scription III is consistent with a flat $\Sigma^- p$ cross section which is roughly equal to the $\Sigma^+ p$ cross section; the other two prescriptions seem to imply a slowly rising $\Sigma^- p$ cross section roughly $\frac{1}{3}$ in magnitude of the $\Sigma^+ p$ cross section in the same momentum range. The only data consist of a few events at very low energy[12] and do not really confront the sum rules. Flat baryon-baryon cross sections are predicted by exchange-degeneracy models[13]; and perhaps one intuitively feels that isospin would lead to $\Sigma^- p$ and $\Sigma^+ p$ cross sections nearly equal at high energies. But a true test must await the hyperon-beam experiments. It is clear that even rough data on the $\Sigma^- p$ cross section would provide a test of prescription III versus I and II.

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Exact Inequality and Test of Chiral SW(3) Theory in K_{l3} Decay Problem*

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Assuming the chiral SW(3) model, exact inequalities for the K_{l3} scalar form factor $D(t)$ and its derivatives have been studied. We can sharpen our inequalities considerably if we take into account the soft-pion theorem. These inequalities are stringent enough to test the validity of the chiral SW(3) theory.

I. INTRODUCTION AND SUMMARY OF PRINCIPAL RESULTS

The purpose of this paper is to study further applications of the exact inequality which was proved elsewhere[1] for the K_{l3} problem, assuming the validity of the chiral SW(3) model[2] of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg (hereafter referred to as GMORGW). In particular, we estimate upper bounds for higher-order derivatives of the K_{l3} scalar form factor $D(t)$ (see below) and we discuss in some detail consequences of the soft-pion theorem. We find that our inequality is stringent enough to be barely satisfied by the present experimental data, and we suggest a simple form for $D(t)$.

As usual, we define the standard K_{l3} form factor $f_\pm(t)$ by

$$\langle \pi^0(p')|V_\mu^{(4-i5)}(0)|K^+(p)\rangle = -(4p_0p_0'V^2)^{-1/2}(\tfrac{1}{2})^{1/2}[(p+p')_\mu f_+(t) + (p-p')_\mu f_-(t)],$$
$$t=-(p-p')^2 . \quad (1.1)$$

We are especially interested in the scalar form factor $D(t)$ given by

$$D(t) = (m_K^2 - m_\pi^2) f_+(t) + t f_-(t) . \quad (1.2)$$

Moreover, let us set

$$f_\pm(t) = f_\pm(0)\left[1 + \lambda_\pm\left(\frac{t}{m_\pi^2}\right) + \alpha_\pm\left(\frac{t}{m_\pi^2}\right)^2 + O(t^4)\right],$$
$$\xi = f_-(0)/f_+(0) , \quad (1.3)$$
$$\Lambda_0 = m_\pi^2 D'(0)/D(0) = \lambda_+ + \xi m_\pi^2 (m_K^2 - m_\pi^2)^{-1} ,$$
$$\rho = \tfrac{1}{2} m_\pi^4 D''(0)/D(0) = \alpha_+ + \lambda_- \xi m_\pi^2 (m_K^2 - m_\pi^2)^{-1}$$

Then, we can prove the following inequality:

$$|\Lambda_0 - 0.018|^2 + |86.5\rho - 3.55\Lambda_0 + 0.011|^2 \leq \{[3.13 - 0.67/f_+(0)]^2 - 1\} \times 1.32 \times 10^{-4} , \quad (1.4)$$

where the numerical estimate of the right-hand side depends on the experimental value

$$f_K/f_\pi f_+(0) \approx 1.28 . \quad (1.5)$$

In particular, this inequality independent of the value of ρ gives

$$-0.098 \leq \xi + 12.3\lambda_+ \leq 0.54 \quad (1.6)$$

for $f_+(0)$ in the range $f_+(0) \leq 1.0$. We can improve this estimate considerably if we assume the validity of the soft-pion theorem[3]

$$\frac{D(\delta)}{D(0)} = \frac{f_K}{f_\pi f_+(0)}[1 + O(m_\pi^2)] ,$$
$$\delta = m_K^2 - m_\pi^2 . \quad (1.7)$$

In that case, we obtain a stringent bound

$$0.062 \leq \xi + 12.3\lambda_+ \leq 0.234 . \quad (1.8)$$

These bounds are barely satisfied at the lower end by the present world average value:

$$\xi + 12.3\lambda_+ = -0.29 \pm 0.24 . \quad (1.9)$$

On the basis of this analysis, we suggest a simple approximate form of $D(t)$ given by

$$D(t) = \text{const}\,[1 + (t_0 - t)^{1/2}(t_0 - t_1)^{-1/2}]^{-1/2} , \quad (1.10)$$

where t_0 and t_1 are

$$t_0 = (m_K + m_\pi)^2 , \quad t_1 = (m_K - m_\pi)^2 \quad (1.11)$$

The only essential assumption used in the derivation of our inequality Eq. (1.4) is the following: Let us set

$$A_{\alpha\beta} = i\int d^4x \langle 0 | (\partial_\mu A_\mu^{(\alpha)}(x), \partial_\nu A_\nu^{(\beta)}(0))_+ | 0\rangle ,$$
$$V_{\alpha\beta} = i\int d^4x \langle 0 | (\partial_\mu V_\mu^{(\alpha)}(x), \partial_\nu V_\nu^{(\beta)}(0))_+ | 0\rangle , \quad (1.12)$$

where $V_\mu^{(\alpha)}(x)$ and $A_\mu^{(\alpha)}(x)$ $(\alpha = 1, \ldots, 8)$ are the standard vector and axial-vector currents. Then, by means of the Kamefuchi-Umezawa-Lehmann-Källén representation (hereafter KULK), we have

$$A_{33} = \tfrac{1}{2} m_\pi^2 f_\pi^2 + \delta_\pi ,$$
$$A_{44} = \tfrac{1}{2} m_K^2 f_K^2 + \delta_K , \quad (1.13)$$

where δ_π and δ_K are the non-negative contributions of multimeson intermediate states to A_{33} and A_{44}, respectively.

Ordinarily, we neglect δ_π and δ_K. However, for our purpose it is enough to assume a weaker ansatz

$$\delta_\pi \geq \delta_K \geq 0, \quad A_{44} \geq A_{33} \geq 0 . \quad (1.14)$$

Then, for the GMORGW model, we can prove

$$0 \leq V_{44} \leq \tfrac{1}{2}(m_K f_K - m_\pi f_\pi)^2 . \quad (1.15)$$

Equation (1.15) is our only essential assumption. Our method is based on a consideration of the Hilbert space H^2 which consists of all analytic functions inside the unit circle $|z| < 1$ with a suitable definition of the inner product.

II. INEQUALITY FOR K_{l3} PARAMETERS

Before going into details, it is convenient to state some mathematical preliminaries. Let $D(\xi)$ be a real analytic function of the complex variable ξ with a cut on the real axis at $t_0 \leq \xi \leq \infty$. Suppose moreover that for a non-negative weight function $k(\xi)$ defined on the cut $t_0 \leq \xi \leq \infty$, the integral

$$I^2 = \frac{1}{\pi}\int_{t_0}^{\infty} d\xi\, k(\xi) |D(\xi)|^2 \quad (2.1)$$

is given. The reality condition for $D(\xi)$ implies

$$D^*(\xi^*) = D(\xi) \quad (2.2)$$

in the cut plane; therefore, the path of integration in Eq. (2.1) can be taken along either the upper or the lower cut. Now, we ask whether we can give some upper bounds for the function $D(t)$ and its derivatives below the threshold $t < t_0$. The answer is affirmative, as has been shown in (I). As a matter of fact, we can prove a stronger Parseval equality of the form

$$\sum_{n=0}^{\infty} |h_n(t)|^2 = I^2 A^2 , \quad (2.3)$$

where $h_n(t)$ is a linear combination of $D^{(m)}(t)$ given by the form

$$h_n(t) = \sum_{m=0}^{n} \gamma_{nm}(t)(t_0 - t)^m \frac{d^m}{dt^m} D(t) . \quad (2.4)$$

In the above, $\gamma_{nm}(t)$ $(n \geq m)$ and A are calculable in terms of the weight function $k(\xi)$ as follows: First, define a function $w(\theta)$ $(0 \leq \theta \leq 2\pi)$ by

$$w(\theta) = (\xi - t)\left(\frac{\xi - t_0}{t_0 - t}\right)^{1/2} k(\xi) \quad (\xi \geq t_0 \geq t)\ ,$$
$$\xi = t_0 + (t_0 - t)\cot^2(\tfrac{1}{2}\theta)\ . \tag{2.5}$$

Hereafter, we denote by t a fixed real point below the cut, i.e., $t < t_0$, unless otherwise stated. When we set

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{(-ni\theta)} \ln w(\theta) \tag{2.6}$$

for n equal to a natural non-negative integer, then we can calculate A and γ_{nm} (up to n, $m=3$) to be

$$\begin{aligned}
&A = e^{(-b_0/2)}\\
&\gamma_{00} = 1,\quad \gamma_{11} = -4,\quad \gamma_{10} = b_1,\quad \gamma_{22} = 8\ ,\\
&\gamma_{21} = -4(b_1+2),\quad \gamma_{20} = b_2 + \tfrac{1}{2}(b_1)^2\ ,\\
&\gamma_{33} = -\tfrac{32}{3},\quad \gamma_{32} = 8(b_1+4)\ ,\\
&\gamma_{31} = -2[2 + 2b_2 + (b_1+2)^2]\ ,\\
&\gamma_{30} = b_3 + b_1 b_2 + \tfrac{1}{2}(b_1)^3\ .
\end{aligned} \tag{2.7}$$

The derivation of these formulas will be given in the Appendix.

For many physical applications, $k(\xi)$ has the simple form

$$k(\xi) = C\prod_{j=1}^{n}(\xi - t_j)^{\alpha_j} \quad (t_j \leq t_0)\ . \tag{2.8}$$

Then, we find (see Appendix)

$$\begin{aligned}
&A = [4C(t_0 - t)]^{-1/2}\prod_{j=1}^{n}(t_0 - t)^{-\alpha_j/2}(1+\beta_j)^{-\alpha_j}\ ,\\
&b_1 = 2\sum_{j=1}^{n}\alpha_j(1+\beta_j)^{-1} + 2\ ,\\
&b_2 = 2\sum_{j=1}^{n}\alpha_j\beta_j(1+\beta_j)^{-2} + \tfrac{1}{2}\ ,\\
&b_3 = \tfrac{2}{3}\sum_{j=1}^{n}\alpha_j(1+3\beta_j^{\,2})(1+\beta_j)^{-3} + \tfrac{2}{3}\ ,
\end{aligned} \tag{2.9}$$

where β_j is defined by

$$\beta_j = (t_0 - t_j)^{1/2}(t_0 - t)^{-1/2}\ . \tag{2.10}$$

In actual applications, we use only the weaker Bessel inequality

$$\sum_{n=0}^{N}|h_n(t)|^2 \leq I^2 A^2\ , \tag{2.11}$$

where N is an arbitrary non-negative integer. Equation (2.11) is obviously a special case of Eq. (2.3). As noted before,[1] the case $N=0$ gives immediately the result obtained originally by Meiman.[4]

Now, let us apply this formula to the K_{l3} problem. We identify $D(\xi)$ with the scalar form factor defined in Eq. (1.2). Furthermore, we choose

$$k(\xi) = (3/64\pi)\,\xi^{-2}(\xi - t_0)^{1/2}(\xi - t_1)^{1/2}\ , \tag{2.12}$$

where t_0 and t_1 are given by (1.11). The reason for this choice is as follows. If we set

$$\Delta(q^2) = \tfrac{1}{2}i\int d^4x\, e^{iq(x-y)} \times \langle 0|(\partial_\mu V_\mu^{(4-i5)}(x), \partial_\nu V_\nu^{(4-i5)}(y))_+|0\rangle\ , \tag{2.13}$$

then the KULK representation gives us

$$\Delta(q^2) = \int_{t_0}^{\infty} dt\,\frac{\rho(t)}{t+q^2}\ , \tag{2.14}$$

where the spectral weight function $\rho(t)$ is expressed as

$$\rho(-q^2) = \tfrac{1}{2}(2\pi)^3\sum_n |\langle 0|\partial_\mu V_\mu^{(4-i5)}(0)|n\rangle|^2\delta^4(p_n - q)\ . \tag{2.15}$$

Because of positivity, we can omit all intermediate states other than π-K states in Eq. (2.15), to find an inequality

$$I^2 \leq \Delta(0) = V_{44}\ , \tag{2.16}$$

if we choose the form (2.12); this was first noted by Li and Pagels.[5] Hence from Eqs. (2.11) and (2.9), we find

$$\sum_{n=0}^{N}|h_n(t)|^2 \leq I^2A^2 \leq K^2\ , \tag{2.17}$$

where K is given by

$$\begin{aligned}
&K = 4[\tfrac{1}{3}\pi\Delta(0)]^{1/2}(1+\beta_2)^2(1+\beta_1)^{-1/2}\ ,\\
&\beta_1 = (t_0 - t_1)^{1/2}(t_0 - t)^{-1/2}\ ,\\
&\beta_2 = t_0^{\,1/2}(t_0 - t)^{-1/2}\ .
\end{aligned} \tag{2.18}$$

We remark that if the π-K intermediate state dominates the spectral weight function $\rho(t)$ of (2.15), then we should have the equality $IA = K$ in Eq. (2.17).

Now we evaluate $\Delta(0)$ in the following model-independent way. In the GMORGW model, we must have an inequality,[6]

$$[\Delta(0)]^{1/2} = (V_{44})^{1/2} \leq |(A_{44})^{1/2} - (A_{33})^{1/2}|\ , \tag{2.19}$$

as was noted in (I). Hence, if we express A_{33} and A_{44} as in Eq. (1.13) and if we assume the validity of Eq. (1.14), we find

$$[\Delta(0)]^{1/2} \leq (m_K f_K - m_\pi f_\pi)/\sqrt{2} \tag{2.20}$$

independently of the explicit values of δ_π and δ_K. Notice that the inequality $\delta_\pi \geq \delta_K$ has been suggested by Olshansky and Kang[7] and by Acharya[8] from the consideration of asymptotic symmetry. At any rate, we expect both δ_π and δ_K to be relatively small,[9] and Eq. (2.20) will be approximately valid even if the above arguments should fail.

For comparison, we mention an estimate for $\Delta(0)$ given elsewhere.[10]

$$[\Delta(0)]^{1/2} \approx 1.01 m_\pi f_\pi\,, \quad f_+(0) = 0.85\,. \qquad (2.21)$$

This value is one-half smaller than that of Eq. (2.20). However, since several calculations[1,5,11,12] have already been completed on the basis of Eq. (2.21), we will not consider this estimate unless otherwise so stated. At any rate, we believe that Eq. (2.20) is likely to give an over-estimate for $\Delta(0)$.

From Eqs. (2.17) and (2.20), we find

$$\sum_{n=0}^{N} |h_n(t)|^2 \leq R^2\,, \qquad (2.22)$$

where R is given by

$$R = 4[(6/\pi)(1+\beta_1)]^{-1/2}(1+\beta_2)^2(m_K f_K - m_\pi f_\pi)\,. \qquad (2.23)$$

We remark that in deriving Eq. (2.22) we made use of several estimates, various contributions being dropped in each step. First, we omitted contributions to $\sum_n |h_n(t)|^2$ for $n \geq N+1$ (see Eq. 2.3); second, we used the inequality $IA \leq K$ (see Eq. 2.16); finally, we exploited $K \leq R$ (see Eq. 2.20). Therefore, there is ample reason to believe that the upper bound R^2 in Eq. (2.22) is an over-estimate. Nevertheless, we shall see shortly that Eq. (2.22) gives a very stringent inequality.

First, let us consider the case $N=0$ and set $t=0$. Then Eq. (2.22) gives us

$$|f_+(0)| \leq M\,, \qquad (2.24)$$

where M is computed to be

$$M = 16[(6/\pi)(1+\beta_1)]^{-1/2}(m_K{}^2 - m_\pi{}^2)^{-1}(m_K f_K - m_\pi f_\pi)\,,$$
$$\beta_1 = (4 m_K m_\pi)^{1/2}(m_K + m_\pi)^{-1}\,. \qquad (2.25)$$

Using the experimental value $f_K/f_\pi f_+(0) = 1.28$, we find the numerical estimate

$$M = 3.13 f_+(0) - 0.67\,, \qquad (2.26)$$

where we assumed $f_+(0) \geq 0$ in conformity with the SU(3) limit value of $f_+(0)=1$. Then, Eqs. (2.24) and (2.26) give the lower bound

$$f_+(0) \geq 0.31\,, \qquad (2.27)$$

which is not so good since the Ademollo-Gatto theorem[13] leads us to expect $f_+(0) \approx 1$.

Next, let us consider the cases $N=1$ and $N=2$. Setting $t=0$, Eq. (2.22) gives us

$$|\Lambda_0 - C_1| \leq X[(M/f_+(0))^2 - 1]^{1/2}\,, \qquad (2.28)$$

$$|\Lambda_0 - C_1|^2 + |(\rho/X) - (2+b_1)\Lambda_0 + C_2|^2 \leq X^2[(M/f_+(0))^2 - 1]\,, \qquad (2.29)$$

where Λ_0 and ρ are defined by Eq. (1.3) and we have used the simplifying notation:

$$\begin{aligned} X &= \tfrac{1}{4} m_\pi{}^2 (m_K + m_\pi)^{-2}\,, \\ C_1 &= b_1 X \approx 0.018\,, \\ C_2 &= (2\beta_1 + \tfrac{3}{2})(1+\beta_1)^{-2} X \approx 0.011\,, \\ b_1 &= 1 + (1+\beta_1)^{-1} \approx 1.55\,. \end{aligned} \qquad (2.30)$$

Inserting numerical values of C_1, C_2, and X, Eq. (2.29) reduces to Eq. (1.4) of the Introduction. Similarly, for values of $f_+(0) \leq 1.0$, we find from Eq. (2.28)

$$-0.008 \leq \Lambda_0 \leq 0.044 \qquad (2.31)$$

which gives the estimate Eq. (1.6). The best world average value[14] for Λ_0 is given by

$$\Lambda_0 = -0.024 \pm 0.02 \qquad (2.32)$$

so that our inequality Eq. (2.31) is roughly satisfied, near the lower bound. We may remark that, if we had used the stronger estimate Eq. (2.21), we would have obtained

$$0.014 \leq \Lambda_0 \leq 0.022 \qquad (2.33)$$

which is outside the experimental error of Eq. (2.32). At any rate, these considerations, together with another estimate to be given in Sec. III, strongly suggest $\Lambda_0 \approx 0$. Then Eq. (2.29) leads to

$$-3.5 \times 10^{-4} \leq \rho \leq 0.5 \times 10^{-4}\,. \qquad (2.34)$$

Chien *et al.*[15] find $\lambda_+ = 0.026 \pm 0.006$ and $\alpha_+ = 0.0045 \pm 0.0015$ from K_{e3} analysis. If these values are accepted, then Eq. (2.34) together with $\Lambda_0 \approx 0$ gives us

$$\begin{aligned} -0.41 &\leq \xi \leq 0.21\,, \\ -0.060 &\leq \lambda_- \xi \leq -0.055\,. \end{aligned} \qquad (2.35)$$

For $\xi = -0.4$, this gives a very large value for λ_-:

$$0.15 \geq \lambda_- \geq 0.14\,.$$

So far we have investigated the case $t=0$. Now, we shall proceed to a consideration of the inequalities for $t \neq 0$. First, at the soft-pion point $t = \delta = m_K{}^2 - m_\pi{}^2$, Eq. (2.22) with $N=0$ is easily seen[1] to be satisfied by Eq. (2.27), and we will not discuss it further. Since the left-hand side of Eq. (2.22) is a quadratic form in $D^{(n)}(t)$ [see Eq. (2.4)], we can minimize the expression with respect to unwanted variables. For example, consider the case $N=2$ and take the minimum over variation of $D'(t)$. In this way, we find

$$[1+(b_1+2)^2]|D(t)|^2 + |8(t_0-t)^2 D''(t) + \tilde{\gamma}_{20} D(t)|^2 \leq I^2 A^2 [1+(b_1+2)^2]\,,$$
$$\tilde{\gamma}_{20} \equiv \gamma_{20} - b_1(b_1+2)\,. \qquad (2.36)$$

Furthermore, we can minimize the left-hand side of this expression by varying $D(t)$; we thus obtain

$$8(t_0-t)^2|D''(t)| \le IA[1+(b_1+2)^2+(\tilde{\gamma}_{20})^2]^{1/2} . \tag{2.37}$$

From Eq. (2.36) we can estimate

$$-0.011 \le m_\pi^{\ 2}D''(0) \le 0.037 \tag{2.38a}$$

for $f_+(0)=0.9$, while we find

$$-0.012 \le m_\pi^{\ 2}D''(0) \le 0.042 \tag{2.38b}$$

for $f_+(0)=1.0$.

Also, from Eq. (2.37), we estimate

$$\begin{aligned} &m_\pi^{\ 2}\operatorname*{Max}_{|t|\le t_1}|D''(t)| \le 0.115 \text{ or } 0.132, \\ &m_\pi^{\ 2}\operatorname*{Max}_{|t|\le \delta}|D''(t)| \le 0.375 \text{ or } 0.430 \end{aligned} \tag{2.39}$$

for values of $f_+(0)=0.9$ and 1.0, respectively. Writing

$$D(t)=D(0)+D'(0)t+\epsilon_1 , \tag{2.40}$$

we have of course

$$|\epsilon_1| \le (1/2!)\,\text{Max}\,|D''(t)t^2| .$$

Therefore, the error in neglecting the quadratic term ϵ_1 is bounded by

$$\begin{aligned} &|\epsilon_1|/D(0) \le 0.27 \text{ for } |t|\le t_1 , \\ &|\epsilon_1|/D(0) \le 2.64 \text{ for } |t|\le \delta \end{aligned} \tag{2.41}$$

for values of $f_+(0)=0.9-1.0$. This implies that the neglect of the quadratic terms even in the physical region $m_l^{\ 2} \le t \le t_1$ could cause a maximum error of 27%. But, for the test of the soft-pion theorem at $t=\delta$, the linear approximation could cause a very large error of more than 200%. The possible importance of the quadratic term in testing the soft-pion theorem Eq. (1.7) has been emphasized by many authors.[8,12,14,16]

Similarly we can estimate the third-order derivative. Setting

$$\begin{aligned} &K_1=\gamma_{11}\gamma_{22}\gamma_{33} , \\ &K_2=(\gamma_{11}\gamma_{22})^2+(\gamma_{11}\gamma_{32})^2+(\gamma_{22}\gamma_{31}-\gamma_{32}\gamma_{21})^2 , \\ &K_3=\gamma_{11}\gamma_{22}\gamma_{30}-\gamma_{11}\gamma_{32}\gamma_{20}-\gamma_{22}\gamma_{10}\gamma_{31}+\gamma_{32}\gamma_{21}\gamma_{10} , \end{aligned} \tag{2.42}$$

we obtain

$$\begin{aligned} &K_2|D(t)|^2+|K_1(t_0-t)^3D'''(t)+K_3D(t)|^2 \le K_2 I^2A^2 , \\ &(t_0-t)^3|K_1||D'''(t)| \le [K_2+(K_3)^2]^{1/2} IA . \end{aligned} \tag{2.43}$$

From these, we find for $f_+(0)=0.9$

$$-0.0016 \le m_\pi^{\ 4}D'''(0) \le 0.0046 \tag{2.44}$$

as well as

$$m_\pi^{\ 4}\,\text{Max}\,|D'''(t)| = \begin{cases} 0.021, & |t|\le t_1 \\ 0.108, & |t|\le \delta . \end{cases} \tag{2.45}$$

The bound Eq. (2.44) for $D'''(0)$ can be quite restrictive as is the case in the model of Mathur and Yang.[16] Expanding

$$D(t)=D(0)+D'(0)t+\tfrac{1}{2}D''(0)t^2+\epsilon_2 , \tag{2.46}$$

we calculate

$$\begin{aligned} &|\epsilon_2|/D(0) \le 11\% \text{ for } |t|\le t_1 , \\ &|\epsilon_2|/D(0) \le 300\% \text{ for } |t|\le \delta . \end{aligned} \tag{2.47}$$

Hence we conclude that even the inclusion of the quadratic term will not improve the error at all if we neglect the effects of third- and higher-order terms. This strongly suggests that the perturbation expansion of $D(t)$ in terms of t is a very slowly converging one, at least near the soft-pion point $t=\delta$.

Up to now, we have considered only the spin-zero part $D(t)$. We shall briefly consider the spin-one part $f_+(t)$. First, let us set

$$\begin{aligned} &[\delta_{\mu\nu}+(1/t)k_\mu k_\nu]\rho^{(1)}(t)+(1/t)k_\mu k_\nu\,\rho^{(0)}(t) \\ &\quad=\tfrac{1}{2}(2\pi)^3\sum_n \langle 0|V_\mu^{(4-i5)}(0)|n\rangle\langle n|V_\nu^{(4+i5)}(0)|0\rangle\,\delta^4(p_n-k) , \\ &\qquad t=-k^2 . \end{aligned} \tag{2.48}$$

Then, the standard positivity argument gives us for $t\ge t_0$

$$\rho^{(1)}(t) \ge (1/128\,\pi^2)t^{-2}(t-t_0)^{3/2}(t-t_1)^{3/2}|f_+(t)|^2 \tag{2.49}$$

when we consider only π-K intermediate-state contributions to $\rho^{(1)}(t)$.

Now, we identify $D(\xi)$ with $f_+(\xi)$ and choose $k(\xi)$ to be

$$k(\xi)=(1/128\,\pi)\xi^{-(n+2)}(\xi-t_0)^{3/2}(\xi-t_1)^{3/2} . \tag{2.50}$$

Then, we find

$$I^2=\frac{1}{\pi}\int_{t_0}^{\infty} d\xi\, k(\xi)|f_+(\xi)|^2 \le \Delta_n , \tag{2.51}$$

where Δ_n is given by

$$\Delta_n=\int_{t_0}^{\infty} d\xi\,\xi^{-n}\rho^{(1)}(\xi) . \tag{2.52}$$

Unfortunately, we have no exact inequality for Δ_n. We take the view that the K^*-dominance approximation for Δ_n will be reasonable although it may not be a good one for three-point vertex functions. Then, if we set

$$\langle 0|V_\mu^{(4-i5)}(0)|K^{*+}(q)\rangle=(2q_0V)^{-1/2}\sqrt{2}\,g_{K^*}\epsilon_\mu(q) , \tag{2.53}$$

we estimate

$$\Delta_n \approx g_{K^*}{}^2 (m_{K^*})^{-2n} \ .$$

Moreover, assuming[17] the validity of the Weinberg sum rule as well as the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin formula, we find

$$g_{K^*}{}^2/m_{K^*}{}^2 \approx g_\rho{}^2/m_\rho{}^2 \approx f_\pi{}^2 \ .$$

Therefore, we obtain

$$I \lesssim f_\pi (m_{K^*})^{-(n-1)} \ . \qquad (2.54)$$

On the basis of this estimate together with the general formulas, we find

$$-0.039 \le m_\pi{}^2 f_+'(0) \le 0.115 \ , \qquad -0.007 \le m_\pi{}^4 f_+''(0) \le 0.013 \qquad (2.55a)$$

for the case $n=1$, while we have

$$-0.14 \le m_\pi{}^2 f_+'(0) \le 0.174 \ , \qquad -0.013 \le m_\pi{}^4 f_+''(0) \le 0.015 \qquad (2.55b)$$

for $n=3$. Our estimate of λ_+ given by Eq. (2.55) is not precise enough. We note that Mathur[18] derived a better bound

$$m_\pi{}^2 f_+'(0) \lesssim 0.06 \qquad (2.56)$$

by taking a more elaborate choice of $k(\xi)$.

III. USE OF SOFT-PION THEOREM

As we have seen in Sec. II, the testing of the soft-pion theorem by means of the linear extrapolation of $D(t)$ can be very dangerous and untrustworthy. Here, we shall utilize the theorem as input information. In principle, this is new information, and hence it will improve the bound of Sec. II. Mathematically, we want to find a bound for $D(t)$ and its derivatives using not only the integral I but also $D(\delta)$ ($\delta = m_K{}^2 - m_\pi{}^2$) as known quantities. In this section, we shall report a new bound which exploits the soft-pion theorem Eq. (1.7). Although it may not be the best inequality we can derive, we do find that the results become more stringent as we expected.

To this end, it is convenient to map the cut ξ plane onto the inside of a unit circle $|z|<1$ by the following conformal transformation (see Appendix):

$$(\xi - t_0)^{1/2} = i(t_0 - t)^{1/2} \frac{1+z}{1-z} \quad (t<t_0) \ . \qquad (3.1)$$

In terms of this transformation, let us set

$$F(z) \equiv D(\xi) \ . \qquad (3.2)$$

Moreover, for a non-negative function $\rho(\theta)$ ($0 \le \theta \le 2\pi$) to be determined shortly, let us set

$$g(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\rho(\theta)\right) \ . \qquad (3.3)$$

Then, it is known[19] that $g(z)$ satisfies

$$|g(e^{i\theta})| = \rho(\theta) \qquad (3.4)$$

under suitable conditions. Now setting

$$H(z) = g(z)[F(z)]^2 \qquad (3.5)$$

and applying the Cauchy theorems to $H(z)$, we find

$$H(z) - H(0) = \frac{z}{2\pi}\int_0^{2\pi} d\theta \frac{1}{e^{i\theta}-z} H(e^{i\theta}) \ . \qquad (3.6)$$

Now, choosing z to be real, we find

$$|H(z)-H(0)| \le \frac{|z|}{2\pi}\int_0^{2\pi} d\theta [1+z^2-2z\cos\theta]^{-1/2} \rho(\theta)|F(e^{i\theta})|^2 \qquad (3.7)$$

where we have used Eq. (3.4). Choosing $\rho(\theta)$ to be

$$\rho(\theta) = (1+z^2-2z\cos\theta)^{1/2} w(\theta) \qquad (3.8)$$

for a fixed real value of z, Eq. (3.7) is rewritten as

$$|g(z)[F(z)]^2 - g(0)[F(0)]^2| \le |z| I^2 \ , \qquad (3.9)$$

where I^2 is the same quantity as that defined in Sec. II, i.e. (see Appendix),

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, w(\theta) |F(e^{i\theta})|^2 = \frac{1}{\pi}\int_{t_0}^{\infty} d\xi\, k(\xi) |D(\xi)|^2 \ . \qquad (3.10)$$

Again, if we choose $k(\xi)$ in the form of Eq. (2.8), then we compute

$$g(z) = 4C(t_0 - t)\left(\frac{1+z}{1-z}\right)^2 \prod_{j=1}^{n} (t_0 - t)^{\alpha_j} \left(\frac{1+z}{1-z} + \beta_j\right)^{2\alpha_j} \qquad (3.11)$$

as we shall prove in the Appendix.

For the K_{l3} problem we are considering, $k(\xi)$ has the explicit form Eq. (2.12); therefore, we have

$$g(z) = \frac{3\pi}{(16\pi)^2}(1+\beta_1) G(z) \ ,$$
$$G(z) = \left(1 + \frac{1-\beta_1}{1+\beta_1} z\right)(1+z)^3 \ , \qquad (3.12)$$
$$\beta_1 = (t_0 - t_1)^{1/2} t_0{}^{-1/2} \ ,$$

where for simplicity we have chosen $t=0$. Equation (3.9) becomes then

$$|f_+(0)| |1 - G(z)[D(\xi)/D(0)]^2|^{1/2} \le |z|^{1/2} M \ , \qquad (3.13)$$

where M is given by Eq. (2.25). We remark that this equation reduces to Eq. (2.24) when we set $z=-1$. Now, choose ξ to be the soft-pion point $\xi = \delta = m_K{}^2 - m_\pi{}^2$, so that the corresponding value of z is given by

$$z=\frac{\beta_s-1}{\beta_s+1}, \quad \beta_s=\left(\frac{t_0-\delta}{t_0}\right)^{1/2} . \tag{3.14}$$

Using the soft-pion theorem Eq. (1.7) together with Eq. (2.26) gives a lower bound for $f_+(0)$ indistinguishable from that given by Eq. (2.27),

$$f_+(0) \geq 0.31 . \tag{3.15}$$

Next, let us consider the first-order-derivative equation

$$H(z)-H(0)-zH'(0)=\frac{z^2}{2\pi}\int_0^{2\pi} d\theta \frac{1}{e^{i\theta}-z} H(e^{i\theta}) . \tag{3.16}$$

Then, using the same technique with $t=0$, we obtain

$$|f_+(0)| \left| G(z)\left[\frac{D(\xi)}{D(0)}\right]^2 - 1 + 8zt_0\left[\frac{D'(0)}{D(0)}\right] - zG'(0)\right|^{1/2} \leq |z| M . \tag{3.17}$$

Using the soft-pion theorem and setting $f_+(0) \approx 0.9$ leads to

$$0.005 \leq m_\pi^2 D'(0)/D(0) \leq 0.019 \tag{3.18}$$

or, equivalently,

$$0.062 \leq \xi + 12.3\lambda_+ \leq 0.234 \tag{3.19}$$

which improves considerably the bound of Sec. II. This condition is barely satisfied at the lower bound by the present experimental data.

IV. CONCLUDING REMARKS

As we have seen in the previous sections, our inequality is stringent enough to test the chiral SW(3) model. In particular, the estimate $[\Delta(0)]^{1/2} \approx 1.01 m_\pi f_\pi$ may be in conflict with experiment. A similar remark is also applicable to the combination of the maximal estimate Eq. (2.20) and the soft-pion theorem. At any rate, our sum rule Eq. (2.22) appears to be already nearly saturated with $N=1$. If this is so, then it implies first that just π-K intermediate states are enough to dominate the spectral weight function $\rho(t)$ of $\Delta(q^2)$. Second, the value of $\Delta(0)$ must be near the maximal value

$$\Delta(0) \approx (m_K f_K - m_\pi f_\pi)/\sqrt{2} . \tag{4.1}$$

If this is really the case, then it follows[1] from the derivation of the inequality Eq. (2.19) that the parameter of Ref. 10 satisfies $a \approx b$. This is rather unwelcome since on physical grounds we expect to have $a \approx -1$ and $b \approx 0$. However, in view of the present experimental uncertainty, this conjecture is perhaps premature. However, without assuming any specific form for the Hamiltonian, such as the GMORGW model, the value of $\Delta(0)$ is usually found[1] to be less than the maximal value Eq. (4.1). This is understandable for the following reason. Independent of any specific Hamiltonian, we expect to have

$$\Delta(0)=V_{44} \approx \tfrac{1}{2} m_\kappa{}^2 f_\kappa{}^2 \tag{4.2}$$

if we use κ dominance to evaluate V_{44}. Hence, assuming $m_\kappa \approx 1100\,\text{MeV}$ and $f_\kappa \approx 0.3\, f_\pi$, we obtain

$$[\Delta(0)]^{1/2} \approx 1.70\, m_\pi f_\pi \tag{4.3}$$

which is certainly smaller than our maximal value in Eq. (4.1). Hence, the approximate saturation of our inequality for $N=1$ will still be plausible in any model. If we believe this, then Eq. (2.3) implies $h_n(0) \equiv 0$ for $n \geq 2$. As we shall prove in the Appendix, this is possible only if $D(\xi)$ has the form

$$D(\xi)=(a+bz)(1+z)^{-1}\left(\beta_1+\frac{1+z}{1-z}\right)^{-1/2} , \tag{4.4}$$

where a and b are constants and

$$\frac{1+z}{1-z}=\left(\frac{t_0-\xi}{t_0}\right)^{1/2}, \quad \beta_1=\left(\frac{t_0-t_1}{t_0}\right)^{1/2} . \tag{4.5}$$

At $\xi=t_0$ or $z=-1$, Eq. (4.4) tells us that $D(\xi)$ will be divergent. Since this is unlikely, we set $b=a$; therefore, $D(\xi)$ is given by

$$D(\xi)=a\left[\beta_1+\left(\frac{t_0-\xi}{t_0}\right)^{1/2}\right]^{-1/2} . \tag{4.6}$$

This reproduces Eq. (1.10) of the Introduction. $D(\xi)$, as given in Eq. (4.6), has the right analytic property with respect to ξ. Also, it follows that

$$D(\delta)/D(0) \approx 1.11 \tag{4.7}$$

at the soft-pion point. This is smaller by 15%, compared to the value 1.28 given by the exact soft-pion theorem Eq. (1.7). Similarly, we find

$$m_\pi{}^2 D'(0)/D(0)=m_\pi{}^2(4t_0)^{-1}(1+\beta_1)^{-1} \approx 0.0063 . \tag{4.8}$$

This value is consistent with Eq. (3.18). Also, it is amusing to see that it gives a value similar to that of the Dashen-Weinstein relation[20]

$$D'(0)=\tfrac{1}{2}[(f_K/f_\pi)-(f_\pi/f_K)]+O(\epsilon^2) , \tag{4.9}$$

where ϵ is the SW(3)-violating parameter. Hence, Eq. (4.8) will give an approximate expression for $D(t)$ provided that our estimate Eq. (4.8) turns out to be experimentally correct.

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APPENDIX

Here, we shall prove many relations used in the text. Although Eqs. (2.1)–(2.11) are more or less given in (I), we shall briefly sketch the proof in order to make this paper as self-contained as possible. Besides, in the course of the proof, we shall discover several other useful relations to be used in some other proofs.

Let $f(z)$ and $g(z)$ be analytic functions inside the unit circle $|z|<1$ such that they have finite boundary values almost everywhere on the boundary. We shall define an inner product (g,f) by

$$(g,f)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, g^*(e^{i\theta})f(e^{i\theta})\ . \tag{A1}$$

All such analytic functions $f(z)$ with the finite norm $(f,f)^{1/2}$ are known[19,21] to form a Hilbert space denoted conventionally by H^2. Notice that the polynomials

$$g_n(z)=z^n\quad (n=0,1,2,\ldots) \tag{A2}$$

form an orthonormal set in this space since we have by an elementary calculation

$$(g_n,g_m)=\delta_{n,m}\quad (n,m=0,1,2,\ldots)\ . \tag{A3}$$

Moreover, the set composed of all polynomials $g_n(z)$ is complete[19,21] in the H^2 space. Hence, any function $f(z)$ belonging to H^2 can be expanded in terms of g_n:

$$f=\sum_{n=0}^{\infty}a_n g_n\,,\quad a_n=(g_n,f)\ . \tag{A4}$$

Then, the standard Parseval equality gives us

$$(f,f)=\sum_{n=0}^{\infty}|(g_n,f)|^2\ . \tag{A5}$$

If follows easily from the Cauchy formula that

$$(g_n,f)=(1/n!)f^{(n)}(0)\ . \tag{A6}$$

Therefore, Eq. (A5) can now be rewritten as

$$I^2=\sum_{n=0}^{\infty}\left|\frac{1}{n!}f^{(n)}(0)\right|^2\ , \tag{A7}$$

where we have set

$$I^2=(f,f)=\frac{1}{2\pi}\int_0^{2\pi}d\theta|f(e^{i\theta})|^2\ . \tag{A8}$$

In particular, this gives the Bessel inequality

$$I^2\geq\sum_{n=0}^{N}\left|\frac{1}{n!}f^{(n)}(0)\right|^2 \tag{A9}$$

for any non-negative integer N.

Now, let $w(\theta)$ $(0\leq\theta\leq 2\pi)$ be a non-negative function defined on the unit circle such that both $w(\theta)$ and $\ln w(\theta)$ are summable on the circle. Then, the function defined by

$$\varphi(z)=\exp\left(\frac{1}{4\pi}\int_0^{2\pi}d\theta\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln w(\theta)\right) \tag{A10}$$

is known[19] to belong to H^2 with the boundary-value condition

$$|\varphi(e^{i\theta})|^2=w(\theta) \tag{A11}$$

almost everywhere on the unit circle. Further, when we set

$$f(z)=\varphi(z)F(z)\ , \tag{A12}$$

then Eq. (A11) implies

$$I^2=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, w(\theta)|F(e^{i\theta})|^2\ . \tag{A13}$$

Also, Eq. (A7) becomes

$$I^2=\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\varphi^{(n-m)}(0)F^{(m)}(0)\right|^2. \tag{A14}$$

Before going into details, it is convenient to define the parameters A and b_n by

$$A=[\varphi(0)]^{-1}\ ,\qquad b_n=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{-ni\theta}\ln w(\theta)\ ; \tag{A15}$$

then we can show

$$b_0=2\ln\varphi(0)\ ,\qquad b_n=\frac{1}{n!}\left(\frac{d^n}{dz^n}\ln\varphi(z)\right)_{z=0}\quad (n\geq 1)\ ,\qquad A=e^{-b_0/2} \tag{A16}$$

so that, for example,

$$\varphi'(0)=b_1\varphi(0)\ ,\qquad \varphi''(0)=(2b_2+b_1^2)\varphi(0)\ ,\qquad \varphi'''(0)=[6(b_3+b_1b_2)+b_1^3]\,\varphi(0)\ . \tag{A17}$$

Now, let $D(\xi)$ be a real analytic function in the complex ξ plane with a cut on the real axis $t_0\leq\xi\leq\infty$. Then, we define the conformal transformation

$$(\xi-t_0)^{1/2}=i(t_0-t)^{1/2}\frac{1+z}{1-z}\ , \tag{A18}$$

where t is a real fixed value satisfying $t<t_0$. By this mapping, the whole cut ξ plane is mapped inside the unit circle $|z|<1$. In addition, upper and lower cuts at $t_0\leq\xi\leq\infty$ are mapped into the lower and upper semicircle $|z|=1$, respectively, and the three points $\xi=\infty$, t_0, and t are transformed into $z=1$, -1, and 0, respectively. In terms of our mapping, we shall set

$$F(z)\equiv D(\xi)\ . \tag{A19}$$

Then, if we define (on the cut)

$$w(\theta)=\left(\frac{\xi-t_0}{t_0-t}\right)^{1/2}(\xi-t_0)k(\xi)\quad(\;\geq t_0)\,,$$
$$\xi-t_0=(t_0-t)\cot^2(\tfrac{1}{2}\theta)\,, \tag{A20}$$

we find

$$I^2=\frac{1}{\pi}\int_{t_0}^{\infty}d\xi\,k(\xi)|D(\xi)|^2 \tag{A21}$$

if we utilize the reality condition

$$D^*(\xi^*)=D(\xi)\,. \tag{A22}$$

Moreover, we note

$$F(0)=D(t)\,,$$
$$F'(0)=-4(t_0-t)D'(t)\,,$$
$$F''(0)=16(t_0-t)^2D''(t)-16(t_0-t)D'(t)\,, \tag{A23}$$
$$F'''(0)=-64(t_0-t)^3D'''(t)+192(t_0-t)^2D''(t)-72(t_0-t)D'(t)\,.$$

The relations Eqs. (A14), (A16), (A21), and (A23), reproduce Eq. (2.1), (2.3), (2.4), and (2.7).

Now, suppose that $k(\xi)$ has the form

$$k(\xi)=C\prod_{j=1}^{n}(\xi-t_j)^{\alpha_j}\quad(t_j\leq t_0)\,. \tag{A24}$$

Then Eq. (A20) leads to

$$\ln w(\theta)=\ln[4C(t_0-t)]+\ln|1+z|-3\ln|1-z|+\sum_{j=1}^{n}\alpha_j\left(\ln(t_0-t)+\ln|1-\beta_j^2|-2\ln|1-z|+\ln\left|1+\frac{1+\beta_j}{1-\beta_j}z\right|+\ln\left|1+\frac{1-\beta_j}{1+\beta_j}z\right|\right), \tag{A25}$$

where we have set

$$z=e^{i\theta}\,,\qquad \beta_j=(t_0-t_j)^{1/2}(t_0-t)^{-1/2}\,. \tag{A26}$$

We can evaluate the explicit form of $\varphi(z)$ by means of the Jensen-Poisson integral formula. Suppose that $G(z)$ is analytic inside the closed unit circle $|z|\leq 1$. Then, it has the representation[22]

$$G(z)=a\left(\prod_n\frac{\lambda_n-z}{1-\lambda_n^*z}\right)\exp\left(\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln|G(e^{i\theta})|\right), \tag{A27}$$

where a is a constant with unit modulus (i.e., $|a|=1$) and the λ_n are the zero points of $G(z)$ inside the open circle $|z|<1$. By means of this formula and Eqs. (A10) and (A25), we can show that $\varphi(z)$ is given by

$$\varphi(z)=[4C(t_0-t)]^{1/2}(1+z)^{1/2}(1-z)^{-3/2}\times\prod_{j=1}^{n}(t_0-t)^{\alpha_j/2}\left(\beta_j+\frac{1+z}{1-z}\right)^{\alpha_j}\,. \tag{A28}$$

The validity of Eq. (A11) can be easily checked. This formula gives ($l\geq 1$)

$$A=[4C(t_0-t)]^{-1/2}\prod_{j=1}^{n}(t_0-t)^{-\alpha_j/2}(1+\beta_j)^{-\alpha_j}\,, \tag{A29}$$

$$b_l=\frac{1}{l}\left\{\tfrac{1}{2}[3-(-1)^l]+\sum_{j=1}^{n}\alpha_j\left[1-(-1)^l\left(\frac{1-\beta_j}{1+\beta_j}\right)^l\right]\right\}$$

as in Eqs. (2.9). For the K_{l3} problem, $k(\xi)$ has the form Eq. (2.12). Then we calculate

$$\varphi(z)=(3/16\pi)^{1/2}(1+z)(1-z)^{-2}\times\left(\beta_1+\frac{1+z}{1-z}\right)^{1/2}\left(\beta_2+\frac{1+z}{1-z}\right)^{-2}\,,$$
$$\beta_1=(t_0-t_1)^{1/2}(t_0-t)^{-1/2}\,, \tag{A30}$$
$$\beta_2=t_0^{1/2}(t_0-t)^{-1/2}\,,$$

where t_0 and t_1 are now given by Eq. (1.11).

When we set $t=0$, then $g(z)$, given by Eqs. (3.3) and (3.8), is related to our $\varphi(z)$ by

$$g(z)=(1-z^2)[\varphi(z)]^2\,. \tag{A31}$$

This immediately reproduces Eq. (3.11).

From our derivation, it is clear that our equality is the best one unless other information on $D(\xi)$ is explicitly given as in Sec. III.

Also, if we have $f^{(n)}(0)\equiv 0$ ($n\geq N+1$), then $f(z)$ must be a polynomial in z of the order N. Hence, if our sum rule is already saturated for $N=1$ as we suggested in Sec. IV, then we must have $f(z)=a+bz$, where a and b are constants. Then, we must have

$$D(\xi)=F(z)=(a+bz)/\varphi(z)\,, \tag{A32}$$

and this reproduces Eq. (4.6) for $t=0$.

Finally, in the above derivation we have assumed that both $w(\theta)$ and $\ln w(\theta)$ are summable. In some applications, $w(\theta)$ may have singularities at $\theta=0$ and $\theta=\pi$ so that it may not be summable. In such cases, let us set

$$F(z)=(1-z)^{\alpha}(1+z)^{\beta}\tilde{F}(z)\,,$$
$$\tilde{w}(\theta)=|2\sin\tfrac{1}{2}\theta|^{2\alpha}|2\cos\tfrac{1}{2}\theta|^{2\beta}w(\theta) \tag{A33}$$

for some real non-negative constants α and β. Then $\tilde{w}(\theta)$ may become summable for some choices of α and β. If we notice

$$I^2=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\tilde{w}(\theta)|\tilde{F}(z)|^2\,, \tag{A34}$$

then we can use $\tilde{w}(\theta)$ and $\tilde{F}(\theta)$ instead of $w(\theta)$ and $F(z)$. Then, after the final calculation, we vary

α and β over their allowable values so as to minimize the desired inequality. In our K_{l3} problem, it is sufficient to choose $\beta = 0$ and $\alpha > 0$ from the beginning. Then, after the final calculation, we let $\alpha \to 0$. This does not affect[1] the final expression at all and hence, we can ignore this complication from the beginning. In general, the summability condition for $w(\theta)$ can be dispensed with by a similar limiting procedure as long as $\ln w(\theta)$ is summable.

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