combination of $\mathcal{H}_4$ with $\mathcal{H}_3$ where the two g_{4i}'s are contracted. Then if we pick up the extra term in this contraction, we will get

$$-2i\vec{b}_i\times(\vec{b}_4\times\vec{b}_i)\cdot\partial_3^{-2}(\vec{g}_{4j}\times\vec{b}_j)\,, \qquad (A16)$$

which cancels (A15).

Another class of extra terms could arise if two b_4's coming from two different $\mathcal{H}_5$'s were contracted. The extra term would be $\vec{b}_i\times(\vec{b}_i\times\vec{b}_4)\cdot\partial_3^{-2}[\vec{b}_j\times(\vec{b}_j\times\vec{b}_4)]$. But there is a corresponding term in the expansion of $T(\mathcal{H}_3(x)\mathcal{H}_3(x_2)\mathcal{H}_4(x_3))$ where the two g_{4i}'s in $\mathcal{H}_4$ are contracted with g_{4i}'s in different $\mathcal{H}_3$'s. It is easy to see that the extra term is exactly equal (with opposite sign) to the above one and therefore they cancel.

These things hold in the presence of other Wick contractions and therefore are true to all orders. In fact, proceeding this way, one can show that all the unwanted terms cancel, and we do not have any extra terms left except those required to give the rules suggested in Eqs. (23) and (24).

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[1]R. N. Mohapatra, Phys. Rev. D 4, 378 (1971) (hereafter referred to as I); R. N. Mohapatra, Phys. Rev. D 4, 1007 (1971) (hereafter referred to as II). (An extensive list of references can be found in either of these papers.)

[2]Notation is the same as in papers I and II: $\mu,\nu,\alpha,\beta,\ldots$ each stands for four Lorentz indices; $i,j,k,\ldots$ stand for two space indices 1 and 2; $a,b,c,d,\ldots$ stand for isospin indices. An arrow over any symbol denotes that it is an isovector. Our metric is $x^2=x_1^2+x_2^2+x_3^2+x_4^2$, $x_4=ict$.

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Covariance and the Feynman Rules of a Massive Gauge Theory*

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If the interaction Lagrangian contains derivatives, or if the spins of the interacting particles ≥ 1, then both the propagators and the interaction Hamiltonian contain normal-dependent terms. Lee and Yang showed that if the noncovariant part of the propagator is dropped, then the finite interaction Hamiltonian is modified. In addition, there appears a term that is divergent, anti-Hermitian, and generally noncovariant if it does not vanish, namely, $\delta H=\frac{1}{2}i\hbar\delta^4(0)\times\ln\det\underline{g}$, where $\underline{g}$ depends on the structure of the theory. For massive gauge theories we have shown that $\delta H=0$ and that $H_{\text{int}}=-L_{\text{int}}$. These theories are manifestly covariant and the ordinary Feynman rules hold. Renormalization is not discussed.

I. INTRODUCTION

When the massive Yang-Mills field is discussed in an arbitrary gauge, the parameters of the local group appear as new field variables that give rise to fictitious scalar particles after quantization.[1,2] Although scalar particles also appear in the massless case, there is a fundamental difference between the massive and massless cases: Only the latter is locally gauge-invariant, while there exists for the massive field a privileged gauge in which the spinless part has been completely eliminated from the Lagrangian. We shall discuss the quantization in this privileged gauge; the formal SU(2) problem then becomes very similar to the one that arises when a charged vector field interacts with the electromagnetic field.

Lee and Yang[3] have investigated the electromagnetic interaction of charged vector mesons with arbitrary magnetic moment. They showed that some of the propagators contain noncovariant (normal-dependent) terms, and that one may drop these terms to obtain covariant propagators only at the price of modifying the interaction Hamiltonian and also adding an expression of the following type:

$$\delta H=\tfrac{1}{2}\,i\hbar\delta^4(0)\ln\det\underline{g}\,, \qquad (1.1)$$

where $\underline{g}$ is a matrix defined by the structure of the theory. Although the propagators used to compute the S matrix are now covariant, the new Hamiltonian is divergent, noncovariant, and non-Hermitian. To deal with this situation, Lee and Yang introduced the ξ-limiting process.

It has also been noticed, however, that there is another way of making the theory manifestly covariant and unitary; namely, by the addition of a direct pair-pair coupling between the charged particles in the original Lagrangian.[4,5] If this pair-pair interaction is added with such a coupling constant that the resulting Lagrangian is symmetric in the three fields (photon, and two charged vectors), then one finds that $\det \underline{g}=1$ and, therefore, $\delta H=0$. The theory arrived at in this way is formally identical with the description of the SU(2) massive gauge field, as Nakamura pointed out.[4] The connection of the massive gauge field with the theory of charged vector mesons interacting with the electromagnetic field had been discussed in an earlier work[6] and has been pursued further in more recent investigations.[7,8]

Since the simplicity of this result evidently depends upon the additional symmetries of the gauge field, one would like to have a direct proof that $\det\underline{g}=1$ for the gauge field SU(2). We shall give such a proof here and also generalize the result to any massive gauge field.

Recently, Eq. (1.1) has also turned up in the investigation of the perturbation theory for effective Lagrangians that are chirally invariant. Just as in the Lee-Yang case it has been shown[9,10] that the noncovariant part of the propagators may be dropped if one modifies the Hamiltonian and adds a δH that is again given by (1.1). It has also been shown that there is a particular choice of perturbation theory that permits use of the simple Feynman rules and at the same time preserves the general current-algebra theorems order by order.[9-11] This particular choice again corresponds to the condition $\det\underline{g}=1$. It also happens that the gauge theories and the chiral theories are described not only by Eq. (1.1) but by related Lagrangians as well.

II. THE ORIGIN OF δH

Before connecting with the work of Lee and Yang, we shall first establish Eq. (1.1) in a very elementary and general way with the aid of Feynman's representation of the transition matrix, namely,

$$\langle q_1'\cdots q_f' t' \,|\, q_1\cdots q_f t\rangle=\sum_{[p]} R_p e^{iS_p/\hbar}\,, \tag{2.1}$$

where the $q_1\cdots q_f$ are f generalized coordinates, S_p is the action computed for the path (p), R_p is the weight or normalization, and $[p]$ indicates a sum over all paths.

If t' differs from t by Δt, then[12]

$$\langle \vec{q}'t+\Delta t|\vec{q}t\rangle \cong Re^{iL\Delta t/\hbar}\,, \tag{2.2}$$

where $\vec{q}$ is written for the complete set of generalized coordinates, and R is the normalization determined by the condition

$$\lim_{\Delta t\to 0}\langle \vec{q}'t+\Delta t|\vec{q}t\rangle=\delta(\vec{q}'-\vec{q})\,. \tag{2.3}$$

Assume now that

$$L=\tfrac{1}{2}\sum g_{mn}\dot{q}_m\dot{q}_n - V(q_1,\ldots) \tag{2.4}$$

or

$$L=\tfrac{1}{2}\sum G_{mn}\dot{Q}_m\dot{Q}_n - V(Q_1,\ldots)\,, \tag{2.5}$$

where

$$Q_m=\sum_n C_{mn}q_n \tag{2.6}$$

and the matrices $\underline{G}$ and $\underline{g}$ are connected by

$$\underline{G}=\underline{C}^T\underline{g}\,\underline{C}\,. \tag{2.7}$$

Now choose $\underline{C}$ so that $\underline{G}$ is diagonal and $\det\underline{C}=1$. Then

$$L=\tfrac{1}{2}\sum G_r\dot{Q}_r^{\,2} - V(Q_1,\ldots)\,, \tag{2.8}$$

$$\det\underline{g}=\det\underline{G}=\prod_1^f G_r \tag{2.9}$$

and

$$\exp\left(\frac{i}{\hbar}L\Delta t\right)=\exp\left[\frac{i}{\hbar}\left(\sum_r G_r\frac{(\Delta Q_r)^2}{\Delta t}-V\Delta t\right)\right],$$

then

$$\lim_{\Delta t\to 0}\left[\exp\left(\frac{i}{\hbar}L\Delta t\right)\right]=\lim_{\Delta t\to 0}\left[\exp\left(\frac{i}{\hbar}\sum_r G_r\frac{(\Delta Q_r)^2}{\Delta t}\right)\right]. \tag{2.10}$$

In order that (2.3) hold, R must be

$$R(t)=\prod_{r=1}^f R_r(t)\,, \tag{2.11}$$

where

$$R_r(t)=[G_r(t)/2\pi i\hbar\Delta t]^{1/2} \tag{2.11a}$$

and the product is extended over all f degrees of freedom. By the unitary composition law, (2.1) may be expressed in terms of (2.2) in such a way that the functional integration over paths may be written as the limit of a finite number of ordinary integrations:

$$\langle \vec{q}'t'|\vec{q}t\rangle=\lim_{N\to\infty}\langle \vec{q}'t'|\vec{q}t\rangle_N\,,$$

where

$$\langle \vec{q}'t'|\vec{q}t\rangle_N = \int\cdots\int\langle \vec{q}'t'|\vec{q}_{N-1}t_{N-1}\rangle d\vec{q}_{N-1} \times\langle \vec{q}_{N-1}t_{N-1}|\vec{q}_{N-2}t_{N-2}\rangle\cdots d\vec{q}_1\langle \vec{q}_1t_1|\vec{q}t\rangle \tag{2.12}$$

and the time $t'-t$ has been broken into N equal subintervals. The separate integrations are associated with these intermediate times.

To apply this formula to the field case we may imagine space-time divided into cells $d\vec{x}dt$. Let the independent field components be ϕ_α. Then $\phi_\alpha(\vec{x},t)$ are the generalized coordinates associated with the cell $(\vec{x},t)$, and $\langle \vec{q}'t'|\vec{q}_{N-1}t_{N-1}\rangle$ in (2.12) becomes

$$(2\pi i\hbar\Delta t)^{-f}\prod_{(\alpha,\vec{x})}\exp\left(\frac{i}{\hbar}S(\vec{x},t_{N-1})\right)G^{1/2}(\alpha,\vec{x},t_{N-1}),$$

where the product runs over all spatial cells at one time as well as over all field components in each cell. Here $S(t_{N-1})$ is the total action between t_{N-1} and t_N.

Therefore,

$$\langle\phi't'|\phi t\rangle_N = (2\pi i\hbar\Delta t)^{-Nf}\prod_{(\alpha,\vec{x})}\int\exp\left(\frac{i}{\hbar}S(\vec{x},t_{N-1})\right)G^{1/2}(\alpha,\vec{x},t_{N-1})d\phi(\alpha,\vec{x},t_{N-1})\cdots. \tag{2.13}$$

The complete integrand is by (2.9)

$$\exp\left(\frac{i}{\hbar}S'\right) = \left[\exp\left(\frac{i}{\hbar}\sum_{(\vec{x},t)}S(\vec{x},t)\right)\right]\prod_{(\vec{x},t)}[g(\vec{x},t)]^{1/2}, \tag{2.14}$$

where $g(\vec{x},t)$ is the subproduct belonging just to the index α and localized at $(\vec{x},t)$. Here g is $\det\underline{g}$. Therefore,

$$S' = \sum_{(\vec{x},t)}S(\vec{x},t) - \tfrac{1}{2}i\hbar\sum_{(\vec{x},t)}\ln g(\vec{x},t)$$

$$=\sum_k\left(L(x_k) - \tfrac{1}{2}i\frac{\hbar}{\Delta^4x_k}\ln g(\vec{x},t)\right)\Delta^4x_k, \tag{2.15}$$

where $L(x_k)$ is the Lagrangian density, $x=(\vec{x},t)$, $\Delta^4x=\Delta\vec{x}\Delta t$, and k refers to the discrete cells. If we now go to the limit, we find

$$S' = \int L'd^4x, \tag{2.16a}$$

$$L' = L - \tfrac{1}{2}i\hbar\delta^4(0)\ln g, \tag{2.16b}$$

$$\delta L = -\tfrac{1}{2}i\hbar\delta^4(0)\ln g. \tag{2.16c}$$

This is the relation (1.1) that we wish to establish although it is written in terms of L rather than H. We may also write

$$L = \tfrac{1}{2}\sum g_{mn}\dot{\phi}_m\dot{\phi}_n - V, \tag{2.17}$$

where V is the part of the Lagrangian density that does not contain two time derivatives. The foregoing derivation has the advantage of providing an easy way to isolate the part of the Lagrangian that defines the $\underline{g}$ appearing in Eq. (1.1). The result is that the g correct for (2.16) is

$$g = \det\underline{g}, \tag{2.18}$$

where g_{mn} is defined by the structure of the Lagrangian according to (2.17). In other words the significant part of the Lagrangian density consists of those terms that contain two time derivatives and therefore dominate as $\Delta t\to 0$.

One expects δL may be noncovariant if it does not vanish, since it is obtained by a noncovariant procedure. It will be shown, however, that δL does vanish for the massive gauge field.

III. THE MASSIVE GAUGE FIELD

The Lagrangian is

$$L = \tfrac{1}{4}\operatorname{Tr}P_{\mu\lambda}P^{\mu\lambda} - \tfrac{1}{2}M^2\operatorname{Tr}P_\mu P^\mu, \tag{3.1}$$

where

$$P_{\mu\lambda} = (\partial_\mu + P_\mu)P_\lambda - (\partial_\lambda + P_\lambda)P_\mu. \tag{3.2}$$

The P_μ are anti-Hermitian fields with the matrix expansions

$$P_\mu = \sum_a P_{\mu a}F_a, \tag{3.3}$$

where the F_a are the group generators satisfying the structure relations:

$$(F_a, F_b) = if_{abs}F_s. \tag{3.4}$$

Therefore,

$$L = L_0 + L_1, \tag{3.5}$$

$$L_0 = \tfrac{1}{4}\operatorname{Tr}(\partial_\mu P_\lambda - \partial_\lambda P_\mu)(\partial^\mu P^\lambda - \partial^\lambda P^\mu) - \tfrac{1}{2}M^2\operatorname{Tr}P_\mu P^\mu, \tag{3.6a}$$

$$L_1 = \tfrac{1}{4}\operatorname{Tr}[2(\partial_\mu P_\lambda - \partial_\lambda P_\mu)(P^\mu, P^\lambda) + (P_\mu, P_\lambda)(P^\mu, P^\lambda)]. \tag{3.6b}$$

Now introduce the Hermitian fields V_λ, $V_{\lambda\mu}$ by writing

$$P_\lambda = iV_\lambda, \quad P_{\lambda\mu} = iV_{\lambda\mu} \tag{3.7}$$

and calculate the indicated traces. Then

$$L_0 = -\tfrac{1}{4}\sum(\partial_\mu V_{\lambda a} - \partial_\lambda V_{\mu a})(\partial^\mu V_a^\lambda - \partial^\lambda V_a^\mu) + \tfrac{1}{2}M^2 \sum_a V_{\mu a} V_a^\mu \,, \tag{3.8a}$$

$$L_1 = \tfrac{1}{4}\sum_{abc} 2f_{abc}(\partial_\mu V_{\lambda a} - \partial_\lambda V_{\mu a}) V_b^\mu V_c^\lambda - \tfrac{1}{4}\sum_{abcd} f_{abs} f_{cds} V_a^\mu V_b^\lambda V_{\mu c} V_{\lambda d} \,. \tag{3.8b}$$

Introduce the $4N$-component field

$$\phi_{\mu a} = \{\partial_0 V_{ka} - \partial_k V_{0a}, M V_{0a}\}, \quad a = 1, \ldots, N, \tag{3.9}$$

where N is the number of parameters of the gauge group. Then terms bilinear in $\phi_{\mu a}$ contain two time indices. We shall assume that these terms correspond to the "kinetic-energy" part of (2.17) since V_{0a} may be expressed as follows:

$$V_{0a} = -M^{-2}[\partial^k V_{k0a} + i(V^k, V_{k0})_a] \,. \tag{3.10}$$

The total Lagrangian may now be written

$$L = \tfrac{1}{2} \sum_{a,b;\, \mu,\lambda = 0,(0k)} \phi_{\mu a} \langle \mu a | g | \lambda b \rangle \phi_{\lambda b} - V \,. \tag{3.11}$$

Here a is the gauge index and the space-time index (μ, λ) is either $(0k)$ or 0. Then V is a remainder that depends only on the space indices, (kl) and k.

The matrix that enters (1.1) or (2.16) has non-vanishing elements

$$\langle 0, a | g | 0, b \rangle = \delta_{ab} + \frac{1}{M^2}\sum_{cds} f_{acs} f_{bds} V_{kc} V_{kd} \,, \tag{3.12a}$$

$$\langle 0k, a | g | 0l, b \rangle = \delta_{ab}\delta_{kl} \,, \tag{3.12b}$$

$$\langle 0k, a | g | 0, b \rangle = \langle 0, b | g | 0k, a \rangle = -f_{abs} V_{ks}/M \,. \tag{3.12c}$$

Therefore the matrix $\underline{g}$ has the following structure:

$$\underline{g} = \begin{vmatrix} \underline{I} & \underline{\alpha}^T \\ \underline{\alpha} & \underline{\beta} \end{vmatrix} \tag{3.13a}$$

where $\underline{I}$ and $\underline{\beta}$ are square matrices with dimensions $3N \times 3N$ and $N \times N$, respectively, while $\underline{\alpha}$ is a rectangular matrix of dimensions $3N \times N$.

Here

$$\langle 0, a | \beta | 0, b \rangle = \langle 0, a | g | 0, b \rangle = \delta_{ab} + \sum f_{acs} f_{bds} V_{kc} V_{kd}/M^2 \,, \tag{3.13b}$$

$$\langle 0, a | \alpha | 0k, b \rangle = \langle 0, a | g | 0k, b \rangle = f_{abc} V_{kc}/M \,,$$

$$\langle 0k, a | \alpha^T | 0, b \rangle = \langle 0k, a | g | 0, b \rangle = -f_{abc} V_{kc}/M \,. \tag{3.13c}$$

In Appendix A it is shown that the determinant of the $4N \times 4N$ matrix $\underline{g}$ is

$$\det \underline{g} = \det(\underline{\beta} - \underline{\alpha}\underline{\alpha}^T) \,, \tag{3.14}$$

where $\underline{\beta} - \underline{\alpha}\underline{\alpha}^T$ has the dimensions $N \times N$. Therefore, in a theory with a $\underline{g}$ matrix like (3.13a)

$$\delta L = -\tfrac{1}{2} i\hbar \delta^4(0) \ln \det \underline{M} \,, \tag{3.15}$$

where

$$\underline{M} = \underline{\beta} - \underline{\alpha}\underline{\alpha}^T \,. \tag{3.15a}$$

In a gauge theory $\underline{\alpha}$ and $\underline{\beta}$ are simply related. By (3.13c)

$$\begin{aligned} \langle 0, b | \alpha\alpha^T | 0, c \rangle &= \sum \langle 0, b | \alpha | 0k, m \rangle \langle 0k, m | \alpha^T | 0, c \rangle \\ &= \sum \langle 0, b | \alpha | 0k, m \rangle \langle 0, c | \alpha | 0k, m \rangle \\ &= \sum f_{bsm} f_{ctm} V_{ks} V_{kt}/M^2 \,. \end{aligned} \tag{3.16}$$

Therefore, by (3.13b) and (3.15a)

$$\underline{M} = \underline{\beta} - \underline{\alpha}\underline{\alpha}^T = \underline{I} \tag{3.17}$$

and

$$\delta L = 0 \,. \tag{3.18}$$

The key step has been the decomposition (3.11) according to which the matrix $\underline{g}$ is determined by only those components containing a time index or by the part of the "kinetic" energy that dominates as $\Delta t \to 0$.

IV. THE HAMILTONIAN FORMULATION

To connect more closely with the argument of Lee and Yang, let us write the theory in the Hamiltonian form. Let the momenta conjugate to (V_{ia}, V_{0a}) be (Π_a^i, Π_a^0). Then

$$\Pi_a^i = \frac{\partial L}{\partial \dot{V}_{ia}} = V_a^{i0} \,, \tag{4.1a}$$

$$\Pi_a^0 = 0 \,, \tag{4.1b}$$

and

$$\dot{V}_{ia} = -\Pi_{ia} + \partial_i V_{0a} + f_{abc} V_{0b} V_{ic} \,. \tag{4.2}$$

The Hamiltonian density is (up to a perfect divergence)

$$H = \Pi_a^i \dot{V}_{ai} - L \tag{4.3}$$

$$= \vec{\Pi}_a \cdot (\vec{\Pi}_a + \vec{\partial} V_{0a} - f_{abc} V_{0b} \vec{V}_c) - (\tfrac{1}{2}\vec{\Pi}_a \cdot \vec{\Pi}_a - \tfrac{1}{2}\vec{H}_a \cdot \vec{H}_a - \tfrac{1}{2}M^2 \vec{V}_a \cdot \vec{V}_a + \tfrac{1}{2}M^2 V_{0a} V_{0a}) \,, \tag{4.4}$$

where the repeated index is now summed, and

$$H_{ka} = \tfrac{1}{2}\epsilon_{klm} V_{lma} \,. \tag{4.5}$$

Then

$$H = H' + \Delta \,, \tag{4.6}$$

where

$$H' = \tfrac{1}{2}(\vec{\Pi}_a\cdot\vec{\Pi}_a + \vec{H}_a\cdot\vec{H}_a) + \tfrac{1}{2}M^2(\vec{V}_a\cdot\vec{V}_a + V_{0a}V_{0a}) \qquad (4.7)$$

and

$$\Delta = \vec{\Pi}_a\cdot(\vec{\partial}V_{0a} - f_{abc}V_{0b}\vec{V}_c) - M^2 V_{0a}V_{0a}. \qquad (4.8)$$

But Δ can be shown to be a perfect divergence in view of the equations of motion:

$$\partial_\mu V^{\mu\lambda} + i(V_\mu, V^{\mu\lambda}) + M^2 V^\lambda = 0, \qquad (4.9)$$

including

$$\vec{\partial}\cdot\vec{\Pi}_a = -M^2 V_{0a} - i(V_k, V^{k0})_a$$
$$= -M^2 V_{0a} + f_{abc}V_{kb}V_c^{k0}. \qquad (4.10)$$

Therefore,

$$\Delta = \vec{\Pi}_a\cdot(\vec{\partial}V_{0a}) + V_{0a}(\vec{\partial}\cdot\vec{\Pi}_a) = \vec{\partial}\cdot(V_{0a}\vec{\Pi}). \qquad (4.11)$$

Hence one may adopt the new Hamiltonian density (4.7) where

$$V_{0a} = -M^{-2}(\vec{\partial}\cdot\vec{\Pi}_a + f_{abc}\vec{V}_b\cdot\vec{\Pi}_c). \qquad (4.12)$$

It then follows that the interaction Hamiltonian is entirely contained in

$$\tfrac{1}{2}(\vec{H}_a\cdot\vec{H}_a + M^2 V_{0a}V_{0a}) \qquad (4.13)$$

and is simply

$$H_1 = \frac{1}{2M^2}[(f_{abc}\vec{V}_b\cdot\vec{\Pi}_c)^2 + 2(\vec{\partial}\cdot\vec{\Pi}_a)f_{abc}\vec{V}_b\cdot\vec{\Pi}_c]$$
$$+ \tfrac{1}{2}[(f_{abc}\vec{V}_b\times\vec{V}_c)^2 + 2f_{abc}\vec{V}_b\times\vec{V}_c\cdot\vec{\partial}\times\vec{V}_a]. \qquad (4.14)$$

The interacting fields (lower case) satisfy free-field equations, so that

$$-M^2 v_{0a} = \vec{\partial}\cdot\vec{\pi}_a, \qquad (4.15a)$$

$$\pi_{ia} = v_{i0a}. \qquad (4.15b)$$

Therefore,

$$H_1 = f_{abc}v_{0a}v_{ib}v_{0ic} + (1/2M^2)f_{abc}f_{ade}v_{ib}v_{0ic}v_{jd}v_{0je} - C, \qquad (4.16a)$$

where

$$-C = \tfrac{1}{2}[(f_{abc}\vec{v}_b\times\vec{v}_c)^2 + 2f_{abc}\vec{v}_b\times\vec{v}_c\cdot\vec{\partial}\times\vec{v}_a]. \qquad (4.16b)$$

Introduce the $4N$-component field

$$\hat{\phi}_{\mu a} = \{v_{0ia}, Mv_{0a}\} \qquad (4.17)$$

as in (3.9). Then

$$H_1 = -\tfrac{1}{2}\sum_{a,b;\,\mu,\lambda=0,0k}\hat{\phi}_{\mu a}\langle\mu a|h|\lambda b\rangle\hat{\phi}_{\lambda b} - C, \qquad (4.18)$$

where

$$\langle 0,a|h|0i,b\rangle = \frac{1}{M}f_{abs}v_{is} = \langle 0,a|\,\alpha\,|0i,b\rangle, \qquad (4.19a)$$

$$\langle 0i,a|h|0j,b\rangle = -(1/M^2)f_{asm}f_{bsn}v_{im}v_{jn}$$
$$= -\langle 0i,a|\,\alpha^T\alpha\,|0j,b\rangle. \qquad (4.19b)$$

Note that the fields $\hat{\phi}$ have noncovariant propagators. The result (Lemma 2, Appendix C) of Lee and Yang now tells us that the S matrix obtained from H_1 with the use of noncovariant propagators is identical to the S matrix obtained from the following interaction Hamiltonian:

$$\tilde{H} = -\tfrac{1}{2}\sum\hat{\phi}'_{\mu a}\langle\mu a|(h/I-h)|\lambda b\rangle\hat{\phi}'_{\lambda b} - C + \delta H, \qquad (4.20)$$

where now $\hat{\phi}'$ has covariant propagators and

$$\delta H = \tfrac{1}{2}i\hbar\delta^4(0)\,\mathrm{Tr}\ln\frac{1}{\underline{I}-\underline{h}} = \tfrac{1}{2}i\hbar\delta^4(0)\ln\det\frac{1}{\underline{I}-\underline{h}}. \qquad (4.21)$$

Since C arises from $\frac{1}{4}v_{ija}v_a^{ij}$ it is clear that C is identical with the part of L_1 depending only on space indices.

Next note that $\underline{I}-\underline{h}$ has the following structure:

$$\underline{I}-\underline{h} = \begin{vmatrix} \underline{I}+\underline{\alpha}^T\underline{\alpha} & -\underline{\alpha}^T \\ -\underline{\alpha} & \underline{I} \end{vmatrix} \qquad (4.22)$$

while $\underline{g}$ is by (3.13a) and (3.17)

$$\underline{g} = \begin{vmatrix} \underline{I} & \underline{\alpha}^T \\ \underline{\alpha} & \underline{I}+\underline{\alpha}\underline{\alpha}^T \end{vmatrix}. \qquad (4.23)$$

By direct multiplication we find the matrix equation

$$(\underline{I}-\underline{h})\underline{g} = \underline{I}. \qquad (4.24)$$

By (4.24) and (4.21)

$$\delta H = \tfrac{1}{2}i\hbar\delta^4(0)\ln\det\underline{g} \qquad (4.25)$$

and according to (2.16c) and (3.18)

$$\delta H = -\delta L = 0. \qquad (4.26)$$

In addition, by (4.24),

$$\underline{h}(\underline{I}-\underline{h})^{-1} = \underline{g}-\underline{I}. \qquad (4.27)$$

But $\underline{g}-\underline{I}$ is the part of the Lagrangian interaction matrix depending on time indices. The rest of the interaction matrix is contained in $-C$, and only changes its sign between the Lagrangian and Hamiltonian. Therefore, by (4.20),

$$\tilde{H} = -L_{\mathrm{int}}(\hat{\phi}') + \tfrac{1}{2}i\hbar\delta^4(0)\ln\det\underline{g} \qquad (4.28a)$$
$$= -L_{\mathrm{int}}(\hat{\phi}'). \qquad (4.28b)$$

Lee and Yang proved their theorem by a direct comparison of terms in the perturbation expansion of the S matrix. In that context, the feature which distinguished the ϕ from other fields is only that their propagators contain noncovariant δ-function parts.

In the transition-matrix approach of Sec. II, however, the significant property of ϕ (which determines g and δH) is that they are velocities in some general sense, as seen, for example, in the constraint equation (4.10).

We see, then, that we can identify the fields ϕ by noting either that they are "velocities" or that they have noncovariant propagators.

V. THE PERTURBATION EXPANSION

The complete transition matrix is

$$\langle \phi' t' | \phi t \rangle = \sum_{\text{histories}} R_h e^{iS_h} , \qquad (5.1)$$

where the sum over histories again stands for a functional integration. R_h is the weight of a particular history (h) and contains the factor

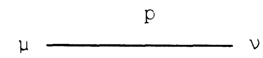
$$\prod_{(\vec{x},t)} [g(\vec{x},t)]^{1/2} \qquad (5.2)$$

of (2.14) which is usually noncovariant. If $\sqrt{g}=1$, however, then its cofactor in $\sum_h R_h$ is also manifestly covariant.

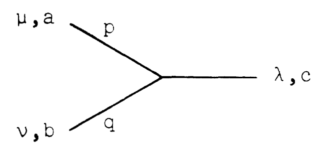
In the perturbation expansion the phase is expanded as

$$e^{iS} = e^{iS_0}(1 + iS_{\text{int}} + \cdots). \qquad (5.3)$$

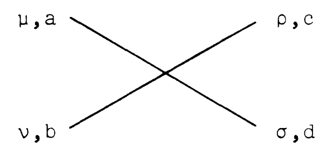
The usual treatment of this expansion leads to the time-ordered product denoted by T^* which is completely covariant in contrast to the T product that arises in the Hamiltonian expansion. (See Appendix B.) We may then understand that this expansion (5.3) is described by simple Feynman rules constructed for the $L_{\text{int}}(\hat{\phi}')$ appearing in (4.28) as asserted in the theorem of Lee and Yang.[13]



$$S_{\mu\nu}(p) = (-g_{\mu\nu} + m^{-2} p_\mu p_\nu)(p^2 - m^2 + i\varepsilon)^{-1}$$



$$V_{\mu\nu\lambda,abc}(p,q) = g f_{abc}[(2p-q)_\nu g_{\mu\lambda} + (2q-p)_\mu g_{\nu\lambda} - (p+q)_\lambda g_{\mu\nu}]$$



$$\begin{aligned} U_{\mu\nu\rho\sigma,abcd} = {} & ig^2 f_{abs} f_{cds}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \\ & + ig^2 f_{acs} f_{bds}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho}) \\ & + ig^2 f_{ads} f_{cbs}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}) \end{aligned}$$

FIG. 1. Feynman rules for general gauge group with structure constants f_{abc}. Diagrams apply to Hermitian fields with group indices a, b, c, d.

The Feynman rules for a general gauge group are given in Fig. 1. Special cases of these rules are given by Nakamura and Tzou and have been further discussed by Barnebey for the case SU(2).

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APPENDIX A

Note that

$$\det\begin{vmatrix} 1 & 0 & \cdots & a_{11} & a_{12} & \cdots \\ 0 & 1 & & a_{21} & a_{22} & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ b_{11} & b_{12} & \cdots & c_{11} & c_{12} & \cdots \\ b_{21} & b_{22} & & c_{21} & c_{22} & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \end{vmatrix} = \det\begin{vmatrix} 1 & 0 & \cdots & 0 & a_{12} & \cdots \\ 0 & 1 & & a_{21} & a_{22} & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ b_{11} & b_{12} & \cdots & c_{11} - b_{11}a_{11} & c_{12} & \cdots \\ b_{21} & b_{22} & & c_{21} - b_{21}a_{11} & c_{22} & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \\ \cdot & & & \cdot & & \end{vmatrix} .$$

By repeating this operation we may eliminate all matrix entries in the upper-right corner without changing the determinant. Therefore,

$$\det\begin{vmatrix} \underline{I} & \underline{A} \\ \underline{B} & \underline{C} \end{vmatrix} = \det\begin{vmatrix} \underline{I} & \underline{0} \\ \underline{B} & \underline{C} - \underline{B}\underline{A} \end{vmatrix}$$

and by Laplace's expansion

$$\det\begin{vmatrix} \underline{I} & \underline{A} \\ \underline{B} & \underline{C} \end{vmatrix} = \det(\underline{C} - \underline{B}\underline{A}).$$

By (3.13a) it follows that

$$\det\underline{g} = \det(\underline{\beta} - \underline{\alpha}\underline{\alpha}^T).$$

In the case considered by Lee and Yang and by Tzou, $\underline{g}$ is 11×11 and $\underline{\beta}$ is 2×2. One may then use the following notation:

$$\underline{g} = \begin{vmatrix} & & & R_1 & S_1 \\ & & & \cdot & \cdot \\ & \underline{I} & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & R_n & S_n \\ R_1 & \cdots & R_n & \beta_{11} & \beta_{12} \\ S_1 & \cdots & S_n & \beta_{21} & \beta_{22} \end{vmatrix}$$

and

$$M_{11} = \beta_{11} - R^2\,, \quad M_{12} = \beta_{12} - \vec{\mathrm{R}}\cdot\vec{\mathrm{S}}\,,$$

$$M_{22} = \beta_{22} - S^2\,, \quad M_{21} = \beta_{21} - \vec{\mathrm{R}}\cdot\vec{\mathrm{S}}\,.$$

The matrix entries given by Tzou are

$$\beta_{11} = 1 + \frac{e^2}{m^2}a^2 + \zeta\mathcal{R}^2\,, \quad \vec{\mathrm{R}} = \left(\kappa\vec{\mathcal{R}}\,, 0, \frac{e}{m}\vec{\mathrm{a}}\right),$$

$$\beta_{22} = 1 + \frac{e^2}{m^2}a^2 + \zeta\mathcal{J}^2\,, \quad \vec{\mathrm{S}} = \left(\kappa\vec{\mathcal{J}}\,, \frac{-e}{m}\vec{\mathrm{a}}\,, 0\right),$$

$$\beta_{12} = \zeta\vec{\mathcal{R}}\cdot\vec{\mathcal{J}} = \beta_{21}\,.$$

One then finds immediately that $M = I$ if $\zeta = \kappa^2$, and therefore $\det\underline{g} = 1$. In the Lee and Yang case, $\zeta = 0$ and therefore $\det\underline{g} \neq 1$.

APPENDIX B

For completeness, the results of Lee and Yang referred to in Sec. I are rephrased here by employing the Hamiltonian path-integral method[14,15] in which the transition matrix is expressed as follows:

$$\langle\vec{\mathrm{q}}'t'|\vec{\mathrm{q}}\,t\rangle = \int\cdots\int\left[\prod_\tau d\vec{\mathrm{Q}}(\tau)d\vec{\mathrm{P}}(\tau)\right]\exp\left[\frac{i}{\hbar}\int_t^{t'}d\tau[\vec{\mathrm{P}}\cdot\dot{\vec{\mathrm{Q}}} - H(\vec{\mathrm{P}},\vec{\mathrm{Q}})]\right] \tag{B1}$$

for $\vec{\mathrm{q}} = (q_1, \ldots, q_f)$ and $\vec{\mathrm{q}}' = (q_1', \ldots, q_f')$. The right-hand side of this equation denotes[16] as usual the limit $(N\to\infty)$ of

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{j=1}^{N-1}d\vec{\mathrm{Q}}(t_j)\prod_{j=1}^{N}d\vec{\mathrm{P}}(t_j)\right)\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\Delta t_j[\vec{\mathrm{P}}(t_j)\cdot\dot{\vec{\mathrm{Q}}}(t_j) - H(\vec{\mathrm{P}}(t_j),\vec{\mathrm{Q}}(t_j))]\right] \tag{B2}$$

for progressively finer partitions $t = t_0 < \cdots < t_N = t'$. For a simple Hamiltonian of the form $H = \frac{1}{2}\vec{\mathrm{P}}\cdot\vec{\mathrm{P}} + V(\vec{\mathrm{Q}})$, Garrod has shown that by first integrating over the "momenta" $\vec{\mathrm{P}}$ the transition matrix (B1) becomes

$$\langle\vec{\mathrm{q}}'t'|\vec{\mathrm{q}}t\rangle = \int\cdots\int\left(\prod_\tau R(\tau)d\vec{\mathrm{Q}}(\tau)\right)\exp\left[\frac{i}{\hbar}\int_t^{t'}d\tau[\tfrac{1}{2}\dot{\vec{\mathrm{Q}}}\cdot\dot{\vec{\mathrm{Q}}} - V(\vec{\mathrm{Q}})]\right] \tag{B3}$$

as given by the Lagrangian theory of Feynman,[17] with the prescribed normalization factor

$$R(t_j)=(2\pi i\hbar\Delta t_j)^{-f/2}. \tag{B4}$$

Consider now the more complex dynamical system of f harmonic oscillators whose Lagrangian and Hamiltonian are

$$\begin{aligned} L&=L_0+L_I=(\tfrac{1}{2}\dot{\vec{Q}}\cdot\dot{\vec{Q}}-\tfrac{1}{2}\vec{Q}\cdot\vec{Q})+(\tfrac{1}{2}\dot{\vec{Q}}\cdot\underline{A}\dot{\vec{Q}}+\tfrac{1}{2}\dot{\vec{Q}}\cdot\vec{B}+\tfrac{1}{2}\vec{B}\cdot\dot{\vec{Q}}+C)\,,\\ H&=H_0+H_I=(\tfrac{1}{2}\vec{P}\cdot\vec{P}+\tfrac{1}{2}\vec{Q}\cdot\vec{Q})+[\tfrac{1}{2}(\vec{P}-\vec{B})\cdot(\underline{I}+\underline{A})^{-1}(\vec{P}-\vec{B})-\tfrac{1}{2}\vec{P}\cdot\vec{P}-C]\,. \end{aligned} \tag{B5}$$

Here $\underline{A}=\underline{A}^T$, $\vec{B}$, and C are $(f\times f)$, $(1\times f)$, and (1×1) matrix quantities, respectively, which depend explicitly on $\vec{Q}$ only. This system was studied by Lee and Yang as a nonrelativistic prototype of the field theory of charged vector-particle electrodynamics, having an interaction term which is quadratic in the velocities.

Using the Hamiltonian path-integral method (B1) and the Hamiltonian (B5), the perturbation series is represented by

$$\langle \vec{q}'t'|\vec{q}t\rangle=\int\cdots\int\Big(\prod_\tau d\vec{Q}(\tau)d\vec{P}(\tau)\Big)\Big(1-\frac{i}{\hbar}\int_t^{t'}d\tau(H_I)+\cdots\Big)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau(\vec{P}\cdot\dot{\vec{Q}}-H_0)\Big]. \tag{B6}$$

Then (B6) becomes in the usual way the perturbation expansion of the S matrix. If we integrate first over the $\vec{P}$'s before forming (B6), we find

$$\langle \vec{q}'t'|\vec{q}t\rangle=\int\cdots\int\Big(\prod_\tau\hat{R}(\tau)d\vec{Q}(\tau)\Big)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau[L(\dot{\vec{Q}},\vec{Q})]\Big]. \tag{B7}$$

By explicit calculation (see Salam and Strathdee[18] for a field-theoretic example), the normalizing factor $\hat{R}$ for this system is found to differ from R of (B4) by a multiplicative factor

$$\hat{R}=R\exp(-i\delta H/\hbar)\,, \tag{B8}$$

where δH is, as expected, the quantity discussed in Sec. II, with $\underline{g}=\underline{I}+\underline{A}$. For the Lagrangian theory, then, the perturbation series is

$$\langle \vec{q}'t'|\vec{q}t\rangle=\int\cdots\int\Big(\prod_\tau R(\tau)d\vec{Q}(\tau)\Big)\Big(1-\frac{i}{\hbar}\int_t^{t'}d\tau(-L_I+\delta H)+\cdots\Big)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau(L_0)\Big]. \tag{B9}$$

It remains now to determine the "propagators" appropriate to the two formulations (B6) and (B9). In both cases, the development of the perturbation series is exactly parallel to the operator-algebra version, with time orderings of the interaction terms (call the respective ordering operators T_H and T_L) which can be recast, in analogy with Wick's theorem, into the familiar normal-ordered products and (free) ground-state expectation values. It is known[17,19] that T_L is the T^* operator; this is true because, for example,

$$\begin{aligned} {}_0\langle \vec{q}'t'|T_L(\dot{Q}_r(t_1)\dot{Q}_s(t_2))|\vec{q}t\rangle_0&=\int\cdots\int\Big(\prod_\tau R(\tau)\Big)\dot{Q}_r(t_1)\dot{Q}_s(t_2)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau(L_0)\Big]\\ &=\frac{d}{dt_1}\frac{d}{dt_2}\int\cdots\int\Big(\prod_\tau R(\tau)d\vec{Q}(\tau)\Big)Q_r(t_1)Q_s(t_2)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau(L_0)\Big]\\ &=\frac{d}{dt_1}\frac{d}{dt_2}\,{}_0\langle \vec{q}'t'|T_L(Q_r(t_1)Q_s(t_2))|\vec{q}t\rangle_0. \end{aligned} \tag{B10}$$

In the Hamiltonian version (B6), it is $\vec{P}$ rather than $\dot{\vec{Q}}$ which appears in the interaction terms. Since the functional-integration kernel still involves the free-action function, however, the paths which contribute significantly to the integration lie in a tube surrounding the free classical orbit in phase space (for which $\vec{P}=\dot{\vec{Q}}$). Hence we must compare (B10) with the properties of the Hamiltonian counterpart

$${}_0\langle \vec{q}'t'|T_H(P_r(t_1)P_s(t_2))|\vec{q}t\rangle_0. \tag{B11}$$

For this purpose, we construct the generating functional for matrix elements of time-ordered products of $\vec{P}$'s:

$$Z_{H_0}(\vec{F})=\int\cdots\int\Big(\prod_\tau d\vec{Q}(\tau)d\vec{P}(\tau)\Big)\exp\Big[\frac{i}{\hbar}\int_t^{t'}d\tau[\vec{P}\cdot\dot{\vec{Q}}-H_0(\vec{P},\vec{Q})-i\hbar\vec{P}\cdot\vec{F}]\Big]. \tag{B12}$$

Then (B11) is represented by

$$ {}_0\langle \vec{q}'t' | T_H(P_r(t_1)P_s(t_2)) | \vec{q}t\rangle_0 = \frac{\delta}{\delta F_r(t_1)} \frac{\delta}{\delta F_s(t_2)} Z_{H_0}(\vec{F}) \Big|_{\vec{F}=\vec{0}} $$

$$ = \int \cdots \int \left(\prod_\tau d\vec{Q}(\tau) d\vec{P}(\tau) \right) P_r(t_1)P_s(t_2) \exp\left[\frac{i}{\hbar} \int_t^{t'} d\tau (\vec{P}\cdot\dot{\vec{Q}} - H_0) \right]. \tag{B13} $$

In order to make the desired comparison with the Lagrangian theory, we reexpress (B12) by first integrating over the momenta. Notice that (B12) differs from the zeroth-order term in the perturbation series (B6) only by the replacement $\dot{\vec{Q}} \to \dot{\vec{Q}} - i\hbar\vec{F}$. Therefore, the result of the integration over the $\vec{P}$'s should simply be the zeroth-order term of (B9) with the same replacement. The expression (B12) is then equivalent to

$$ Z_{H_0}(\vec{F}) = \int \cdots \int \left(\prod_\tau R(\tau) d\vec{Q}(\tau) \right) \exp\left[\frac{i}{\hbar} \int_t^{t'} d\tau [L_0(\dot{\vec{Q}} - i\hbar\vec{F}, \vec{Q})] \right] $$

$$ = \int \cdots \int \left(\prod_\tau R(\tau) d\vec{Q}(\tau) \right) \exp\left[\frac{i}{\hbar} \int_t^{t'} d\tau [L_0(\dot{\vec{Q}}, \vec{Q}) - i\hbar\dot{\vec{Q}}\cdot\vec{F} - \tfrac{1}{2}\hbar^2 \vec{F}\cdot\vec{F}] \right]. \tag{B14} $$

Performing the indicated functional differentiation, then, we find

$$ {}_0\langle \vec{q}'t' | T_H(P_r(t_1)P_s(t_2)) | \vec{q}t\rangle_0 = {}_0\langle \vec{q}'t' | [T_L(\dot{Q}_r(t_1)\dot{Q}_s(t_2)) - i\hbar\delta_{rs}\delta(t_1 - t_2)] | \vec{q}t\rangle_0 . \tag{B15} $$

In terms of ground-state expectation values, if

$$ {}_0\langle \Phi | T_L(Q_r(t_1)Q_s(t_2)) | \Phi\rangle_0 = \tfrac{1}{2}\hbar\delta_{rs} S(t_1 - t_2) = \tfrac{1}{2}\hbar\delta_{rs} \exp(-i\,|t_1 - t_2|), $$
$$ {}_0\langle \Phi | T_L(\dot{Q}_r(t_1)\dot{Q}_s(t_2)) | \Phi\rangle_0 = \hbar\delta_{rs} \frac{d}{dt_1}\frac{d}{dt_2} S(t_1 - t_2), \tag{B16} $$

then we have

$$ {}_0\langle \Phi | T_H(P_r(t_1)P_s(t_2)) | \Phi\rangle_0 = \tfrac{1}{2}\hbar\delta_{rs} \frac{d}{dt_1}\frac{d}{dt_2} S(t_1 - t_2) - i\hbar\delta_{rs}\delta(t_1 - t_2). \tag{B17} $$

Thus T_H is the ordinary T product. These results are identical to those of Lee and Yang for the nonrelativistic system (B5), and can be extended by the methods outlined in Sec. II to the field-theoretic system which was their principal concern: electromagnetic interactions of charged vector particles. In the language of path-integrals, however, "δH" appears to have a more natural role as part of the functional-integration measure (ensuring unitarity) than as a part of the kernel; such a (divergent) normalizing factor arises even for the simple theory described in (B3). Thus even when $\delta H \neq 0$, the relation between the Hamiltonian and the Lagrangian path-integral theories described above seems to reflect the principal features of what has been called "Matthew's rule": Namely, that one may replace $\mathcal{H}_{\rm int}$ by $-\mathcal{L}_{\rm int}$ in calculating S-matrix elements, provided that the T product is replaced by the T^* product. From this viewpoint the result of our paper may be expressed by saying that the massive gauge theories have the same functional measure as the "simple" theories originally covered by Matthew's result.

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[16]We define

$$\vec{Q}(t_0)=\vec{q},\quad \vec{Q}(t_N)=\vec{q}\,';$$
$$d\vec{Q}(t_j)=dQ_1(t_j)\cdots dQ_f(t_j),\quad \text{similarly for } d\vec{P}(t_j);$$
$$\Delta t_j=t_j-t_{j-1},\quad \dot{\vec{Q}}(t_j)=[\vec{Q}(t_j)-\vec{Q}(t_{j-1})]/\Delta t_j.$$

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Behavior of Commutator Matrix Elements at Small Distances. III. Sum Rules for Form Factors and Their Derivatives at $t=0$

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From general principles of quantum field theory (especially locality and Poincaré invariance), the small-distance behavior of commutator matrix elements, and partial conservation of axial-vector current (smoothness of the matrix elements in the meson masses), sum rules between the four weak isovector form factors and their first derivatives at zero momentum transfer are derived.

I. INTRODUCTION

In two recent publications[1,2] we started a systematic investigation of the behavior of commutator matrix elements at small distances on the basis of general principles of quantum field theory. These investigations gain some physical significance[3] in connection with Gell-Mann's program of current-generated commutator algebras.[4,5]

In the present article we give an application of these ideas and derive from the current-density algebras, especially from the nonoccurrence of q-number Goto-Imamura-Schwinger terms and partial conservation of axial-vector currents (PCAC), a sum rule between the four weak isovector form factors.

We assume the equal-time commutation relations for the current densities,[6]

$$\lim_{\delta\to 0}\langle\Psi|[j^0_\alpha(\varphi_\delta;h)_a, j^\nu_\beta(0)_b]\Phi\rangle^T=\epsilon_{[\alpha,\beta]}{}^\gamma\langle\Psi|j^\nu_\gamma(0)_c|\Phi\rangle h(\vec{0}),\tag{1}$$

to hold for all C^∞ functions $h(\vec{x})$, all state vectors Ψ,Φ from a certain domain D of the Hilbert space of physical states $\mathfrak{H}$ which is stable under the Poincaré group, and all symmetric δ-sequences $\varphi_\delta(x^0)=:(1/\delta)\varphi(x^0/\delta)$. Here $\varphi(x^0)$ is a C^∞ function with compact support contained in $[-a,a]$ and normalized according to[7]

$$\int dx^0\varphi(x^0)=1.\tag{2}$$

(The equality sign with the colon means "is by definition.") Of course, we assume the currents to be members of a complete local Wightman field theory.[8-10]

T means subtraction of the vacuum expectation value before taking the limit.

$$\langle\Psi|[\cdots]|\Phi\rangle^T=\langle\Psi|[\cdots]|\Phi\rangle-\langle 0|[\cdots]|0\rangle\langle\Psi||\Phi\rangle.\tag{3}$$

A consequence of (1) and Poincaré symmetry of the theory are the following two relations[2]:

$$\lim_{\delta\to 0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[\partial_\mu j^\mu_\alpha(x)_a, j^\nu_\beta(0)_b]|\Phi\rangle^T=0\tag{4}$$

and

$$\lim_{\delta\to 0}\int d^4x\,\varphi_\delta(x^0)x^k h(\vec{x})\langle\Psi|[j^0_\alpha(x)_a,\partial_\nu j^\nu_\beta(0)_b]|\Phi\rangle^T=0\quad(k=1,2,3).\tag{5}$$

In other words Eqs. (4) and (5) are necessary (but not sufficient) conditions for the absence of q-number