GIS terms, since the second and third term of (6) do not show up any more in (8) or (9).

In Secs. II and III we derive from Eqs. (1), (4), and (5) [taken for one vector – and one axial-vector – current, external one-nucleon states, and $h(\vec{x})\equiv1$] and the PCAC assumption in the form of smoothness of matrix elements in the meson-mass variables, the following sum rules between the four weak isospin form factors:

$$\left(\frac{d^n}{dt^n}-\frac{n}{m_\rho^2}\frac{d^{n-1}}{dt^{n-1}}\right)\left(G_A(0)[F^V_e(t)+2\mu^VF^V_m(t)]-(1+2\mu^V)G_A(t)\right)\Big|_{t=0}$$
$$+\frac{n2\mu^V}{m^2}\left\{\left[1-\left(\frac{m}{m_\rho}\right)^2\right]\frac{d^{n-1}}{dt^{n-1}}G_A(t)-\frac{m^2}{2M}\left(\frac{d^{n-1}}{dt^{n-1}}-\frac{n-1}{m^2}\frac{d^{n-2}}{dt^{n-2}}\right)F_P(t)\right\}\Big|_{t=0}\approx0,\qquad(10)$$

where $F^V_e(t)$ and $F^V_m(t)$ are the isovector parts of the electric and magnetic form factors, $G_A(t)$ is the axial-vector form factor, $F_P(t)$ is the induced pseudoscalar form factor, and finally μ^V is half the difference of the anomalous magnetic moments of the proton and neutron:

$$\mu^V=\tfrac{1}{2}(\mu_p-\mu_n)=1.85.$$

M is the nucleon mass, m the pion mass, and m_ρ the ρ-meson mass. The normalizations are

$$F^V_e(0)=F^V_m(0)=1,\quad G_A(0)=1.2,\quad n=1,2,\ldots,n_{\max}.$$

The number $n_{\max}$ in (10) depends on the smoothness of certain matrix elements in the meson-mass variables. Roughly, it is the number of derivatives with respect to the meson masses being sufficiently small at zero momentum transfer (see end of Sec. III).

In Sec. II we derive from Eqs. (1), (4), and (5) the corresponding Low equations. In Sec. III we apply dispersion theory and PCAC to get our final result.

II. LOW EQUATIONS

Our next step is to replace the generalized charges, respectively, their first moments in (1), (4), and (5) by four-dimensional integrals over the corresponding current divergences or, more generally, to replace the zeroth components of the currents as far as possible by current divergences. In general, this is done by means of Gauss's theorem.[11,12] In the present simple cases of charge moments of at most order one, we can considerably simplify this procedure in the form of a division problem in momentum space.

We consider matrix elements between one-nucleon states $|M, \vec{k}, \delta\rangle$ of mass M, four-momentum k, and internal quantum numbers (spin, isospin) δ. In short we denote these states often by $|k\rangle$.

Furthermore, let us introduce the notation

$$\begin{aligned} F^{\mu\nu}_{j_\alpha j_\beta}(x) &= \langle k_1 | [j^\mu_\alpha(\tfrac{1}{2}x); j^\nu_\beta(-\tfrac{1}{2}x)] | k_2\rangle^T \\ &= e^{-i\Delta x}\langle k_1 | [j^\mu_\alpha(x); j^\nu_\beta(0)] | k_2\rangle^T, \\ F^{\nu}_{D_\alpha j_\beta}(x) &= \langle k_1 | [\partial_\mu j^\mu_\alpha(\tfrac{1}{2}x); j^\nu_\beta(-\tfrac{1}{2}x)] | k_2\rangle^T, \text{ etc.,} \end{aligned} \tag{11}$$

with

$$P = \tfrac{1}{2}(k_1 + k_2), \quad \Delta = \tfrac{1}{2}(k_1 - k_2). \tag{12}$$

The Fourier transforms of these (generalized) functions,

$$\tilde{F} \because\because (q) = \frac{1}{(2\pi)^{5/2}} \int d^4 q\, e^{iqx} F \because\because (x), \tag{13}$$

satisfy the following divergence relations:

$$\begin{aligned} \tilde{F}^{\nu}_{D_\alpha j_\beta}(q) &= -i(q-\Delta)_\mu \tilde{F}^{\mu\nu}_{j_\alpha j_\beta}(q), \\ \tilde{F}^{\mu}_{j_\alpha D_\beta}(q) &= i(q+\Delta)_\nu \tilde{F}^{\mu\nu}_{j_\alpha j_\beta}(q), \\ \tilde{F}_{D_\alpha D_\beta}(q) &= -i(q-\Delta)_\mu \tilde{F}^{\mu}_{j_\alpha D_\beta}(q) \\ &= i(q+\Delta)_\nu \tilde{F}^{\nu}_{D_\alpha j_\beta}(q) \\ &= (q-\Delta)_\mu (q+\Delta)_\nu \tilde{F}^{\mu\nu}_{j_\alpha j_\beta}(q). \end{aligned} \tag{14}$$

Let us first consider the left-hand side of (1) for the special test function $h(\vec{x}) \equiv 1$:

$$\langle k_1 | [j^0_\alpha(\varphi_\delta; 1); j^\nu_\beta(0)] | k_2\rangle^T = (2\pi)^{3/2} \int d\omega\, \tilde{F}^{0\nu}_{j_\alpha j_\beta}(\omega + \Delta^0, \vec{\Delta})\tilde{\varphi}_\delta(\omega), \tag{15}$$

with

$$\tilde{\varphi}_\delta(\omega) = \int dx^0 e^{-i\omega x^0}\varphi_\delta(x^0). \tag{16}$$

We want to use the divergence equations (14) to replace j^0 by $\partial_\mu j^\mu$:

$$\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega + \Delta^0, \vec{\Delta}) = -i\omega \tilde{F}^{0\nu}_{j_\alpha j_\beta}(\omega + \Delta^0, \vec{\Delta}). \tag{17}$$

For the *unique* solution of this equation (division by ω) we need the exact behavior of all functions in the neighborhood of $\omega = 0$. This can be obtained from the spectrum of possible intermediate states in the commutator matrix elements.

Besides the discrete one-nucleon states we have a continuous spectrum of many-particle states starting with the two-particle pion-nucleon states. The support of the latter contributions is given by

$$\text{supp}\tilde{F} \because\because (q)_{II} \subseteq \{q\colon (q^0 \geq -p^0 + [(M+m)^2 + (\vec{p}+\vec{q})^2]^{1/2}) \cup (q^0 \leq p^0 - [(M+m)^2 - (\vec{p}-\vec{q})^2]^{1/2})\}. \tag{18}$$

Here m is the pion mass. At the point $q = \omega + \Delta$ we therefore have

$$\tilde{F} \because\because (\omega + \Delta^0, \vec{\Delta})_{II} = 0 \quad \text{for} \quad \omega \in \mathfrak{M} \tag{19}$$

with the region $\mathfrak{M}$ defined by

$$\mathfrak{M} = \{\omega\colon\ k^0_2 - [(M+m)^2 + \vec{k}_2^{\,2}]^{1/2} < \omega < k^0_1 + [(M+m)^2 + \vec{k}_1^{\,2}]^{1/2}, \ \ k^0_i = \omega(\vec{k}_i) = [M^2 + \vec{k}_i^{\,2}]^{1/2}\}. \tag{20}$$

The essential point is that there always exists a finite circle of $\vec{\Delta} = 0$ with $\mathfrak{M} \neq \varnothing$. In the following we restrict $\vec{\Delta}$ to such a circle without mentioning it.

Next we calculate the one-nucleon contributions to the intermediate spectrum:

$$\tilde{F}^{\mu\nu}_{\cdot\cdot\cdot}(q)_{\rm I} = (2\pi)^{3/2}\sum_{\gamma}[\Theta(p^0+q^0)\delta((p+q)^2-M^2)\langle k_1|j^\mu_\alpha(0)|M,\vec{p}+\vec{q},\gamma\rangle\langle\gamma,\vec{p}+\vec{q},M|j^\nu_\beta(0)|k_2\rangle$$
$$-\Theta(p^0-q^0)\delta((p-q)^2-M^2)\langle k_1|j^\nu_\beta(0)|M,\vec{p}-\vec{q},\gamma\rangle\langle\gamma,\vec{p}-\vec{q},M|j^\mu_\alpha(0)|k_2\rangle]. \tag{21}$$

Now, it easily follows from (14) and (19)–(21) together with $\omega\delta(\omega)=0$ that for $\omega\in\mathfrak{M}$

$$\tilde{F}^{\mu\nu}_{\cdot\cdot\cdot}(\omega+\Delta^0,\vec{\Delta}) = (2\pi)^{3/2}\delta(\omega)\sum_{\gamma}\left[\frac{1}{2\omega(k_1)}\langle k_1|j^\mu_\alpha(0)|k_1,\gamma\rangle\langle\gamma,k_1|j^\nu_\beta(0)|k_2\rangle - \frac{1}{2\omega(k_2)}\langle k_1|j^\nu_\beta(0)|k_2,\gamma\rangle\langle\gamma_1 k_2|j^\mu_\alpha(0)|k_2\rangle\right], \tag{22}$$

$$\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta}) = \tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta}) = 0. \tag{23}$$

In view of (22) and (23), Eq. (17) has the unique solution[13]

$$\tilde{F}^{0\nu}_{j_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta}) = \frac{i}{\omega}\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta}) + (2\pi)^{3/2}\delta(\omega)\sum_{\gamma}\left[\frac{1}{2\omega(k_1)}\langle k_1|j^0_\alpha(0)|k_1,\gamma\rangle\langle\gamma,k_1|j^\nu_\beta(0)|k_2\rangle - \frac{1}{2\omega(k_2)}\langle k_1|j^\nu_\beta(0)|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)|k_2\rangle\right]. \tag{24}$$

For the special case $\nu=0$ we can go one step further and divide once more. In this case we get, from the divergence equations (14),

$$\omega\tilde{F}^{0}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta}) = -i[\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta}) - i\,2\Delta_\nu\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta})]. \tag{25}$$

According to the boundary condition (23), the unique solution is

$$\tilde{F}^{0}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta}) = -\frac{i}{\omega}[\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta}) - i2\Delta_\nu\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta})]. \tag{26}$$

Inserting (15), (24), and (26) into the commutation relations (1), we get the Low equations

$$i(2\pi)^{3/2}\lim_{\delta\to 0}\int\frac{d\omega}{\omega}\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta})_{ab}\tilde{\varphi}_\delta(\omega)$$
$$+(2\pi)^3\sum_{\gamma}\left[\frac{1}{2\omega(k_1)}\langle k_1|j^0_\alpha(0)_a|k_1,\gamma\rangle\langle\gamma,k_1|j^\nu_\beta(0)_b|k_2\rangle - \frac{1}{2\omega(k_2)}\langle k_1|j^\nu_\beta(0)_b|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)_a|k_2\rangle\right]$$
$$=\epsilon_{[\alpha,\beta]}{}^{\gamma}\langle k_1|j^\nu_\gamma(0)_c|k_2\rangle; \tag{27}$$

or, for $\nu=0$,

$$(2\pi)^{3/2}\lim_{\delta\to 0}\int\frac{d\omega}{\omega^2}[\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})_{ab} - i2\Delta_\nu\tilde{F}^{\nu}_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta})_{ab}]\tilde{\varphi}_\delta(\omega)$$
$$+(2\pi)^3\sum_{\gamma}\left[\frac{1}{2\omega(k_1)}\langle k_1|j^0_\alpha(0)_a|k_1,\gamma\rangle\langle\gamma,k_1|j^0_\beta(0)_b|k_2\rangle - \frac{1}{2\omega(k_2)}\langle k_1|j^0_\beta(0)_b|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)_a|k_2\rangle\right]$$
$$=\epsilon_{[\alpha,\beta]}{}^{\gamma}\langle k_1|j^0_\gamma(0)_c|k_2\rangle. \tag{28}$$

We want to mention here that the explicit one-particle contributions in (27) and (28) are exactly the contributions of the infinite timelike surfaces in the usual application of Gauss's theorem.[11,12]

The derivation of the Low equations corresponding to the equal-time limits (4) and (5) is done along the same lines as above. However, there we will face a new problem. We have to give a unique definition for the product of the two distributions $1/\omega$ and $\tilde{F}(\omega)$ with the latter having point support in $\omega=0$. We will demonstrate this in the case of Eq. (5).

$$\int d^4x\,\varphi_\delta(x^0)x^l\langle k_1|[j^0_\alpha(x),\partial_\nu j^\nu_\beta(0)]|k_2\rangle^T = i(2\pi)^{3/2}\int d\omega\left[\frac{\partial}{\partial q^l}\tilde{F}^0_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\bigg|_{\vec{q}=\vec{\Delta}}\right]\tilde{\varphi}_\delta(\omega). \tag{29}$$

From the divergence equations (14), we obtain for $q=\omega+\Delta$

$$\omega\left[\frac{\partial}{\partial q^l}\tilde{F}^0_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\right]\bigg|_{\vec{q}=\vec{\Delta}} = i\left[\frac{\partial}{\partial q^l}\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\bigg|_{\vec{q}=\vec{\Delta}} - i\tilde{F}^l_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})\right]. \tag{30}$$

Again we need the behavior of the square brackets in the neighborhood of the origin as a boundary condition for the division by ω. From Eqs. (19)–(21), we get for $\omega \in \mathfrak{M}$

$$\frac{\partial}{\partial q^l}\tilde{F}^{\mu}_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\Big|_{\vec{q}=\vec{\Delta}} = (2\pi)^{3/2}\sum_\gamma \Big\{ \delta(\omega)\Big[\frac{\partial}{\partial q^l}\Big(\frac{1}{2\omega(\vec{q})}\langle k_1|j^\mu_\alpha(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\nu j^\nu_\beta(0)|k_2\rangle\Big)\Big|_{\vec{q}=\vec{k}_1}$$

$$+\frac{\partial}{\partial q^l}\Big(\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\nu j^\nu_\beta(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^\mu_\alpha(0)|k_2\rangle\Big)\Big|_{\vec{q}=\vec{k}_2}\Big]$$

$$-\frac{d}{d\omega}\delta(\omega)\Big(\frac{k_1^l}{2\omega(k_1)^2}\langle k_1|j^\mu_\alpha(0)|k_1,\gamma\rangle\langle\gamma,k_1|\partial_\gamma j^\nu_\beta(0)|k_2\rangle$$

$$-\frac{k_2^l}{2\omega(k_2)^2}\langle k_1|\partial_\nu j^\nu_\beta(0)|k_2\gamma\rangle\langle\gamma,k_2|j^\mu_\alpha(0)|k_2\rangle\Big)\Big\} \tag{31}$$

and, furthermore,

$$\Big(\frac{\partial}{\partial q^l}\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\Big|_{\vec{q}=\vec{\Delta}} - i\tilde{F}^l_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})\Big) = -i(2\pi)^{3/2}\delta(\omega)\sum_\gamma\Big(\frac{k_1^l}{2\omega(k_1)^2}\langle k_1|j^0_\alpha(0)|k_1,\gamma\rangle\langle\gamma,k_1|\partial_\nu j^\nu_\beta(0)|k_2\rangle$$

$$-\frac{k_2^l}{2\omega(k_2)^2}\langle k_1|\partial_\nu j^\nu_\beta(0)|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)|k_2\rangle\Big). \tag{32}$$

In contrast to the case before, the inhomogeneous part of Eq. (30) does not vanish at $\omega=0$ but has a δ singularity at this point. Therefore, we have to define the product with $1/\omega$ in such a way that the result is zero for $\omega\in\mathfrak{M}$. This is achieved by the definition

$$\frac{1}{\omega}\circ F(\omega)=\tfrac{1}{2}\lim_{\epsilon\to 0}\Big[\Big(\frac{1}{\omega+\epsilon}+\frac{1}{\omega-\epsilon}\Big)F(\omega)\Big], \tag{33}$$

with the prescription that the limit as $\epsilon\to 0$ has to be taken at the very end of all manipulations; for instance,

$$\frac{1}{\omega}\circ\delta(\omega)=\tfrac{1}{2}\lim_{\epsilon\to 0}\Big(\frac{1}{\epsilon}-\frac{1}{\epsilon}\Big)=0.$$

This prescription should be kept in mind throughout the whole paper. It becomes very significant in the calculation of the one-nucleon contributions to the dispersion integrals in Sec. III.

Now, the unique solution of Eq. (30) with the boundary condition (31) is

$$\frac{\partial}{\partial q^l}\tilde{F}^0_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\Big|_{\vec{q}=\vec{\Delta}} = i\frac{1}{\omega}\circ\Big(\frac{\partial}{\partial q^l}\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\Big|_{\vec{q}=\vec{\Delta}} - i\tilde{F}^l_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})\Big)$$

$$+(2\pi)^{3/2}\sum_\gamma\Big\{\delta(\omega)\Big[\frac{\partial}{\partial q^l}\Big(\frac{1}{2\omega(\vec{q})}\langle k_1|j^0_\alpha(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\nu j^\nu_\beta(0)|k_2\rangle\Big)\Big|_{\vec{q}=\vec{k}_1}$$

$$+\frac{\partial}{\partial q^l}\Big(\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\nu j^\nu_\beta(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^0_\alpha(0)|k_2\rangle\Big)\Big|_{\vec{q}=\vec{k}_2}\Big]$$

$$-\frac{d}{d\omega}\delta(\omega)\Big[\frac{k_1^l}{2\omega(k_1)^2}\langle k_1|j^0_\alpha(0)|k_1,\gamma\rangle\langle\gamma,k_1|\partial_\nu j^\nu_\beta(0)|k_2\rangle$$

$$-\frac{k_2^l}{2\omega(\vec{k}_2)^2}\langle k_1|\partial_\nu j^\nu_\beta(0)k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)|k_2\rangle\Big]\Big\}. \tag{34}$$

Inserting Eqs. (29) and (34) into Eq. (5) we get, by means of $\lim_{\epsilon\to 0}\tilde{\varphi}_\epsilon(\omega)=1$, the Low equation:

$$-(2\pi)^{3/2}\lim_{\delta\to 0}\int d\omega\,\frac{1}{\omega}\circ\left(\frac{\partial}{\partial q^l}\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})\Big|_{\vec{q}=\vec{\Delta}}-i\tilde{F}^l_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})\right)\tilde{\varphi}_\delta(\omega)$$
$$+i(2\pi)^3\sum_\gamma\left[\frac{\partial}{\partial q^l}\left(\frac{1}{2\omega(\vec{q})}\langle k_1|j^0_\alpha(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\nu j^\nu_\beta(0)|k_2\rangle\right)\Big|_{\vec{q}=\vec{k}_1}+\frac{\partial}{\partial q^l}\left(\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\nu j^\nu_\beta(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^0_\alpha(0)|k_2\rangle\right)\Big|_{\vec{q}=\vec{k}_2}\right]=0. \quad (35)$$

In the same way we obtain the Low equation corresponding to (4):

$$\lim_{\delta\to 0}(2\pi)^{3/2}\int d\omega\frac{1}{\omega}\circ\left(i\tilde{F}^l_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{\Delta})+\frac{\partial}{\partial q^l}\left[\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})-i2\Delta_\nu\tilde{F}^\nu_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{q})\right]\Big|_{\vec{q}=\vec{\Delta}}\right)\tilde{\varphi}_\delta(\omega)$$
$$+i(2\pi)^3\sum_\gamma\left[\frac{\partial}{\partial q^l}\left(\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\mu j^\mu_\alpha(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^0_\beta(0)|k_2\rangle\right)\Big|_{\vec{q}=\vec{k}_1}+\frac{\partial}{\partial q^l}\left(\frac{1}{2\omega(\vec{q})}\langle k_1|j^0_\beta(0)|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\mu j^\mu_\alpha(0)|k_2\rangle\right)\Big|_{\vec{q}=\vec{k}_2}\right]=0. \quad (36)$$

Instead of this equation we will use in the following the difference between (36) and (27) for $\nu=l$.

III. DISPERSION THEORY AND PCAC

Our first task in this section is to transform the Low equations of Sec. II into covariant dispersion relations. We define smooth retarded and advanced commutators $\tilde{F}^{\cdots}_{\cdots}(q,\varphi_\delta)^\pm$ by

$$\tilde{F}^{\cdots}_{\cdots}(q,\varphi_\delta)^\pm=-\frac{1}{2\pi}\int\frac{d\omega}{\omega-q^0\mp i\epsilon}\tilde{F}^{\cdots}_{\cdots}(\omega,\vec{q})\tilde{\varphi}_\delta(\omega-q^0)$$
$$=\mp\frac{i}{(2\pi)^{5/2}}\int d^4x\int dy^0e^{iqx}\Theta(\pm y^0)\varphi_\delta(x^0-y^0)F^{\cdots}_{\cdots}(x), \quad (37)$$

$$\tilde{F}^{\cdots}_{\cdots}(q,\varphi_\delta)^+-\tilde{F}^{\cdots}_{\cdots}(q,\varphi_\delta)^-=-i\tilde{F}^{\cdots}_{\cdots}(q). \quad (38)$$

Furthermore, we make the technical assumption[14] that the limit $\delta\to 0$ of (37) exists uniformly in q from a neighborhood of $q=\Delta$:

$$\lim_{\delta\to 0}\tilde{F}^{\cdots}_{\cdots}(q,\varphi_\delta)^\pm=\tilde{F}^{\cdots}_{\cdots}(q)^\pm. \quad (39)$$

Our main demand, the absence of q-number GIS terms in the equal-time commutators, implies that these limits define Lorentz-covariant sharp retarded and advanced commutators (see Appendix).

With the shorthand notation,

$$\tilde{F}^{\cdots}_{\cdots}(q)^P=\tfrac{1}{2}i[\tilde{F}^{\cdots}_{\cdots}(q)^++\tilde{F}^{\cdots}_{\cdots}(q)^-], \quad (40)$$

the Low equations (27), (28), and (35) read

$$-(2\pi)^{5/2}\lim_{\omega\to 0}\tilde{F}^\nu_{D_\alpha j_\beta}(\omega+\Delta)^P_{ab}+(2\pi)^3\sum_\gamma\left(\frac{1}{2\omega(\vec{k}_1)}\langle k_1|j^0_\alpha(0)_a|k_1,\gamma\rangle\langle\gamma,k_1|j^\nu_\beta(0)_b|k_2\rangle-\frac{1}{2\omega(\vec{k}_2)}\langle k_1|j^\nu_\beta(0)_b|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)_a|k_2\rangle\right)=\epsilon_{[\alpha,\beta]}{}^\gamma\langle k_1|j^\nu_\gamma(0)_c|k_2\rangle, \quad (41)$$

$$i(2\pi)^{5/2}\lim_{\omega\to 0}\frac{\partial}{\partial\omega}[\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta)^P_{ab}-2i\Delta_\nu\tilde{F}^\nu_{D_\alpha j_\beta}(\omega+\Delta)^P_{ab}]+(2\pi)^3\sum_\gamma\left(\frac{1}{2\omega(k_1)}\langle k_1|j^0_\alpha(0)_a|k_1,\gamma\rangle\langle\gamma,k_1|j^0_\beta(0)_b|k_2\rangle-\frac{1}{2\omega(k_2)}\langle k_1|j^0_\beta(0)_b|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)_a|k_2\rangle\right)=\epsilon_{[\alpha,\beta]}{}^\gamma\langle k_1|j^0_\gamma(0)_c|k_2\rangle, \quad (42)$$

$$-i(2\pi)^{5/2}\,{}^{\circ}\lim_{\omega\to 0}\left[\frac{\partial}{\partial q^l}\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})^P\bigg|_{\vec{q}=\vec{\Delta}} - i\tilde{F}^l_{j_\alpha D_\beta}(\omega+\Delta^0,\vec{\Delta})^P\right]_{ab}$$
$$+i(2\pi)^3\sum_\gamma\left\{\frac{\partial}{\partial q^l}\left[\frac{1}{2\omega(\vec{q})}\langle k_1|j^0_\alpha(0)_a|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\nu j^\nu_\beta(0)_b|k_2\rangle\right]\bigg|_{\vec{q}=\vec{k}_1}\right.$$
$$\left.+\frac{\partial}{\partial q^l}\left[\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\nu j^\nu_\beta(0)_b|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^0_\alpha(0)_a|k_2\rangle\right]\bigg|_{\vec{q}=\vec{k}_2}\right\}=0. \tag{43}$$

Finally, the difference between Eqs. (36) and (27) for $\nu=l$ reads

$$i(2\pi)^{5/2}\,{}^{\circ}\lim_{\omega\to 0}\frac{\partial}{\partial q^l}\left\{\tilde{F}_{D_\alpha D_\beta}(\omega+\Delta^0,\vec{q})^P_{ab}-2i\Delta_\nu\tilde{F}^\nu_{D_\alpha j_\beta}(\omega+\Delta^0,\vec{q})^P_{ab}\right\}\bigg|_{\vec{q}=\vec{\Delta}}$$
$$+i(2\pi)^3\sum_\gamma\left\{\frac{\partial}{\partial q^l}\left[\frac{1}{2\omega(\vec{q})}\langle k_1|\partial_\mu j^\mu_\alpha(0)_a|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|j^0_\beta(0)_b|k_2\rangle\right]\bigg|_{\vec{q}=\vec{k}_1}\right.$$
$$\left.+\frac{\partial}{\partial q^l}\left[\frac{1}{2\omega(\vec{q})}\langle k_1|j^0_\beta(0)_b|M,\vec{q},\gamma\rangle\langle\gamma,\vec{q},M|\partial_\mu j^\mu_\alpha(0)_a|k_2\rangle\right]\bigg|_{\vec{q}=\vec{k}_2}\right\}$$
$$+(2\pi)^3\sum_\gamma\left\{\frac{1}{2\omega(\vec{k}_1)}\langle k_1|j^0_\alpha(0)_a|k_1,\gamma\rangle\langle\gamma,k_1|j^l_\beta(0)_b|k_2\rangle\right.$$
$$\left.-\frac{1}{2\omega(\vec{k}_2)}\langle k_1|j^l_\beta(0)_b|k_2,\gamma\rangle\langle\gamma,k_2|j^0_\alpha(0)_a|k_2\rangle\right\}=\epsilon_{[\alpha,\beta]}{}^\gamma\langle k_1|j^l_\gamma(0)_c|k_2\rangle. \tag{44}$$

In the last two relations the symbol $\circ$ indicates the limit prescription introduced in Eq. (33):

$${}^{\circ}\lim_{\omega\to 0}\tilde{F}(\omega+\Delta)^P=\tfrac{1}{2}\lim_{\omega\to 0}[\tilde{F}(\omega+\Delta)^P+\tilde{F}(-\omega+\Delta)^P]. \tag{45}$$

We decompose the (retarded/advanced) matrix elements into invariant functions. In this decomposition we write explicitly, besides the invariant energy $q\cdot p$, the invariant momentum transfer $t=4\Delta^2$, also the (off-mass-shell) boson mass variables $\mu=(q-\Delta)^2$ and $\xi=(q+\Delta)^2$:

$$\tilde{F}^\nu_{D_\alpha j_\beta}(q)^{(\pm)}_5=i\sum_{r=1}^4\bar{u}^{\sigma_1}(k_1)\gamma^5T^\nu_r(q)(b^r_{D_\alpha j_\beta}+\not{q}\,\bar{b}^r_{D_\alpha j_\beta})((q-\Delta)^2,(q+\Delta)^2,t\,|\,q\cdot p)^{(\pm)}_5u^{\sigma_2}(k_2), \tag{46}$$

with

$$T^\nu_1(q)=\gamma^\nu,\quad T^\nu_2(q)=\Delta^\nu,\quad T^\nu_3(q)=q^\nu,\quad T^\nu_4(q)=P^\nu \tag{47}$$

The commutator $\tilde{F}^\mu_{j_\alpha D_\beta}(q)_5$ can be described by the same invariants as in (47) by means of the commutator relation:

$$\tilde{F}^\mu_{j_\alpha D_\beta}(q)_5=-\tilde{F}^\mu_{D_\beta j_\alpha}(-q)_5. \tag{48}$$

For the sake of technical simplicity, we assume unsubtracted dispersion relations in $\nu=qp$ for the invariant functions. The results do not depend on this assumption. They remain true if we admit an arbitrary finite number of subtractions, since the PCAC assumption is made anyway for the real parts of the amplitudes.

$$b\vdots\vdots(\mu,\xi,t\,|\,q\cdot p)^{(\pm)}_5=-\frac{1}{2\pi}\int\frac{d\nu}{\nu-(q\cdot p\pm i\epsilon)}\,b\vdots\vdots(\mu,\xi,t\,|\,\nu)_5, \tag{49}$$

with a similar expression for $\bar{b}\vdots\vdots(\mu,\xi,t|q\cdot p)^{(\pm)}_5$ in terms of $\bar{b}\vdots\vdots(\mu,\xi,t\,|\,\nu)_5$. We split the invariant functions $b(\cdots)_5$ $[\bar{b}(\cdots)_5]$ into the one-nucleon contribution $b(\cdots)^{\rm I}_5$ $[\bar{b}(\cdots)^{\rm I}_5]$ and the many-particle remainder $b(\cdots)^{\rm II}_5$ $[\bar{b}(\cdots)^{\rm II}_5]$.

The one-nucleon part can be calculated from Eq. (21). With the decomposition of the vertices into invariant form factors,[15]

$$\langle k_1|j^\mu_\alpha(0)_V|k_2\rangle=\bar{u}^{\sigma_1}(k_1)[\gamma^\mu G^1_\alpha(t)+P^\mu G^2_\alpha(t)]\,u^{\sigma_2}(k_2), \tag{50}$$

$$\langle k_1|j^\mu_\alpha(0)_A|k_2\rangle=\bar{u}^{\sigma_1}(k_1)\gamma^5[\gamma^\mu G^1_\alpha(t)_5+\Delta^\mu G^2_\alpha(t)_5]\,u^{\sigma_2}(k_2), \tag{51}$$

and the abbreviations

$$F_\alpha(t) = G^1_\alpha(t) + MG^2_\alpha(t)\,, \tag{52}$$

$$F^5_\alpha(t) = 2MG^1_\alpha(t)_5 - \tfrac{1}{2}tG^2_\alpha(t)_5\,, \tag{53}$$

we get[15]

$$b_{D_\alpha j_\beta}(\mu,\,\xi,\,t\,|\,\nu)^{\rm I}_5 = -(2\pi)^{3/2}\sum_{n,s=1}^{2}\big[\Theta(p^0+q^0)\delta(2\nu-\tfrac{1}{2}(t-\mu-\xi))c^r_{ns}(\mu)G^n_\alpha(\mu)_5G^s_\beta(\xi) - \Theta(p^0-q^0)\delta(2\nu+\tfrac{1}{2}(t-\mu-\xi))\,d^r_{ns}(\mu)G^n_\beta(\xi)G^s_\alpha(\mu)_5\big]\,, \tag{54}$$

and a similar expression for $\bar b_{D_\alpha j_\beta}$ in terms of $\bar c^r_{ns}$ and $\bar d^r_{ns}$.

The invariant coefficients c, $\bar c$, d, and $\bar d$ are given in Table I.

Inserting Eqs. (46), (47), (50), and (54) into (41)–(44) and performing the necessary differentiations and taking the necessary limits, we obtain the sum rules (the numbers in the square brackets indicate the Low equations from which they follow):

$$\begin{aligned}-(2\pi)^{3/2}\bar u^{\sigma_1}(k_1)\gamma^5\int_{a(0,t)}^{\infty}\frac{d\nu}{\nu}\{[\gamma^\mu b^1+\Delta^\mu(\bar b^1+b^2+b^3-M(\bar b^2+\bar b^3))+P^\mu(\bar b^1+b^4-M\bar b^4)]_{D_\alpha j_\beta}(0,\,t,\,t\,|\,\nu)^{\rm II}_5-(\nu\to-\nu)\}u^{\sigma_2}(k_2)\\ +(2\pi)^3\bar u^{\sigma_1}(k_1)\gamma^5\{\gamma^\mu[G^1_\alpha(0)_5G^1_\beta(t)-G^1_\beta(t)G^1_\alpha(0)_5]+P^\mu[G^1_\alpha(0)_5G^2_\beta(t)+G^2_\beta(t)G^1_\alpha(0)_5]\}u^{\sigma_2}(k_2)\\ =\epsilon_{[\alpha,\beta]}{}^\gamma\bar u^{\sigma_1}(k_1)\gamma^5[\gamma^\mu G^1_\gamma(t)_5+\Delta^\mu G^2_\gamma(t)_5]u^{\sigma_2}(k_2),\end{aligned}$$

(55) [41]

$$\begin{aligned}2(2\pi)^{3/2}\bar u^{\sigma_1}(k_1)\,\gamma^5\Big\{\gamma^\nu\int_{a(0,t)}^{\infty}\frac{d\nu}{\nu}[\Delta^2(\bar b^2+\bar b^3)_{D_\alpha j_\beta}(0,\,t,\,t\,|\,\nu)^{\rm II}_5-(\nu\to-\nu)]\\ -P^\nu M\int_{a(0,t)}^{\infty}\frac{d\nu}{\nu^2}\Big[\Big(b^1+\frac{\nu}{M}\bar b^1-\frac{\Delta^2}{M}[\bar b^1+b^2+b^3-M(\bar b^2+\bar b^3)]\Big)_{D_\alpha j_\beta}(0,\,t,\,t\,|\,\nu)^{\rm II}_5+(\nu\to-\nu)\Big]\\ +\Delta^\nu\Big[\int_{a(0,t)}^{\infty}\frac{d\nu}{\nu}[(\bar b^1+b^3-M\bar b^3)_{D_\alpha j_\beta}(0,\,t,\,t\,|\,\nu)^{\rm II}_5-(\nu\to-\nu)]\\ -4\frac{\partial}{\partial\xi}\int_{a(0,\xi)}^{\infty}\frac{d\nu}{\nu}\{[Mb^1-\Delta^2(\bar b^1+b^2+b^3-M(\bar b^2+\bar b^3))]_{D_\alpha j_\beta}(0,\,\xi,\,t\,|\,\nu)^{\rm II}_5|_{\xi=t}-(\nu\to-\nu)\}\Big]\Big\}u^{\sigma_2}(k_2)\\ +(2\pi)^3\bar u^{\sigma_1}(k_1)\gamma^5\Big\{\gamma^\nu[G^1_\alpha(0)_5F_\beta(t)-F_\beta(t)G^1_\alpha(0)_5]+P^\nu[G^1_\alpha(0)_5G^2_\beta(t)+G^2_\beta(t)G^1_\alpha(0)_5]\\ +\Delta^\nu\times 8M\frac{d}{dt}[G^1_\alpha(0)_5G^1_\beta(t)-G^1_\beta(t)G^1_\alpha(0)_5]\Big\}u^{\sigma_2}(k_2)=\epsilon_{[\alpha,\beta]}{}^\gamma\bar u^{\sigma_1}(k_1)\gamma^5[\gamma^\nu G^1_\gamma(t)_5+\Delta^\nu G^2_\gamma(t)_5]u^{\sigma_2}(k_2),\end{aligned}$$

(56) [42, 44]

$$\begin{aligned}(2\pi)^{3/2}\bar u^{\sigma_1}(k_1)\gamma^5\int_{a(0,t)}^{\infty}\frac{d\nu}{\nu}\{[\gamma^i b^1+P^i(b^4-\bar b^1+M\bar b^4)+\Delta^i(b^2-b^3-\bar b^1+M(\bar b^2-\bar b^3))]_{D_\beta j_\alpha}(t,\,0,\,t\,|-\nu)^{\rm II}_5-(\nu\to-\nu)\}u^{\sigma_2}(k_2)\\ +(2\pi)^3\bar u^{\sigma_1}(k_1)\gamma^5\Big[\tfrac{1}{2}\gamma^i[G^2_\alpha(0)F^5_\beta(t)-F^5_\beta(t)G^2_\alpha(0)]-4\Delta^i\Big(F_\alpha(0)\frac{d}{dt}F^5_\beta(t)-\frac{d}{dt}F^5_\beta(t)F_\alpha(0)\Big)\Big]u^{\sigma_2}(k_2)=0.\end{aligned}$$

(57) [43]

$a(\mu,\,\xi)$ is determined by the pion-nucleon threshold of intermediate states in the commutator

$$a(\mu,\,\xi)=\tfrac{1}{2}[(M+m)^2-M^2-\tfrac{1}{2}(\mu+\xi-t)]\,. \tag{58}$$

The invariant functions appearing in the relations (55)–(57) are not independent of each other. They are connected by the divergence condition (14),

$$(q+\Delta)_\nu\tilde F^\nu_{D_\alpha j_\beta}(q)_5=0,$$

which leads to the two conditions

$$[-Mb^1+\tfrac{1}{4}(\mu+3\xi-t)(\bar{b}^1+b^3)+\tfrac{1}{4}(\xi-\mu+t)b^2+\nu(b^4-\bar{b}^1)]_{D_\alpha j_\beta}(\mu,\ \xi,\ t\,|\,\nu)_5=0, \tag{59}$$

$$[b^1+\tfrac{1}{4}(\xi-\mu+t)\bar{b}^2+\tfrac{1}{4}(\mu+3\xi-t)\bar{b}^3+\nu\bar{b}^4]_{D_\alpha j_\beta}(\mu,\ \xi,\ t\,|\,\nu)_5=0, \tag{60}$$

with the following solutions in terms of six independent functions:

$$\begin{aligned}
&b^1_{D_\alpha j_\beta}(\mu,\ \xi,\ t\,|\,\nu)_5=\nu E^1_{D_\alpha j_\beta}(\mu,\ \xi,\ t\,|\,\nu)_5,\quad \bar{b}^1=-E^4,\\
&b^2=\nu E^2,\quad \bar{b}^2=\nu E^5,\\
&b^3=\nu E^3+E^4,\quad \bar{b}^3=\nu E^6,\\
&b^4=ME^1-E^4-\tfrac{1}{4}(\xi-\mu+t)E^2-\tfrac{1}{4}(3\xi+\mu-t)E^3,\qquad \bar{b}^4=-E^1-\tfrac{1}{4}(\xi-\mu+t)E^5-\tfrac{1}{4}(3\xi+\mu-t)E^6.
\end{aligned} \tag{61}$$

Next we specialize the currents to the non-Hermitian combinations:

$$\begin{aligned}
&j^\mu_{+1}(x)_{(5)}=:-(1/\sqrt{2}\,)[\,j^\mu_1(x)_{(5)}+ij^\mu_2(x)_{(5)}\,],\\
&j^\mu_0(x)_{(5)}=:j^\mu_3(x)_{(5)}\,,\\
&j^\mu_{-1}(x)_{(5)}=:(1/\sqrt{2}\,)[\,j^\mu_1(x)_{(5)}-ij^\mu_2(x)_{(5)}\,],
\end{aligned} \tag{62}$$

$$\begin{aligned}
&j^\mu_r(x)^*_{(5)}=j^\mu_r(x)_{(5)}\quad (r=1,\ 2,\ 3),\\
&j^\mu_\alpha(x)^*_{(5)}=(-1)^\alpha j^\mu_{-\alpha}(x)_{(5)}\quad (\alpha=+1,\ 0,\ -1).
\end{aligned} \tag{63}$$

The isospin decomposition of the invariant functions E^i and the form factors is defined by

$$G^i_\alpha(t)_{(5)}=\chi(\delta_1)^T\tau_\alpha\chi(\delta_2)G^i(t)_{(5)}\,, \tag{64}$$

$$E^i_{D_\alpha j_\beta}(\cdots)_5=\chi(\delta_1)^T\{[\tau_\alpha;\tau_\beta]\,E^{i-}_{Dj}(\cdots)_5+\{\tau_\alpha;\tau_\beta\}E^{i+}_{Dj}(\cdots)_5\}\chi(\delta_2)\,. \tag{65}$$

Here $\chi(\delta)$ are the isospin spinors of the nucleons and τ_α are again the isospin-lowering and -raising combinations (62) of the Pauli matrices:

$$\sigma_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad [\tau_\alpha;\tau_\beta]=\epsilon_{[\alpha,\beta]}{}^\gamma\tau_\gamma\,, \tag{66}$$

with

$$\epsilon_{[\alpha,\beta]}{}^\gamma=-\epsilon_{[\beta,\alpha]}{}^\gamma,\quad \epsilon_{[-1,1]}{}^0=\epsilon_{[-1,0]}{}^{-1}=\epsilon_{[0,+1]}{}^1=1;$$

all other components are zero.

TABLE I. The invariant coefficients in Eq. (54).

	n	s \ r	1	2	3	4
c^r_{ns}	1	1	0	$-2M$	$4M$	$2M$
	1	2	0	$-M^2$	M^2	$2M^2$
	2	1	0	$\frac{1}{2}\mu$	$-\mu$	$-\frac{1}{2}\mu$
	2	2	0	$\frac{1}{4}\mu M$	$-\frac{1}{4}\mu M$	$-\frac{1}{2}\mu M$
$\bar{c}^r_{ns}$	1	1	$-2M$	0	0	0
	1	2	0	$-M$	M	$2M$
	2	1	$\frac{1}{2}\mu$	0	0	0
	2	2	0	$\frac{1}{4}\mu$	$-\frac{1}{4}\mu$	$-\frac{1}{2}\mu$
d^r_{ns}	1	1	0	$2M$	0	$2M$
	1	2	0	$-\frac{1}{2}\mu$	0	$-\frac{1}{2}\mu$
	2	1	0	M^2	$-M^2$	$2M^2$
	2	2	0	$-\frac{1}{4}\mu M$	$\frac{1}{4}\mu M$	$-\frac{1}{2}\mu M$
$\bar{d}^r_{ns}$	1	1	$-2M$	0	0	0
	1	2	$\frac{1}{2}\mu$	0	0	0
	2	1	0	M	$-M$	$2M$
	2	2	0	$-\frac{1}{4}\mu$	$\frac{1}{4}\mu$	$-\frac{1}{2}\mu$

From the Hermiticity property, PT-conjugation symmetry, and C-conjugation symmetry,

$$Cj^{\mu}_{\alpha}(x)_{(5)}C^{-1}=\eta^{C}_{(5)}j^{\mu}_{\alpha}(x)^{*}_{(5)}, \quad \eta^{C}=-1, \quad \eta^{C}_{5}=1,$$

$$(PT)j^{\mu}_{\alpha}(x)_{(5)}(PT)^{-1}=-j^{\mu}_{\alpha}(-x)_{(5)},$$

where PT is an antiunitary operator, we find that all invariant functions and form factors are real. Moreover, the invariant functions obey the crossing relations

$$E^{i\mp}_{Dj}(\mu,\ \xi,\ t\,|\,\nu)_5=\pm E^{i\mp}_{Dj}(\mu,\ \xi,\ t\,|-\nu)_5, \quad i=1,\ldots,6. \tag{67}$$

Inserting Eqs. (61) and (64)–(67) into the relations (55)–(57) and introducing the new energy variable

$$s=2\nu+M^2+\tfrac{1}{2}(\mu+\xi-t) \tag{68}$$

and the new invariant functions

$$A(\mu,\ \xi,\ t\,|\,s)=E(\mu,\ \xi,\ t\,|\,\tfrac{1}{2}(s-M^2-\tfrac{1}{2}(\mu+\xi-t))), \tag{69}$$

we obtain the following six sum rules, valid in the common analyticity domain in t of all functions, which contains the origin $t=0$:

$$(2\pi)^{3/2}\int_{(M+m)^2}^{\infty}ds[A^{1-}_{Dj}(t,0,t\,|\,s)_5\mp A^{1-}_{Dj}(0,t,t\,|\,s)_5]+(2\pi)^3[\pm G^1(0)_5G^1(t)+MG^2(0)G^1(t)_5-\tfrac{1}{4}tG^2(0)G^2(t)_5]=\pm G^1(t)_5,$$

(70) [(55) ± (57)]

$$-(2\pi)^{3/2}\int_{(M+m)^2}^{\infty}ds[A^{2-}+A^{3-}-M(A^{5-}+A^{6-})]_{Dj}(0,t,t\,|\,s)_5=G^2(t)_5,$$

(71) [55]

$$\tfrac{1}{2}t(2\pi)^{3/2}\int_{(M+m)^2}^{\infty}ds(A^{5-}+A^{6-})_{Dj}(0,t,t\,|\,s)_5+(2\pi)^3G^1(0)_5F(t)=G^1(t)_5,$$

(72) [56]

$$-4(2\pi)^{3/2}\int_{(M+m)^2}^{\infty}\frac{ds}{s-M^2}\{MA^{1+}-A^{4+}-\Delta^2[A^{2+}+A^{3+}-M(A^{5+}+A^{6+})]\}_{Dj}(0,t,t\,|\,s)_5+(2\pi)^3G^1(0)_5G^2(t)=0,$$

(73) [(55), (56)]

$$(2\pi)^{3/2}\Big\{-8\frac{\partial}{\partial\xi}\int_{(M+m)^2}^{\infty}ds\{MA^{1-}-\tfrac{1}{4}t[A^{2-}+A^{3-}-M(A^{5-}+A^{6-})]\}_{Dj}(0,\xi,t\,|\,s)_5\Big|_{\xi=t}$$
$$-\int_{(M+m)^2}^{\infty}ds[A^{2-}-A^{3-}-M(A^{5-}-A^{6-})]_{Dj}(0,t,t\,|\,s)_5\Big\}+(2\pi)^3 8MG^1(0)_5\frac{d}{dt}G^1(t)=2G^2(t)_5,$$

(74) [(55) + (56)]

$$(2\pi)^{3/2}\int_{(M+m)^2}^{\infty}ds[A^{2-}-A^{3-}+M(A^{5-}-A^{6-})]_{Dj}(t,0,t\,|\,s)_5=(2\pi)^3\times 2F(0)\left[4M\frac{d}{dt}G^1(t)_5-G^2(t)_5-t\frac{d}{dt}G^2(t)_5\right].$$

(75) [57]

These sum rules, as they stand, are relatively useless, since it is (with the exception of the photoproduction sum rule for the nuclear magnetic moments[4]) very difficult to connect the integrals on the left-hand side with measurable quantities.

The situation can be improved by application of the usual philosophy of slow variation of the matrix elements in the meson mass variables, as has been done in the past with some success in current algebra[4,5,11] and the vector-meson dominance model.[16]

In order to get functions which are sufficiently smooth in the meson-mass variables, we have to extract from the functions $A\,{\because}(\mu,\ \xi,\ t\,|\,s)$ the pion pole in μ and the ρ-meson pole in ξ. By means of the well-known relation for the retarded/advanced functions

$$\tilde{F}^{\nu}_{D\alpha j\beta}(q)^{(\pm)}_5=\{m^2-[(q-\Delta)\pm i\epsilon]^2\}^{-1}\{m_\rho{}^2-[(q+\Delta)\pm i\epsilon]^2\}^{-1}[m^2-(q-\Delta)^2][m_\rho{}^2-(q+\Delta)^2]\bar{F}^{\nu}_{D\alpha j\beta}(q)^{(\pm)}_5, \tag{76}$$

we introduce new invariant functions $\hat{A}(\mu,\ \xi,\ t\,|\,s)$ which do not have these poles:

$$A(\mu,\ \xi,\ t\,|\,s)=(m^2-\mu)^{-1}(m_\rho{}^2-\xi)^{-1}\hat{A}(\mu,\ \xi,\ t\,|\,s). \tag{77}$$

Introducing these functions into Eq. (70) and developing all terms in a Taylor series in t, we obtain

$$(2\pi)^{3/2}\sum_{r=1}^{n}\binom{n}{r}\frac{\partial^n}{\partial\mu^r\partial t^{n-r}}\int_{(M+m)^2}^{\infty}ds[\hat{A}^{1-}_{Dj}(\mu,0,t\,|\,s)_5-\hat{A}^{1-}_{Dj}(0,\mu,t\,|\,s)_5]\Big|_{t=0,\,\mu=0}$$
$$+(2\pi)^3m^2m_\rho{}^2\Bigg(\left(\frac{d^n}{dt^n}-\frac{n}{m_\rho{}^2}\frac{d^{n-1}}{dt^{n-1}}\right)\left[G^1(0)_5G^1(t)-\frac{1}{(2\pi)^3}G^1(t)_5+MG^2(0)G^1(t)_5\right]$$
$$-\frac{nM}{m^2}G^2(0)\left\{\left[1-\left(\frac{m}{m_\rho}\right)^2\right]\frac{d^{n-1}}{dt^{n-1}}G^1(t)_5+\frac{m^2}{4M}\left(\frac{d^{n-1}}{dt^{n-1}}-\frac{n-1}{m^2}\frac{d^{n-2}}{dt^{n-2}}\right)G^2(t)_5\right\}\Bigg)\Bigg|_{t=0}=0,$$

$$n=1,2,3\ldots. \quad (78)$$

The essential point is that in the sum on the left-hand side, no terms without derivatives with respect to the meson-mass variables occur. Therefore we may apply the PCAC assumption.

The function $[m^2-(q-\Delta)^2][m_\rho{}^2-(q+\Delta)^2]F^{\nu}_{D_\alpha j_\beta}(q)^{(+)}$ is the off-mass-shell $\pi N\to\rho N$ scattering amplitude. According to the vector-dominance model,[16] this expression is for fixed $(q-\Delta)^2\equiv\mu=m^2$ a smooth function or even a constant in the ρ-meson mass $\xi=(q+\Delta)^2$. Therefore, we expect that the first few or even all derivatives of

$$I_i(m^2,\xi,t)=\int_{(M+m)^2}^{\infty}ds\,\hat{A}^i(m^2,\xi,t\,|\,s)_5$$

with respect to ξ vanish. If this is true in one meson mass there is no reason that it may not be true in the other one (that is, the pion mass μ). Moreover, we know from the Goldberger-Treiman and Adler-Weisberger relations that matrix elements of $\partial_\mu j^\mu(x)_5$ apart from the one-pion pole may indeed be almost constant in the pion-mass variable.

Therefore, it seems not unrealistic to assume that $I_1(\mu,\xi,t)$ is (almost) constant in both meson-mass variables μ and ξ. However, we even do not need this property for $I_1(\mu,\xi,t)$ itself but only for the part antisymmetric in μ and ξ.

Before we write down explicitly our so-called PCAC assumption, let us add a remark on possible subtractions in the dispersion relations which we have suppressed for simplicity. If we replace everywhere the integrals $I_i(\mu,\xi,t)$ by

$$\mathrm{Re}\hat{b}^j(\mu,\xi,t\,|\,0)_5^+=\mathrm{Re}[(m^2-\mu)(m_\rho{}^2-\xi)b^j(\mu,\xi,t\,|\,0)_5^+],$$

then all relations above remain true also with subtractions.

Therefore, the only change due to subtractions is that the PCAC assumption *has* to be made in the real part of the amplitudes. If no subtractions are needed, it would be sufficient to make it in the imaginary part $\hat{A}^1(\cdots)_5$ itself instead of the energy integral $I_1(\cdots)$.

PCAC assumption.[17] We assume that the derivatives with respect to the meson masses satisfy the following condition for some integers $1\le n\le n_{\max}$:

$$\sum_{r=1}^{n}\binom{n}{r}\frac{\partial^n}{\partial\mu^r\partial t^{n-r}}\int_{(M+m)^2}^{\infty}ds[\hat{A}^{1-}_{Dj}(\mu,0,t\,|\,s)_5-\hat{A}^{1-}_{Dj}(0,\mu,t\,|\,s)_5]\Big|_{\mu=0,\,t=0}\approx 0. \quad (79)$$

For $n_{\max}=\infty$ this is equivalent to

$$\int_{(M+m)^2}^{\infty}ds[\hat{A}^{1-}_{Dj}(t,0,t\,|\,s)_5-\hat{A}^{1-}_{Dj}(0,t,t\,|\,s)_5]=0. \quad (79')$$

With this assumption we get from (78) sum rules between the form factors alone:

$$\Bigg(\left(\frac{d^n}{dt^n}-\frac{n}{m_\rho{}^2}\frac{d^{n-1}}{dt^{n-1}}\right)[G^1(0)_5G^1(t)-(2\pi)^{-3}G^1(t)_5+MG^2(0)G^1(t)_5]$$
$$-\frac{nM}{m^2}G^2(0)\left\{\left[1-\left(\frac{m}{m_\rho}\right)^2\right]\frac{d^{n-1}}{dt^{n-1}}G^1(t)_5+\frac{m^2}{4M}\left(\frac{d^{n-1}}{dt^{n-1}}-\frac{n-1}{m^2}\frac{d^{n-2}}{dt^{n-2}}\right)G^2(t)_5\right\}\Bigg)\Bigg|_{t=0}\approx 0,$$

$$n=1,2,3,\ldots,n_{\max}. \quad (80)$$

Introducing the usual physical form factors

$$\begin{aligned} F_I^V(t) &= (2\pi)^3[G^1(t) + MG^2(t)], \\ F_m^V(t) &= -(2\pi)^3 \frac{M}{2\mu^V} G^2(t), \\ G_A(t) &= -(2\pi)^3 G^1(t)_5, \\ F_P(t) &= (2\pi)^3 \tfrac{1}{2} G^2(t)_5, \end{aligned} \tag{81}$$

we finally arrive at the sum rules (10) of Sec. I.

IV. DISCUSSION

All that remains to be done is to check the validity of the sum rule (10). For the case $n=1$ all terms are rather well known. Using the dipole fits[18]

$$\begin{aligned} F_e^V(t) &= \left[1 - (1+2\mu^V)\frac{t}{4M^2}\right]\left[\left(1-\frac{t}{4M^2}\right)\left(1-\frac{t}{m_V^2}\right)^2\right]^{-1}, \\ F_m^V(t) &= \left[\left(1-\frac{t}{4M^2}\right)\left(1-\frac{t}{m_V^2}\right)^2\right]^{-1}, \\ G_A(t) &= G_A(0)\left(1-\frac{t}{m_A^2}\right)^{-2}, \end{aligned} \tag{82}$$

with

$$m_V = 0.84 \text{ BeV}, \quad m_A = 0.8 \text{ BeV},$$

we find that the first line of Eq. (10) vanishes. From the remaining terms of Eq. (10) we get

$$F_P(0) = \frac{2M}{m^2}\left[1 - \left(\frac{m}{m_\rho}\right)^2\right] G_A(0). \tag{83}$$

This is [up to the ρ-meson correction $(m/m_\rho)^2 = 0.03$] the Goldberger-Treiman relation, which is in good agreement with μ-decay experiments.

If we now assume the PCAC assumption (79) to be true for all n, then we get a relation between the four form factors for $t \neq 0$:

$$F_P(t) = \frac{2M}{m^2 - t} G_A(t)\left[1 - \left(\frac{m}{m_\rho}\right)^2\right] + \left(\frac{m}{m_\rho}\right)^2 \frac{M}{\mu^V} \frac{m_\rho^2 - t}{(m^2 - t)t} \{G_A(0)[F_e^V(t) + 2\mu^V F_m^V(t)] - (1+2\mu^V)G_A(t)\}. \tag{84}$$

Since $\sqrt{2}\,(f_\pi m^2)^{-1}\partial_\nu j_I^\nu(0)_5$ may be considered as an interpolating pion field $\varphi_+(0)^*$, we get the following identity[19]:

$$G_A(t) + \frac{t}{2M} F_P(t) = \frac{m^2 f_\pi g_{N\pi}}{\sqrt{2}\, M} \frac{K_{N\pi}(t)}{m^2 - t}, \tag{85}$$

where f_π is the pion decay constant, $g_{N\pi}$ is the strong pion-nucleon coupling constant, and $K_{N\pi}(t)$ is normalized to one for $t = m^2$.

The Goldberger-Treiman relation implies that the form factor $K_{N\pi}(t)$ is a smooth function in t:

$$K_{N\pi}(t) \approx K_{N\pi}(m^2) = 1, \quad 0 \le t \le m^2.$$

By means of the dipole fit (82) we obtain from (84) and (85)

$$K_{N\pi}(t) = K_{N\pi}(0)\left(1 - \frac{t}{m_\rho^2}\right)\left(1 - \frac{t}{m_A^2}\right)^{-2} \approx K_{N\pi}(0)\left(1 - \frac{t}{m_\rho^2}\right)^{-1}$$

or

$$\frac{m^{2n}}{K_{N\pi}(0)} \frac{d^n}{dt^n} K_{N\pi}(t)\bigg|_{t=0} = \left(\frac{m}{m_\rho}\right)^{2n} \approx (0.03)^n.$$

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APPENDIX

We give here a very general and short proof of the fact that a retarded or advanced commutator is Lorentz-covariant if and only if no GIS terms occur in the equal-time limit of the corresponding commutators. For special matrix elements this has been proved earlier by Schroer *et al.*[14] and Dietz *et al.*[19]

The retarded/advanced commutators (37) read, for arbitrary state vectors Ψ and Φ,

$$F^{\dots}_{\dots}(\Psi, x, \Phi)^{\pm} = \lim_{\delta\to 0} F^{\dots}_{\dots}(\Psi, x, \varphi_\delta, \Phi)^{\pm} = \mp i \lim_{\delta\to 0} \Theta^{\delta}_{\pm}(x^0)\langle\Psi|[j^{\dots}_{\dots}(\tfrac{1}{2}x), j^{\dots}_{\dots}(-\tfrac{1}{2}x)]|\Phi\rangle^T, \tag{A1}$$

with

$$\Theta^{\delta}_{+}(x^0) = \int_{-\infty}^{x^0} dy\, \varphi_\delta(y) = \begin{cases} 0 & \text{for } x^0 < -\delta a \\ 1 & \text{for } x^0 > \delta a, \end{cases}$$
$$\Theta^{\delta}_{-}(x^0) = \int_{x^0}^{+\infty} dy\, \varphi_\delta(y) = \begin{cases} 1 & \text{for } x^0 < -\delta a \\ 0 & \text{for } x^0 > \delta a. \end{cases} \tag{A2}$$

φ_δ is the δ sequence introduced in part I. $\Theta^{\delta}_{\pm}(x)$ are C^∞ functions for all $\delta > 0$. Under Lorentz transformations Λ the currents transform according to

$$U(\Lambda) j^\mu(f) U(\Lambda^{-1}) = (\Lambda^{-1})^\mu{}_\nu j^\nu(f_\Lambda), \tag{A3}$$

$$f_\Lambda(x) = f(\Lambda^{-1}x). \tag{A4}$$

Lorentz covariance of the retarded/advanced functions means

$$F_{DD}(\Psi, f, \Phi)^{\pm} = F_{DD}(\Psi_\Lambda, f_\Lambda, \Phi_\Lambda)^{\pm}, \quad \Lambda^\mu{}_\nu F^\nu_{jD}(\Psi, f, \Phi)^{\pm} = F^\mu_{jD}(\Psi_\Lambda, f_\Lambda, \Phi_\Lambda)^{\pm}, \tag{A5}$$

with

$$\Psi_\Lambda = U(\Lambda)\Psi. \tag{A6}$$

Theorem. A necessary and sufficient condition for the Lorentz covariance of F^μ_{jD} and F_{DD} is

$$\lim_{\delta\to 0} \int d^4x\, f(x) x^k \varphi_\delta(x^0)\langle\Psi|[j^\mu(\tfrac{1}{2}x), \partial_\nu j^\nu(-\tfrac{1}{2}x)]|\Phi\rangle^T = 0,$$
$$\lim_{\delta\to 0} \int d^4x\, f(x) x^k \varphi_\delta(x^0)\langle\Psi|[\partial_\mu j^\mu(\tfrac{1}{2}x), \partial_\nu j^\nu(-\tfrac{1}{2}x)]|\Phi\rangle^T = 0 \tag{A7}$$

for all $f \in S_4$. (That means no GIS terms in the corresponding equal-time commutators.[2])

Proof.

$$\Lambda^\mu{}_\nu F^\nu_{jD}(\Psi, f, \Phi)^{\pm} = \mp i \lim_{\delta\to 0} \int d^4x\, f(x)\Theta^{\delta}_{\pm}(x^0)\Lambda^\mu{}_\nu\langle\Psi|[j^\nu(\tfrac{1}{2}x), \partial_\lambda j^\lambda(-\tfrac{1}{2}x)]|\Phi\rangle^T$$
$$= \mp i \lim_{\delta\to 0} \int d^4x\, f(x)\Theta^{\delta}_{\pm}(x^0)\langle\Psi_\Lambda|[j^\mu(\Lambda\tfrac{1}{2}x), \partial_\lambda j^\lambda(-\Lambda\tfrac{1}{2}x)]|\Phi_\Lambda\rangle^T$$
$$= \mp i \lim_{\delta\to 0} \int d^4x\, f_\Lambda(x)\Theta^{\delta}_{\pm}((\Lambda^{-1}x)^0)\langle\Psi_\Lambda|[j^\mu(\tfrac{1}{2}x), \partial_\lambda j^\lambda(-\tfrac{1}{2}x)]|\Phi_\Lambda\rangle^T. \tag{A8}$$

Consider infinitesimal Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad \omega^\mu{}_\nu = -\omega_\nu{}^\mu, \quad \omega^r{}_k = 0 \quad \text{(no rotations)}, \tag{A9}$$

$$\Theta^{\delta}_{\pm}((\Lambda^{-1}x)^0) = \Theta^{\delta}_{\pm}(x^0) - \omega^0{}_k\left(x^k \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial x_k}\right)\Theta^{\delta}_{\pm}(x^0)$$
$$= \Theta^{\delta}_{\pm}(x^0) \mp \omega^0{}_k x^k \varphi_\delta(x^0). \tag{A10}$$

Introducing (A10) into (A8), we get

$$\Lambda^\mu{}_\nu F^\nu_{jD}(\Psi, f, \Phi)^{\pm} = F^\mu_{jD}(\Psi_\Lambda, f_\Lambda, \Phi_\Lambda)^{\pm} + i\omega^0{}_k \lim_{\delta\to 0} \int d^4x\, f_\Lambda(x) x^k \varphi_\delta(x^0)\langle\Psi_\Lambda|[j^\mu(\tfrac{1}{2}x), \partial_\lambda j^\lambda(-\tfrac{1}{2}x)]|\Phi_\Lambda\rangle^T.$$

However, from this equation our theorem follows at once.

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[6]The distribution and use of indices at the currents throughout this paper are as follows:

(i) Upper Greek indices μ, ν, $\lambda \ldots = 0, 1, 2, 3$ to the left of the argument(s) indicate tensor properties with respect to the Lorentz group, the corresponding Latin indices $k, l, r \ldots = 1, 2, 3$ their restriction to the space parts.

(ii) Lower Latin indices $a, b, c = V, A$ to the right of the argument(s) differentiate between vectors (V) and axial vectors (A). In the commutation relations we have the connection $a \neq b \rightarrow c = A$ and $a = b \rightarrow c = V$.

(iii) Lower Greek indices α, β, γ to the left of the argument(s) refer to the internal broken-symmetry group with structure constants $\epsilon_{[\alpha,\beta]}{}^{\gamma}$. The usual summation convention for double indices is used. The Minkowski metric is (+1, −1, −1, −1).

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Analyticity, Covariance, and Unitarity in Indefinite-Metric Quantum Field Theories*

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The general properties of analyticity, covariance, and unitarity are studied in quantum field theories regularized by finite-mass, indefinite-norm states. After reviewing the general status of indefinite-metric theories, a relativistic scalar model is analyzed for covariance and analyticity. This model shows that a commonly accepted prescription for treating the negative-norm states is not covariant, and more sophisticated methods are required. The technique of shadow states developed elsewhere is reviewed and applied to this problem.

I. INTRODUCTION

The problem of constructing finite quantum field theories has received considerable attention in recent years. Although it was invented by Dirac[1] as early as 1941 and has been the subject of careful studies for the past decade, the possibility of using quantized fields which are linear operators in a space whose metric is not positive definite is only gradually coming to be widely appreciated. Some of the confusion was due to the misconception that such a quantum field theory with indefinite metric is the same as the recipe of regularization introduced by Pauli and Villars in 1949.[2] The conceptual framework, including the question of probability interpretation and the need to select out a subset of states to be the physical states, has been reviewed by one of the present authors in his report to the 14th Solvay Congress.[3]

The main conclusions of the above-mentioned investigations are the following:

(i) Every order of perturbation theory yields