PHYSICAL REVIEW D VOLUME 4, NUMBER 7 1 OCTOBER 1971

Coulomb and Mass-Difference Corrections to K^-p Scattering

G. C. Oades
Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England

and

G. Rasche
Institut für Theoretische Physik der Universität Zürich, Schönberggasse 9, CH-8001 Zürich
(Received 3 May 1971)

We present a treatment of low-energy K^-p reactions which includes nonrelativistic electromagnetic mass-difference effects and the effect of the Coulomb potential. We show how the corrections to the charge-independent parameters can be calculated in a first-order perturbation treatment. Our work extends the usual K-matrix treatment of these corrections by including both inner and outer corrections. Furthermore, we include these effects in the reaction channels as well as in the elastic channel.

I. INTRODUCTION

Data on the low-energy $\bar{K}N$ scattering and absorption processes have been the subject of many coupled-channel analyses[1] which aim to extract information on the charge-independent nuclear interactions of the $\bar{K}N$, $\pi\Sigma$, and $\pi\Lambda$ systems. It was soon realized that such analyses should include some allowance for electromagnetic effects, both via the attractive K^-p Coulomb interaction and via the relatively large mass difference between the K^-p and $\bar{K}^0n$ systems. The effects of these terms outside the range of the nuclear interaction have been taken into account via a zero-range K-matrix formalism by Jackson and Wyld[2] and via a more general K-matrix treatment by Dalitz and Tuan,[3] the formulas of the latter paper being used directly in most analyses.

Recently we have considered the problem of the Coulomb interaction and mass-difference effects for the coupled π^-p, π^0n systems[4] and now we extend these methods to the coupled $\bar{K}N$, $\pi\Sigma$, and $\pi\Lambda$ systems. We use a nonrelativistic coupled-channel Schrödinger equation to construct the full S matrix in terms of charge-independent parameters, the physically measured scattering and absorption cross sections being directly obtained from these S-matrix elements.

Our results represent an extension of the treatments given in Refs. 2 and 3: We include the effects of the Coulomb interaction in the $\pi^-\Sigma^+$ and $\pi^+\Sigma^-$ channels as well as in the K^-p channel, and we take into account mass differences between the $\pi^-\Sigma^+$, $\pi^+\Sigma^-$, and $\pi^0\Sigma^0$ systems as well as between the K^-p and $\bar{K}^0n$ systems. The Coulomb parameter for the $\pi^\pm\Sigma^\pm$ channels is considerably smaller than that for the K^-p channel, so the additional Coulomb effects may not be significant. However, the mass differences between the $\pi\Sigma$ systems are comparable to those between the $\bar{K}N$ systems, and it is not clear that their effects can be safely neglected. In addition, our treatment of the Coulomb and mass-difference effects covers the nuclear-interaction region where these effects were previously neglected. In πN scattering these inner Coulomb corrections are comparable to the outer corrections[5] and if a similar situation holds for the $\bar{K}N$, $\pi\Sigma$, and $\pi\Lambda$ systems, then the previous formalism will give misleading results due to their neglect.

It is generally accepted that the bulk of low-energy charge-dependent corrections are due to Coulomb-potential and mass-difference effects. This neglects finer details such as the γ processes of radiative capture and of bremsstrahlung. One also assumes that the interaction can be described by the sum of a charge-independent nuclear potential plus a Coulomb potential. This ignores charge-dependent corrections to the nuclear interaction which occur via the electromagnetic interactions of the exchanged particles. In nucleon-nucleon scattering, for example, such corrections occur due to $\pi^\pm$-π^0 mass differences.

A basic difficulty of any nonrelativistic treatment of the mass-difference problem is that there is no Galilean-invariant theory which can describe a scattering process where the sum of the rest masses in the initial state is not equal to the sum of the rest masses in the final state. This can easily be seen by observing that conservation of total 3-momentum is always covariant under Lorentz transformations, but is only covariant under Galilean transformations in the special case where the sum of the rest masses of the initial state is equal to the sum of the rest masses of the final state. Thus, we are faced with the problem that our nonrelativistic treatment is restricted to a

special frame which we choose to be the center-of-momentum frame.

This problem is already present in the treatment of πN mass differences in Ref. 4, but in this case it is less worrying since the mass differences are assumed to be electromagnetic in origin and so vanish for the purely nuclear system. In the case of the K^-p system the problem is more serious due to the "strong" mass differences between the $\bar{K}N$, $\pi\Sigma$, and $\pi\Lambda$ channels which remain even for purely nuclear scattering. However, these mass differences are small compared to the total rest mass in any channel, and so we believe that our nonrelativistic model is suitable for practical calculations in the low-energy region. We would like to emphasize that our aim is to improve the previous nonrelativistic treatments of Refs. 2 and 3 and that we do not claim to tackle the problem in a relativistic manner.

In Sec. II we fix our general notation and in Sec. III we describe the special case of purely nuclear scattering with no electromagnetic effects. In Sec. IV we introduce the Coulomb potential and electromagnetic mass differences within the nuclear-interaction region, and in Sec. V we treat the long-range part of these corrections. In Sec. VI we show how these corrections can be calculated to first order in the fine-structure constant and how they can be included in the analysis of experimental data. We hope later to publish a reanalysis of the low-energy $\bar{K}N$ data using the formalism developed here.

The formalism is developed in such a way that the method can be easily applied to any other strongly interacting multichannel system where mixing between states of the same total isospin is already present in the charge-independent description and where electromagnetic effects change this mixing as well as introducing further mixing between states of different total isospin.

II. GENERAL NOTATION

We will be concerned with pK^- scattering both in the elastic channel and in various reaction channels. Charge conservation requires that they have the same third component of isospin, $I_3 = 0$. This remains true even when electromagnetic interactions are introduced, and so we always work in this subspace of isospin space, and we suppress I_3 in our notation. Since the applicability of our nonrelativistic potential model is restricted to low energies, we consider only the lowest-mass states, i.e., pK^-, $n\bar{K}^0$, $\Sigma^+\pi^-$, $\Sigma^0\pi^0$, $\Sigma^-\pi^+$, and $\Lambda\pi^0$, and we neglect all other channels.

Using the Dirac bra and ket notation, we introduce a notation to deal with the six possible final states in isospin space by writing the physical states as

$$\begin{array}{cccccc} pK^-, & n\bar{K}^0, & \Sigma^+\pi^-, & \Sigma^0\pi^0, & \Sigma^-\pi^+, & \Lambda\pi^0, \\ |1\rangle, & |2\rangle, & |3\rangle, & |4\rangle, & |5\rangle, & |6\rangle, \end{array}$$

where we have written the ket symbol below the corresponding physical 2-particle state. The basis consisting of the $|i\rangle$, $i = 1, \ldots, 6$, we call the charge basis of isospin space.

The physical $\Sigma\pi$ states are linear superpositions of $I = 0, 1, 2$ states, the physical $N\bar{K}$ states are linear superpositions of $I = 0, 1$ states, and the physical $\Lambda\pi^0$ state is a pure $I = 1$ state. We can also introduce the isospin basis of isospin space with kets $|i)$, $i = 1, \ldots, 6$, where we use

$$\begin{array}{cccccc} I=2;\ \Sigma\pi, & & I=1;\ N\bar{K}, & & I=1;\ \Sigma\pi, & \\ |1), & & |2), & & |3), & \\ I=1;\ \Lambda\pi, & & I=0;\ N\bar{K}, & & I=0;\ \Sigma\pi, & \\ |4), & & |5), & & |6). & \end{array}$$

If we write

$$|j) = \sum_{i=1}^{6} |i\rangle\langle i|j)\,,$$

then the unitary transformation matrix is given by[6]

$$\langle i|j) = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 0 & 0 & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 0 & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{1}$$

The inverse transformation matrix $(j|i\rangle$ is given by the transpose of (1).

We will be dealing with the matrix elements of operators (e.g., $\underline{O}$) in isospin space which we write as $\langle i|\underline{O}|j\rangle$ or $(i|\underline{O}|j)$ depending on the basis used. The matrix elements in the two bases are easily related, e.g.,

$$(i|\underline{O}|j) = \sum_{k,l=1}^{6} (i|k\rangle\langle k|\underline{O}|l\rangle\langle l|j)\,. \tag{2}$$

A general state in isospin space is written as

$$|R\} = \sum_{i=1}^{6} \hat{R}_i |i\rangle = \sum_{j=1}^{6} \tilde{R}_j |j)\,,$$

where $\hat{R}_i$ and $\tilde{R}_j$ are the components of $|R\}$ in the charge and isospin bases. They are also easily related, e.g.,

$$\hat{R}_i = \sum_{j=1}^{6} \langle i|j)\tilde{R}_j\,. \tag{3}$$

We will work in the c.m. system and consider the model where the particles obey the nonrelativistic Schrödinger equation. References 2 and 3 also made this nonrelativistic assumption and for that reason both their corrections and the new corrections which we will derive are only expectec to be valid in the sense discussed in the Introduction. For the channel corresponding to the charge-basis state $|i\rangle$ we introduce m_i, the reduced mass, e.g.,

$$m_1 = \frac{m_p m_{K^-}}{m_p + m_{K^-}},$$

k_i, the relative momentum (we use units $\hbar = c = 1$ throughout), and $v_i = k_i/m_i$, the relative velocity. The mass operator $\underline{m}$ is defined so that the matrix elements are given by

$$\langle i|\underline{m}|j\rangle = m_i \delta_{ij}. \tag{4}$$

We also define an operator $\underline{D}$ with matrix elements

$$\langle i|\underline{D}|j\rangle = \delta_{ij}\left(\frac{d^2}{dr^2} + k_i^2\right). \tag{5}$$

Note that in general $\underline{m}$ and $\underline{D}$ are only diagonal in the charge basis.

III. THE PURELY NUCLEAR PROBLEM

For the purely nuclear problem, we neglect electromagnetic mass differences and define the masses as

$$m_K = m_{K^-}, \quad m_\Sigma = m_{\Sigma^+}, \quad m_\Lambda = m_\Lambda,$$
$$m_\pi = m_{\pi^-}, \quad m_N = m_p,$$

where the masses on the right-hand sides of these equations are the physical masses. The corresponding reduced mass for $|i\rangle$ is $\hat{m}_i^N$ and with our choice of the pure nuclear masses we have $\hat{m}_1^N = \hat{m}_2^N = m_1$, $\hat{m}_3^N = \hat{m}_4^N = \hat{m}_5^N = m_3$, and $\hat{m}_6^N = m_6$. In this case $\underline{m}^N$ is diagonal both in the charge and in the isospin bases. We correspondingly introduce operators $\underline{k}^N$ and $\underline{v}^N$ with matrix elements

$$\langle i|\underline{k}^N|j\rangle = \delta_{ij}\hat{k}_i^N,$$
$$\langle i|\underline{v}^N|j\rangle = \delta_{ij}\hat{v}_i^N,$$

where obviously

$$\hat{k}_1^N = \hat{k}_2^N = k_1, \quad \hat{k}_3^N = \hat{k}_4^N = \hat{k}_5^N = k_3, \quad \hat{k}_6^N = k_6,$$
$$\hat{v}_1^N = \hat{v}_2^N = v_1, \quad \hat{v}_3^N = \hat{v}_4^N = \hat{v}_5^N = v_3, \quad \hat{v}_6^N = v_6.$$

Again for this purely nuclear case $\underline{k}^N$ and $\underline{v}^N$ are also diagonal in the isospin basis with matrix elements

$$(i|\underline{k}^N|j) = \delta_{ij}\tilde{k}_i^N,$$
$$(i|\underline{v}^N|j) = \delta_{ij}\tilde{v}_i^N,$$

where

$$\tilde{k}_1^N = \tilde{k}_3^N = \tilde{k}_6^N = k_3, \quad \tilde{k}_2^N = \tilde{k}_5^N = k_1, \quad \tilde{k}_4^N = k_6,$$
$$\tilde{v}_1^N = \tilde{v}_3^N = \tilde{v}_6^N = v_3, \quad \tilde{v}_2^N = \tilde{v}_5^N = v_1, \quad \tilde{v}_4^N = v_6.$$

We also define the operator $\underline{D}^N$ with matrix elements in the charge basis

$$\langle i|\underline{D}^N|j\rangle = \delta_{ij}\left(\frac{d^2}{dr^2} + \hat{k}_i^{N2}\right) \tag{6a}$$

or in the isospin basis

$$(i|\underline{D}^N|j) = \delta_{ij}\left(\frac{d^2}{dr^2} + \tilde{k}_i^{N2}\right). \tag{6b}$$

Finally, since we describe the interaction by a potential, we introduce the charge-independent nuclear potential operator $\underline{U}$ which has nonzero matrix elements $(i|\underline{U}|j)$ only between states with the same total isospin [e.g., $(1|\underline{U}|2) = 0$, $(2|\underline{U}|3) \neq 0$]. To simplify our treatment of the Coulomb corrections, we assume that $\underline{U} = 0$ for $r \geq r_N$. Also for simplicity we consider only the case of s-wave scattering, the results for general l being obtained by a trivial generalization of our s-wave results.

With this notation the purely nuclear Schrödinger equation in the s-wave case has the form

$$\underline{D}^N|R^N\} = 2\underline{m}^N\underline{U}|R^N\}, \quad r \leq r_N$$
$$= 0, \quad r \geq r_N, \tag{7}$$

which corresponds in the charge basis to the following system of six coupled 2nd-order differential equations for the functions $\hat{R}_i^N(r)$:

$$\left(\frac{d^2}{dr^2} + \hat{k}_i^{N2}\right)\hat{R}_i^N(r) = 2\hat{m}_i^N\sum_{j=1}^{6}\langle i|\underline{U}|j\rangle\hat{R}_j^N(r), \quad r \leq r_N, \quad i = 1, \ldots, 6$$
$$= 0, \quad r \geq r_N. \tag{8}$$

$\hat{R}_i^N(r)/r$ are the radial parts of the wave functions in the corresponding channels. The regular solutions of (8) are those which vanish for $r = 0$:

$$\hat{R}_i^N(0) = 0, \quad i = 1, \ldots, 6.$$

The system of six coupled 2nd-order differential equations has six independent regular solutions which we label with a subscript α:

$$|R^N\}_\alpha = \sum_{i=1}^{6}\hat{R}_{i\alpha}^N|i\rangle = \sum_{j=1}^{6}\tilde{R}_{j\alpha}^N|j).$$

Charge independence ensures that in the isospin basis we have an uncoupled equation for $\tilde{R}_1^N$, three coupled equations for $\tilde{R}_2^N$, $\tilde{R}_3^N$, and $\tilde{R}_4^N$, and two coupled equations for $\tilde{R}_5^N$ and $\tilde{R}_6^N$. For this reason it is convenient to specify the regular solutions via the asymptotic behavior of $\tilde{R}_{i\alpha}^N$ rather than of $\hat{R}_{i\alpha}^N$. Guided by our knowledge that the S matrix can be diagonalized by an orthogonal transformation, we

choose our set such that

$$\tilde{R}^N_{i\alpha} = \frac{t^N_{i\alpha}}{(\tilde{v}^N_i)^{1/2}} \sin(\tilde{k}^N_i r + \delta^N_\alpha), \quad r \geq r_N, \quad i = 1, \ldots, 6 . \tag{9}$$

The matrix $t^N_{i\alpha}$ then decomposes into the direct sum of one 1-dimensional, one 3-dimensional, and one 2-dimensional submatrix. The nonvanishing elements can be chosen to be real, since the potential is real. Each solution $|R^N\rangle_\alpha$ can be multiplied by an arbitrary constant, so it is possible to fix the $t^N_{i\alpha}$ so that the following normalization condition holds:

$$\sum_{i=1}^{6} t^N_{i\alpha}\, t^N_{i\alpha} = 1 .$$

We show in Sec. VI that the Wronski conditions applied to the set of regular solutions (9) implies

$$\sum_{i=1}^{6} t^N_{i\alpha}\, t^N_{i\beta} = \delta_{\alpha\beta} . \tag{10}$$

Equation (10) shows that $t^N_{i\alpha}$ is a real orthogonal matrix. Since (10) leaves some sign conventions open, it is always possible to choose the $t^N_{i\alpha}$ so that they form a rotation matrix ($\det t^N = 1$). We can then write the nonvanishing matrix elements explicitly in the following form:

$$\begin{aligned}
t_{11} &= 1 , \\
t_{22} &= \cos\omega \cos\phi \cos\theta - \sin\omega \sin\theta , \\
t_{23} &= -\sin\omega \cos\phi \cos\theta - \cos\omega \sin\theta , \\
t_{24} &= \sin\phi \cos\theta , \\
t_{32} &= \cos\omega \cos\phi \sin\theta + \sin\omega \cos\theta , \\
t_{33} &= -\sin\omega \cos\phi \sin\theta + \cos\omega \cos\theta , \\
t_{34} &= \sin\phi \sin\theta , \\
t_{42} &= -\cos\omega \sin\phi , \\
t_{43} &= \sin\omega \sin\phi , \\
t_{44} &= \cos\phi , \\
t_{55} &= \cos\epsilon , \\
t_{56} &= -\sin\epsilon , \\
t_{65} &= \sin\epsilon , \\
t_{66} &= \cos\epsilon .
\end{aligned} \tag{11}$$

Here θ, ϕ, and ω are the three mixing angles associated with the 3-channel $I = 1$ subsystem. They correspond to the Euler angles used in 3-dimensional configuration space. The mixing angle associated with the 2-channel $I = 0$ subsystem is ϵ. It should be noted that mixing between states with the same total isospin but belonging to different representations of the isospin group occurs in the purely nuclear, charge-independent case for the $N\bar{K}$ system. This does not happen in the $N\pi$ system since in that case only one physically realized representation exists for each value of the total isospin.

A general regular solution can be written as

$$\tilde{R}^N_i = \sum_{\alpha=1}^{6} A_\alpha \tilde{R}^N_{i\alpha} , \quad i = 1, \ldots, 6 ,$$

where A_α are arbitrary constants. We see that this has the expected asymptotic behavior of a superposition of incoming and outgoing waves:

$$\tilde{R}^N_i \underset{r\to\infty}{\sim} \frac{\tilde{B}_i}{(\tilde{v}^N_i)^{1/2}} e^{-i\tilde{k}^N_i r} + \frac{\tilde{C}_i}{(\tilde{v}^N_i)^{1/2}} e^{i\tilde{k}^N_i r} , \quad i = 1, \ldots, 6 ,$$

where

$$\tilde{B}_i = -\frac{1}{2i} \sum_{\alpha=1}^{6} A_\alpha t^N_{i\alpha} e^{-i\delta^N_\alpha} , \quad i = 1, \ldots, 6 , \tag{12}$$

$$\tilde{C}_i = \frac{1}{2i} \sum_{\alpha=1}^{6} A_\alpha t^N_{i\alpha} e^{i\delta^N_\alpha} , \quad i = 1, \ldots, 6 . \tag{13}$$

The purely nuclear S matrix relates $\tilde{B}_i$ and $\tilde{C}_i$ by

$$\tilde{C}_i = -\sum_{j=1}^{6} (i|\underline{S}^N|j)\tilde{B}_j \tag{14}$$

and so, substituting (12) and (13) into (14), we obtain

$$(i|\underline{S}^N|j) = \sum_{\alpha,\beta=1}^{6} t^N_{i\alpha}\, \delta_{\alpha\beta} e^{2i\delta^N_\alpha} (t^{N-1})_{\beta j} . \tag{15}$$

One sees from (15) that $e^{2i\delta^N_\alpha}$ are the eigenvalues of the S matrix in the purely nuclear case, δ^N_α being the eigenphases. These six eigenphases, together with the four mixing angles, are the charge-independent parameters by which $K^- p$ scattering at low energies can be expressed.

IV. THE INTRODUCTION OF INNER CORRECTIONS

We now introduce the Coulomb potential operator $\underline{V}$ with matrix elements

$$\langle i|\underline{V}|j\rangle = \delta_{ij} V_i \tag{16}$$

and we assume that, beyond some charge radius r_c, $\underline{V}$ behaves like the point-charge potential operator

$$\langle i|\underline{V}|j\rangle = \delta_{ij} \frac{\eta_i k_i}{m_i r} , \quad r \geq r_c \tag{17}$$

where η_i is the Coulomb parameter associated with the channel $|i\rangle$

$$\begin{aligned}
\eta_i &= -\frac{e^2 m_i}{k_i} , \quad i = 1, 3, 5 \\
&= 0 , \qquad i = 2, 4, 6 .
\end{aligned} \tag{18}$$

We start by introducing the Coulomb potential and mass-difference effects for $r \leq r_0$, where

$$r_0 = \max(r_N, r_c) .$$

In this case the s-wave Schrödinger equation becomes

$$\begin{aligned} \underline{D}|R^{\rm in}\} &= 2\underline{m}\,\underline{U}|R^{\rm in}\} + 2\underline{m}\,\underline{V}|R^{\rm in}\}, \quad r \le r_0 \\ \underline{D}^N|R^{\rm in}\} &= 0, \qquad\qquad\qquad\quad r \ge r_0 . \end{aligned} \tag{19}$$

This we rewrite as

$$\begin{aligned} \underline{D}^N|R^{\rm in}\} &= 2\underline{m}^N\underline{U}|R^{\rm in}\} + \underline{\Delta}|R^{\rm in}\}, \quad r \le r_0 \\ &= 0, \qquad\qquad\qquad\quad r \ge r_0 , \end{aligned} \tag{20}$$

where

$$\underline{\Delta} = 2\underline{m}\,\underline{V} + 2(\underline{m} - \underline{m}^N)\underline{U} - (\underline{k}^2 - \underline{k}^{N2}) . \tag{21}$$

As before there are six independent regular solutions which we again label with subscript α:

$$|R^{\rm in}\}_\alpha = \sum_{i=1}^{6} \hat{R}^{\rm in}_{i\alpha}|i\rangle = \sum_{j=1}^{6} \tilde{R}^{\rm in}_{j\alpha}|j) .$$

Guided by perturbation theory, we specify these regular solutions by the behavior of $\tilde{R}^{\rm in}_{i\alpha}$ for $r \ge r_0$ which we choose as

$$\tilde{R}^{\rm in}_{i\alpha} = \frac{t^{\rm in}_{i\alpha}}{(\tilde{v}^N_i)^{1/2}} \sin(\tilde{k}^N_i r + \delta^{\rm in}_\alpha), \quad r \ge r_0, \quad i = 1, \ldots, 6 . \tag{22}$$

We can again choose the normalization so that, taking into account the Wronski conditions, we have

$$\sum_{i=1}^{6} t^{\rm in}_{i\alpha}\, t^{\rm in}_{i\beta} = \delta_{\alpha\beta} .$$

The $\delta^{\rm in}_\alpha$, $\alpha = 1, \ldots, 6$, are the new eigenphases and the matrix $t^{\rm in}_{i\alpha}$ specifies the new mixing between states of the same isospin and also the additional mixing which now occurs between states of different isospin. Values for $\delta^{\rm in}_\alpha$ and $t^{\rm in}_{i\alpha}$ can either be obtained by solving the finite-range equations (20) exactly or via perturbation theory as described in Sec. VI. Proceeding exactly as in the purely nuclear case, we can obtain the new S matrix

$$(i|\underline{S}^{\rm in}|j) = \sum_{\alpha,\beta=1}^{6} t^{\rm in}_{i\alpha}\delta_{\alpha\beta} e^{2i\delta^{\rm in}_\alpha} (t^{\rm in\,-1})_{\beta j} . \tag{23}$$

V. DERIVATION OF THE S MATRIX WITH INNER AND OUTER CORRECTIONS

We now consider the problem where the Coulomb potentials and mass-difference effects are included for all r. In this case the s-wave Schrödinger equation becomes

$$\begin{aligned} \underline{D}|R^{\rm tot}\} &= 2\underline{m}\,\underline{U}|R^{\rm tot}\} + 2\underline{m}\,\underline{V}|R^{\rm tot}\}, \quad r \le r_0 \\ &= 2\underline{m}\,\underline{V}|R^{\rm tot}\}, \qquad\qquad r \ge r_0 . \end{aligned} \tag{24}$$

$\underline{V}$ is diagonal in the charge basis so we can write a general solution for $r \ge r_0$ in the form

$$\hat{R}^{\rm tot}_i = \frac{t^{\rm tot}_i}{(v_i)^{1/2}} \left[\cos\delta^{\rm tot}_i F_0(\eta_i, k_i r) + \sin\delta^{\rm tot}_i G_0(\eta_i, k_i r)\right], \quad r \ge r_0 , \tag{25}$$

where $F_0(\eta_i, k_i r)$ and $G_0(\eta_i, k_i r)$ are the s-wave regular and irregular Coulomb radial wave functions.[6] Note that in the channels with $\eta_i = 0$

$$F_0(\eta_i, k_i r) = \sin(k_i r) ,$$
$$G_0(\eta_i, k_i r) = \cos(k_i r) , \quad i = 2, 4, 6 .$$

For $r \le r_0$, $|R^{\rm tot}\}$ satisfies the same set of coupled equations as $|R^{\rm in}\}$ and so we can choose our six independent regular solutions to match onto $|R^{\rm in}\}_\alpha$ at $r = r_0$. Thus we write

$$\hat{R}^{\rm tot}_{i\alpha} = \frac{t^{\rm tot}_{i\alpha}}{(v_i)^{1/2}} \left[\cos\delta^{\rm tot}_{i\alpha} F_0(\eta_i, k_i r) + \sin\delta^{\rm tot}_{i\alpha} G_0(\eta_i, k_i r)\right], \quad r \ge r_0, \quad i = 1, \ldots, 6 , \tag{26}$$

and fix $t^{\rm tot}_{i\alpha}$ and $\delta^{\rm tot}_{i\alpha}$ by the conditions

$$\begin{aligned} &\hat{R}^{\rm tot}_{i\alpha}(r_0) = \hat{R}^{\rm in}_{i\alpha}(r_0) , \\ &\frac{d}{dr}\hat{R}^{\rm tot}_{i\alpha}(r)\Big|_{r=r_0} = \frac{d}{dr}\hat{R}^{\rm in}_{i\alpha}(r)\Big|_{r=r_0}, \quad i = 1, \ldots, 6 . \end{aligned} \tag{27}$$

As before, the general regular solution is expressed as

$$\hat{R}^{\rm tot}_i = \sum_{\alpha=1}^{6} A_\alpha \hat{R}^{\rm tot}_{i\alpha} ,$$

where the A_α are arbitrary constants. Using the asymptotic form of the Coulomb radial wave functions, we see that the asymptotic form of $\hat{R}^{\rm tot}_i$ corresponds to a superposition of incoming and outgoing Coulomb radial waves

$$\begin{aligned} \hat{R}^{\rm tot}_i \underset{r\to\infty}{\sim} &\frac{\hat{B}_i}{(v_i)^{1/2}} \exp[-i(k_i r + \sigma_{i0} - \eta_i \ln 2k_i r)] \\ &+ \frac{\hat{C}_i}{(v_i)^{1/2}} \exp[i(k_i r + \sigma_{i0} - \eta_i \ln 2k_i r)] \\ &\qquad\qquad i = 1, \ldots, 6 , \end{aligned}$$

where

$$\hat{B}_i = -\frac{1}{2i}\sum_{\alpha=1}^{6} A_\alpha t^{\rm tot}_{i\alpha} e^{-i\delta^{\rm tot}_{i\alpha}}, \quad i = 1, \ldots, 6 ,$$

$$C_i = \frac{1}{2i}\sum_{\alpha=1}^{6} A_\alpha t^{\rm tot}_{i\alpha} e^{i\delta^{\rm tot}_{i\alpha}}, \quad i = 1, \ldots, 6 ,$$

and where the s-wave Coulomb phases σ_{i0} are given by

$$\sigma_{i0} = \arg\Gamma(1 + i\eta_i) .$$

If we define $\underline{Q}^{\rm tot}$ and $\underline{Q}^{\rm tot*}$ by

$$\langle i|\underline{Q}^{\rm tot}|j\rangle = t^{\rm tot}_{ij} e^{i\delta^{\rm tot}_{ij}} , \tag{28}$$

$$\langle i|\underline{Q}^{\rm tot*}|j\rangle = t^{\rm tot}_{ij} e^{-i\delta^{\rm tot}_{ij}} , \tag{29}$$

then the S matrix, which relates $\hat{B}_i$ and $\hat{C}_i$ via

$$\hat{C}_i = -\sum_{j=1}^{6} \langle i|\underline{S}^{\rm tot}|j\rangle \hat{B}_j ,$$

is given by

$$\underline{S}^{\rm tot} = \underline{Q}^{\rm tot}(\underline{Q}^{\rm tot*})^{-1} . \tag{30}$$

By making use of the Wronski conditions on the $\hat{R}^{\rm tot}_{i\alpha}$, one can prove that

$$(\underline{Q}^*)^T\underline{Q} = \underline{Q}^T\underline{Q}^*$$

from which the symmetry and unitarity of $\underline{S}^{\rm tot}$ at once follows. Note the changed situation where the $\delta^{\rm tot}_{i\alpha}$ now form a 6×6 matrix and no longer correspond to the six eigenphases of $\underline{S}^{\rm tot}$.

VI. PERTURBATION TREATMENT OF THE CORRECTIONS AND THEIR INCLUSION IN THE SCATTERING AMPLITUDES

From the isospin-basis equivalent of (8), we have for our regular solutions

$$W[\tilde{R}^N_{i\alpha}, \tilde{R}^N_{i\beta}]_{r=r_0} = 2\tilde{m}^N_i \int_0^{r_0} dr \sum_{j=1}^{6} (i|\underline{U}|j)(\tilde{R}^N_{i\alpha}\tilde{R}^N_{j\beta} - \tilde{R}^N_{i\beta}\tilde{R}^N_{j\alpha}) , \tag{31}$$

where W is the usual Wronskian. Using (9) to evaluate the left-hand side of (31) then gives

$$t^N_{i\alpha}t^N_{i\beta} = \frac{2}{\sin(\delta^N_\alpha - \delta^N_\beta)} \times \int_0^{r_0} dr \sum_{j=1}^{6} (i|\underline{U}|j)(\tilde{R}^N_{i\alpha}\tilde{R}^N_{j\beta} - \tilde{R}^N_{i\beta}\tilde{R}^N_{j\alpha}) , \quad \alpha \neq \beta , \tag{32}$$

where we have made use of

$$\tilde{k}^N_i = \tilde{m}^N_i \tilde{v}^N_i .$$

We now use the time-reversal-invariance condition

$$(i|\underline{U}|j) = (j|\underline{U}|i)$$

in (32) to obtain

$$\sum_{i=1}^{6} t^N_{i\alpha}t^N_{i\beta} = 0 , \quad \alpha \neq \beta .$$

If we combine this with the normalization condition

$$\sum_{i=1}^{6} t^N_{i\alpha}t^N_{i\alpha} = 1 ,$$

we finally obtain

$$\sum_{i=1}^{6} t^N_{i\alpha}t^N_{i\beta} = \delta_{\alpha\beta} . \tag{33}$$

We can similarly obtain

$$\sum_{i=1}^{6} t^{\rm in}_{i\alpha}t^{\rm in}_{i\beta} = \delta_{\alpha\beta} . \tag{34}$$

If we now make use of the Wronskian between $\tilde{R}^N_{i\alpha}$ and $\tilde{R}^{\rm in}_{i\beta}$ using the isospin-basis equivalents of (8) and (20), we obtain

$$\sum_{i=1}^{6} t^N_{i\alpha}t^{\rm in}_{i\beta} \sin(\delta^N_\alpha - \delta^{\rm in}_\beta) = \int_0^{r_0} dr \sum_{i,j=1}^{6} \frac{(i|\underline{\Delta}|j)}{\tilde{m}^N_i} \tilde{R}^N_{i\alpha}\tilde{R}^{\rm in}_{j\beta} . \tag{35}$$

We first take the case $\alpha = \beta$ in (35), where we can use (34) to derive the first-order perturbation result

$$\delta^{\rm in}_\alpha = \delta^N_\alpha - \int_0^{r_0} dr \sum_{i,j=1}^{6} \frac{(i|\underline{\Delta}|j)}{\tilde{m}^N_i} \tilde{R}^N_{i\alpha}\tilde{R}^N_{j\alpha} . \tag{36}$$

We now write

$$t^{\rm in}_{i\alpha} = t^N_{i\alpha} + \epsilon_{i\alpha} \tag{37}$$

and use (33) and (34) to obtain, to first order in ϵ,

$$\sum_{i=1}^{6} t^N_{i\alpha}\epsilon_{i\alpha} = 0 . \tag{38}$$

If we substitute (37) into (35) and use (38), we obtain the perturbation result

$$\sum_{i=1}^{6} t^N_{i\alpha}\epsilon_{i\beta} = X_{\alpha\beta} - \delta_{\alpha\beta} , \tag{39}$$

where

$$X_{\alpha\alpha} = 1$$

and

$$X_{\alpha\beta} = \frac{1}{\sin(\delta^N_\alpha - \delta^N_\beta)} \int_0^{r_0} dr \sum_{i,j=1}^{6} \frac{(i|\underline{\Delta}|j)}{\tilde{m}^N_i} \tilde{R}^N_{i\alpha}\tilde{R}^N_{j\beta} , \quad \alpha \neq \beta . \tag{40}$$

Using the fact that $t^N_{i\alpha}$ is an orthogonal matrix then enables us to solve (39) and obtain

$$\epsilon_{i\beta} = \sum_{\alpha=1}^{6} t^N_{i\alpha}(X_{\alpha\beta} - \delta_{\alpha\beta}) . \tag{41}$$

Using these results, we can now calculate the physical S-matrix elements $\langle i|\underline{S}^{\rm tot}|j\rangle$ from the charge-independent nuclear S-matrix elements $(i|\underline{S}^N|j)$ in the following way. We proceed by first using (15) to obtain the charge-independent eigenphases δ^N_α and the charge-independent mixing matrix $t^N_{i\alpha}$ from $(i|\underline{S}^N|j)$. We parametrize $\tilde{R}^N_{i\alpha}$ for $r \leq r_0$ and, knowing δ^N_α and $t^N_{i\alpha}$, adjust the parameters to fit the boundary conditions at $r = r_0$ given by (9).

We now use these values of $\tilde{R}^N_{i\alpha}$ in (36) to obtain first-order perturbation values for $\delta^{\rm in}_\alpha$ and in (40) and (41) to obtain first-order perturbation values for $t^{\rm in}_{i\alpha}$. These values can then be used in (22) to form $\tilde{R}^{\rm in}_{i\alpha}$ for $r \geq r_0$, and we then use (3) to transform to $\hat{R}^{\rm in}_{i\alpha}$ for $r \geq r_0$. Finally from the $\hat{R}^{\rm in}_{i\alpha}$ we obtain $\delta^{\rm tot}_{i\alpha}$ and $t^{\rm tot}_{i\alpha}$ using the boundary conditions (27) at $r = r_0$, and from these values construct the matrix $\underline{Q}$ given by (28), and hence, obtain the matrix

elements $\langle i|\underline{S}^{\rm tot}|j\rangle$ from (30).

We have shown how the physical S-matrix elements can be obtained in the s-wave case. The corresponding general $l, j = l \pm \frac{1}{2}$ values $\langle i|\underline{S}^{\rm tot}_{l\pm}|j\rangle$ can be obtained in a similar way simply by replacing $\sin(k_i r)$ by $k_i r j_l(k_i r)$, $\cos(k_i r)$ by $-k_i r n_l(k_i r)$, $F_0(\eta_i, k_i r)$ by $F_l(\eta_i, k_i r)$, and $G_0(\eta_i, k_i r)$ by $G_l(\eta_i, k_i r)$ where appropriate. Here j_l and n_l are the regular and irregular spherical Bessel functions and F_l and G_l are the regular and irregular Coulomb radial wave functions.

The usual treatment of spin-$\frac{1}{2}$–spin-0 scattering in the presence of the long-range Coulomb potential[7] shows that the scattering amplitude for the process (charge-basis state i) → (charge-basis state f) is given by

$$F_{i\to f} = f^{fi} + i\vec{\sigma}\hat{n}g^{fi},$$

where

$$\hat{n} = \frac{\vec{k}_i \times \vec{k}_f}{|\vec{k}_i \times \vec{k}_f|},$$

$\vec{k}_i$ and $\vec{k}_f$ being the initial and final meson c.m. momenta.

The partial-wave decompositions of f^{fi} and g^{fi} have the form

$$f^{fi} = \delta_{fi} f^{ii}_{\rm Coulomb} + \sum_{l=0}^{\infty} [(l+1)f^{fi}_{l+} + l f^{fi}_{l-}] P_l(\cos\theta),$$

$$g^{fi} = \delta_{fi} g^{ii}_{\rm Coulomb} + \sum_{l=1}^{\infty} [f^{fi}_{l+} - f^{fi}_{l-}] P^1_l(\cos\theta),$$

where θ is the angle between $\vec{k}_i$ and $\vec{k}_f$ and where $f^{ii}_{\rm Coulomb}$ and $g^{ii}_{\rm Coulomb}$ are the non-spin-flip and spin-flip amplitudes corresponding to pure electromagnetic scattering. In our case these correspond to the pure Coulomb amplitudes

$$f^{ii}_{\rm Coulomb} = -\frac{\eta_i}{2k_i \sin^2(\frac{1}{2}\theta)} \exp[-i\eta_i \ln\sin^2(\tfrac{1}{2}\theta) + 2i\sigma_{i0}],$$

$$g^{ii}_{\rm Coulomb} = 0.$$

Magnetic-moment effects and further relativistic corrections can be included,[8] at which stage a non-zero contribution to $g^{ii}_{\rm Coulomb}$ is also obtained.

The partial-wave amplitudes $f^{fi}_{l\pm}$ are given by

$$f^{fi}_{l\pm} = e^{i(\sigma_{il}+\sigma_{fl})} \frac{\langle f|\underline{S}^{\rm tot}_{l\pm} - \underline{1}|i\rangle}{2ik_i},$$

where σ_{il} is the angular momentum l Coulomb phase in the i channel and is given by

$$\sigma_{il} = \arg\Gamma(l+1+i\eta_i).$$

Finally, the differential cross section is given by

$$\frac{d\sigma}{d\Omega_{i\to f}} = |f^{fi}|^2 + |g^{fi}|^2.$$

ACKNOWLEDGMENTS

This paper was written while one of the authors (G.C.O.) was working at the University of Zurich. He would like to thank Professor W. Heitler for the hospitality of his Institute during this visit and the Swiss National Foundation for financial help.

---

[1]For a review of low-energy K-matrix analyses see B. R. Martin, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1970), Vol. 55.

[2]J. D. Jackson and H. W.Wyld, Phys. Rev. Letters 2, 355 (1959); Nuovo Cimento 13, 85 (1959).

[3]R. H. Dalitz and S. F. Tuan, Ann. Phys. (N.Y.) 10, 307 (1960).

[4]G. C. Oades and G. Rasche, Helv. Phys. Acta 44, 5 (1971); 44, 160 (1971). Coulomb corrections for the single-channel case have been considered by many authors. For an interesting comparison of some of the more recent papers, see S. Rae and P. Thurnauer, Phys. Rev. 180, 1387 (1969).

[5]G. Rasche, Nuovo Cimento 62A, 229 (1969).

[6]For a convenient summary of the properties of Coulomb wave functions, see *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards, Applied Mathematics Series, No. 55 (U. S. Government Printing Office, Washington, D. C., 1964), Chap. 14, p. 537.

[7]See, for example, Ref. 4.

[8]For the corresponding treatment of the πN case, see F. T. Solmitz, Phys. Rev. 94, 1799 (1954).