Total Cross Section for Electron-Positron Annihilation and Hadronic Contribution to the Muon Magnetic Moment*

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A class of rigorous inequalities are derived for the integrated cross section for e^-e^+ annihilation into hadrons through one photon. They involve the p-wave $\pi\pi$ cross section and a slope of the electromagnetic form factor of the pion. A lower bound on the hadronic contribution to the muon magnetic moment is estimated in terms of the electromagnetic charge radius r_π. The bound is twice as large as the ρ-meson dominance value if $r_\pi = 0.86$ F as indicated by old data, but no meaningful result is obtained if one substitutes the values for r_π obtained in a new experiment. The inequalities can also be used to test various predictions concerning the asymptotic behavior of the e^-e^+ annihilation cross section. A slowly decreasing behavior is favored over a rapidly decreasing one like $\sigma^{e^+e^-}(t) \propto t^{-3}$, but more accurate data are needed to draw a definite conclusion.

Electron-positron colliding-beam experiments have produced a number of interesting data on the annihilation cross section in the vector-meson region[1] and also some preliminary data in the few-GeV region.[2,3] The purpose of this paper is to investigate relations between the hadronic cross section and the pion form factor. The presently available data for the pion form factor do not lead us to a definite quantitative conclusion, but π-e scattering at Serpukhov, Los Alamos, and the National Accelerator Laboratory in the near future may provide the parameters needed to draw conclusions about the hadronic cross section for electron-positron annihilation into hadrons.

Let us derive an inequality first pointed out by Drell and Zachariasen.[4] With the usual definition of the pion form factor, we obtain an explicit decomposition of the absorptive part as $[F(0)=1]$

$$e\,\mathrm{Im}F(t) = \frac{(2\pi)^4(2p_0)^{1/2}}{t-4\mu^2}\sum_n \delta^{(4)}(p+p'-p_n)\langle p|j(0)|n\rangle \times\langle n|\vec{\mathrm{p}}\cdot\vec{\mathrm{J}}(0)|0\rangle\theta(t-4\mu^2), \quad (1)$$

where μ is the pion mass, $t=-(p+p')^2=-q^2$, $\vec{\mathrm{p}}$ is the pion momentum in the $\pi^+\pi^-$ center-of-mass frame, $\vec{\mathrm{p}}\cdot\vec{\mathrm{J}}(0)$ is a three-dimensional scalar product of $\vec{\mathrm{p}}$ and the electromagnetic current, and $j(0)$ is the pion source function. The Schwarz inequality in the summation over intermediate states n leads us to

$$e^2|\mathrm{Im}F(t)|^2 \leq \pi\left(\frac{t}{t-4\mu^2}\right)^{1/2}\sigma^{\pi\pi}(t)\rho(t), \quad (2)$$

where $\sigma^{\pi\pi}(t)$ is the p-wave $\pi\pi$ scattering cross section at c.m. energy $t^{1/2}$. The isovector spectral function $\rho(t)$, defined by

$$(\delta_\mu{}^\nu - q_\mu q^\nu/q^2)\rho(t) = (2\pi)^3\sum_n \delta^{(4)}(p+p'-p_n) \times\langle 0|J_\mu(0)|n\rangle\langle n|J^\nu(0)|0\rangle, \quad (3)$$

is related to the total cross section for $e^+e^-\to$ hadrons through one isovector photon by

$$\rho(t) = t^2\sigma^{e^+e^-}(t)/4\pi^2\alpha. \quad (4)$$

We use the Schwarz inequality once again, this time in the integral over t, to obtain[5]

$$\left|\frac{F(q_1^2)-F(q_2^2)}{q_1^2-q_2^2}\right|^2 \leq \left|\frac{1}{\pi}\int_{4\mu^2}^\infty dt\,\frac{|\mathrm{Im}F(t)|}{(t+q_1^2)(t+q_2^2)}\right|^2 \leq \frac{1}{16\pi^4\alpha^2}\left|\int_{4\mu^2}^\infty dt\,\sigma^{e^+e^-}(t)f(t)\right|\left|\int_{4\mu^2}^\infty dt\,\frac{t^{5/2}\sigma^{\pi\pi}(t)}{f(t)(t+q_1^2)^2(t+q_2^2)^2(t-4\mu^2)^{1/2}}\right|, \quad (5)$$

where q_1^2 and q_2^2 are two arbitrary points in the spacelike region $q_1^2,\ q_2^2>0$ and $f(t)$ may be an arbitrary non-negative function of t. The second integral on the right-hand side converges if $[f(t)]^{-1} = O(t^2)$ as $t\to+\infty$, because of the unitarity bound on the p-wave $\pi\pi$ cross section $[\sigma^{\pi\pi}(t)\leq 48\pi(t-4\mu^2)^{-1}]$. However, for large values of q_1^2 and q_2^2, the contribution from the large-t region $(t\gg m_\rho^2)$ is important relative to that from the low-t region. Since at present we do not have much knowledge of the p-wave $\pi\pi$ scattering above $t \simeq 1$ GeV^2, Eq. (5) is probably more useful for small values of q_1^2 and q_2^2; for instance, in the case of $q_1^2 = q_2^2 \to 0$:

$$(\tfrac{1}{6} r_\pi^2)^2 \le \frac{1}{16\pi^4\alpha^2} \left| \int_{4\mu^2}^\infty dt\, \sigma^{e^+e^-}(t) f(t) \right| \times \left| \int_{4\mu^2}^\infty dt \frac{\sigma^{\pi\pi}(t)}{f(t) t^{3/2}(t - 4\mu^2)^{1/2}} \right|, \qquad (6)$$

where r_π stands for the electromagnetic charge radius of the pion.

Let us first apply this relation to the muon magnetic moment. The hadronic contribution to $\frac{1}{2}(g-2)$ is given to the order of α^2 by[6]

$$\Delta a_\mu(\text{hadronic}) = \frac{1}{4\pi^3} \int_{4\mu^2}^\infty dt\, \sigma^{e^+e^-}(t) G(t), \qquad (7)$$

where

$$G(t) = \int_0^1 dz \frac{z^2(1-z)}{z^2 + (1-z)t/m_\mu^2}. \qquad (8)$$

Let us choose $f(t) = G(t)$ in (6) (which is positive definite in the integral region). We split the integral involving $\sigma^{\pi\pi}(t)$ into three regions (a) 280 MeV $< \sqrt{t} <$ 500 MeV, (b) 500 MeV $< \sqrt{t} <$ 1000 MeV, and (c) 1000 MeV $< \sqrt{t}$. The region (b), where the ρ meson is located, has been studied by applying the Chew-Low extrapolation to the reaction $\pi^- p \to \pi^- \pi^0 p$. We use the results of Baton *et al.* for this region, reinforcing them with a few other measurements.[7] There is no evidence for anomalous enhancement in the p wave near threshold, so we interpolate between the threshold and 500 MeV with a phase shift $\propto (t - 4\mu^2)^{3/2}$ incorporating the p-wave threshold behavior. We have little knowledge of the region (c) except that there are no conspicuous resonances up to $\sqrt{t} \simeq 2$ GeV. We therefore extrapolate $\sigma^{\pi\pi}(t)$ as a constant, using its value at the upper edge of the region (b) to higher energies until the unitarity bound is reached, and replace $\sigma^{\pi\pi}(t)$ by the unitarity bound above this point. This is very likely to overestimate the contribution from region (c) by a large amount so that it underestimates the lower bound. The lower bound on Δa_μ(hadronic) thus obtained is[8]

$$\Delta a_\mu(\text{hadronic}) > 2.4 r_\pi^4 \times 10^{-7}, \qquad (9)$$

where r_π is the pion charge radius in fermis. There are a few data for r_π, all of which were extrapolated from $\pi^\pm$ electroproduction. Two typical values are the following: (i) $r_\pi = 0.86 \pm 0.14$ F.[9] This was obtained through a dispersion relation of Zagury.[10] The value is considerably larger than the prediction of ρ dominance (0.63 F). (ii) The other is a very new result, $r_\pi = 0.60$ F (dipole fit) or 0.68 F (simple pole fit).[11] This analysis is based on Berends's dispersion relation incorporating the assumption of Born-diagram dominance.[12] The form factor is less accurate for small q^2 than for large q^2. If the value (i) were true, we would have a lower bound

$$\Delta a_\mu(\text{hadronic}) > (13.0^{+11.0}_{-7.6}) \times 10^{-8}, \qquad (10)$$

which should be compared with the ρ-dominance value calculated from Orsay data,[6,13] Δa_μ(hadronic) $= (5.4 \pm 0.3) \times 10^{-8}$. If one recalls our very conservative estimates in deriving the lower bound, (10) indicates that the hadronic contribution to Δa_μ might be much larger than the ρ-dominance value. Our lower bound is toward improving a small discrepancy between theory and experiment.[14] If the value (ii) were true, one could not derive an interesting lower bound from (6). But, as is mentioned in Ref. 11, the new data on the pion form factor are less accurate at low momentum transfers than at high momentum transfers. The slope of the form factor between the highest two points ($q^2 = 0.795$ and 1.188 GeV^2) is steeper by about 30% than that of a simple ρ-pole fit. Therefore, there will be a chance to obtain a meaningful bound from these data points using the inequality (5) when $\pi\pi$ scattering data become available up to the few-GeV region.

As a second application, we consider the possibility of testing some predictions on the asymptotic behavior of $\sigma^{e^+e^-}(t)$ for large t. For instance, let us choose $f(t) = 1$ in (b). The hadron annihilation cross section (sum of those through isovector and isoscalar photons) has been measured at $\sqrt{t} = 644$ MeV and 886 MeV and at three points in between.[1] We extrapolate $\sigma^{e^+e^-}$ at 644 MeV smoothly down to the threshold and evaluate $\int dt\, \sigma^{e^+e^-}$ between the threshold and $\sqrt{t} = 886$ MeV (because of the threshold behavior of $\sigma^{e^+e^-}$, an error introduced here is not so large). The integral involving $\sigma^{\pi\pi}(t)$ is evaluated in the same way as for the muon magnetic moment. Then we obtain a lower bound on the integrated cross section above $\sqrt{t} = 886$ MeV as

$$\int_{(886\text{ MeV})^2}^\infty dt\, \sigma^{e^+e^-}(t) \ge 5.6 r_\pi^4 \times 10^{-3} - \int_{4\mu^2}^{(886\text{ MeV})^2} dt\, \sigma^{e^+e^-}(t) = (5.6 r_\pi^4 - 1.0) \times 10^{-3}, \qquad (11)$$

where r_π is again in units of F. If r_π turns out to be as large as 0.86 F,[8] then (11) would set a lower bound

$$\int_{(886\text{ MeV})^2}^\infty dt\, \sigma^{e^+e^-}(t) \ge 2.1 \times 10^{-3}. \qquad (12)$$

It is always a difficult problem to find at what energy cross sections become asymptotic. If we assume that $\sigma^{e^+e^-}(t)$ becomes asymptotic above $\sqrt{t} \sim 1$ GeV and that $\sigma^{e^+e^-}(t)$ at 886 MeV $< \sqrt{t} < 1$ GeV is equal to the value reached by a smooth extrapolation of a Breit-Wigner shape, the left-hand side is equal to $\simeq 0.10 \times 10^{-3}$ for an asymptotic behavior of $\sigma^{e^+e^-}(t) = O(t^{-2})$ as $t \to \infty$ and about half of this for an asymptotic behavior of $\sigma^{e^+e^-}(t) = O(t^{-3})$.[15] Therefore, an asymptotic behavior like t^{-2} or t^{-3} would be ruled out. If the new data[11] are taken instead of the old ones,[9] we can still obtain a meaningful result from the two highest data. Choosing $q_1^2 = 0.795$ GeV^2 and $q_2^2 = 1.188$ GeV^2 [$F(q_1^2) = 0.400$ and $F(q_2^2) = 0.276$] with $f(t) = t$ in (5), we obtain

$$\int_{4\mu^2}^{\infty} dt\, t\sigma^{e^+e^-}(t) \geq 1.6 \times (\rho\text{-dominance value})$$

or

$$\int_{(886\ \mathrm{MeV})^2}^{\infty} dt\, t\sigma^{e^+e^-}(t) \geq 0.33 \times 10^{-3}\ \mathrm{GeV}^2. \qquad (13)$$

The same argument as above rules out an asymptotic behavior like t^{-3}. Because of the weight function $f(t)$ suppressing the unknown high-energy part of $\sigma^{\pi\pi}(t)$, we have succeeded in deriving a nontrivial result even from the new data.[11] As has been seen in the preceding discussions, if the slope of the pion form factor turns out to be consistent with or smaller than the ρ-dominance value, our whole discussion will collapse.

In principle, we can derive a similar formula for the nucleon charge radius. Although this quantity is known much more accurately, one would have the disadvantage of having to evaluate the large unphysical region between $4\mu^2$ and $4m_N^2$ of the $N\bar{N}$ amplitude.

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[3]Various theoretical models are summarized in the articles by J. J. Sakurai and R. Gatto, *International Symposium on Electron and Photon Interactions at High Energies, Liverpool, England, 1969*, edited by D. W. Braben and R. E. Rand (Daresbury Nuclear Physics Laboratory, Daresbury, Lancashire, England, 1970), pp. 91, 235.

[4]S. D. Drell and F. Zachariasen, Phys. Rev. 119, 463 (1960). A relation somewhat similar to ours was derived by F. Cooper and H. Pagels, Phys. Rev. D 2, 228 (1970). Their relation does not involve the $\pi\pi$ cross section, but is less stringent.

[5]A once-subtracted dispersion relation is assumed for $F(t)$. If the ρ-meson dominance were exact, (5) would become an equality.

[6]See, for instance, S. J. Brodsky and S. D. Drell, Ann. Rev. (N.Y.) Nucl. Sci. 20, 147 (1970). This contains further references.

[7]J. P. Baton *et al.*, Nucl. Phys. B3, 349 (1967). The $\pi^-\pi^0$ cross section and also the p-wave phase shift were determined by the Chew-Low extrapolation. This is in good agreement with other analyses by W. D. Walker *et al.*, Phys. Rev. Letters 18, 603 (1967), and N. N. Biswas *et al.*, Phys. Letters 27B, 513 (1968).

[8]In the narrow-width approximation to $\sigma^{\pi\pi}(t)$, the right-hand side of (9) is $2.7\, r_\pi^4 \times 10^{-7}$. The nonresonant p-wave background below the ρ region enhances the integral of $\sigma^{\pi\pi}(t)$ to push down the lower bound on Δa_μ.

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[14]$\Delta a_\mu^{\mathrm{theor}} - \Delta a_\mu^{\mathrm{exp}} = -(27 \pm 34) \times 10^{-8}$. See Ref. 6.

[15]According to Gatto,[3] $\sigma^{e^+e^-} = O(t^{-3})$ for a finite field algebra and $\sigma^{e^+e^-} = O(t^{-2})$ for a divergent field algebra.