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Equivalence Classes of Minimum-Uncertainty Packets. II

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In a previous paper we have shown that all minimum-uncertainty packets are unitarily equivalent to the coherent states and that coherence may be viewed as stationary minimality. In this note we give some additional information relating to the nature of the unitary-equivalence structure. We also give a new calculation of some matrix elements of the operator that implements the unitary equivalence which is not subject to the shortcoming inherent in the original calculation.

In a recent paper[1] we pointed out that there is an equivalence-class structure to the totality of minimum-uncertainty packets. We demonstrated that any minimum-uncertainty packet (m.u.p.) can be written in the form $U_r|\alpha\rangle$, where $|\alpha\rangle$ is a coherent state and U_r is a special case of the unitary operator $U_z=\exp[\frac{1}{2}(zaa-z^*a^\dagger a^\dagger)]$ which occurs when z is real and equal to r. We also pointed out that of all the m.u.p.'s only the coherent states ($r=0$) *remain minimal* under the influence of the free-field Hamiltonian and hence one may identify the coherent states as those which possess stationary minimality.

We have written this addendum to present some additional information and to remedy an incomplete calculation of the matrix elements of U_z in the coherent-state basis. In Eqs. (15) and (15′) (see Ref. 1) there should be a multiplicative factor of $(\cosh r)^{-1/2}$ on the right-hand sides. The method that we used in Ref. 1 to get Eqs. (15) and (15′) permitted us to determine the matrix elements of U_z only up to a multiplicative function of r. We will now present a different calculation of $\langle\alpha|U_z|\beta\rangle$ which does not have the above-mentioned difficulty. It is sufficient, as we shall see later, to take z to be real, say $z=r$. It is also convenient to express U_r in terms of position and momentum variables x and p. Then we have

$$\langle\alpha|U_r|\beta\rangle=\langle\alpha|e^{ir(px+xp)/2}|\beta\rangle = e^{r/2}\langle\alpha|e^{irxp}|\beta\rangle. \qquad (1)$$

The matrix element in (1) may be evaluated directly by extending a result due to Haag[2] as follows:

$$\langle\alpha|e^{irxp}|\beta\rangle=\int dxdy\langle\alpha|x\rangle\langle x|e^{irxp}|y\rangle\langle y|\beta\rangle, \qquad (2)$$

where $|x\rangle$ and $|y\rangle$ are position eigenstates. It is easily seen that the second matrix element in the above integral is simply $\delta(xe^r-y)$. Using this result to do the y integration in (2), we have

$$\langle\alpha|U_r|\beta\rangle=e^{r/2}\int dx\langle\alpha|x\rangle\langle xe^r|\beta\rangle. \qquad (3)$$

In the notation of Ref. 1 we have

$$\langle\alpha|x\rangle=(\lambda^2/\pi)^{1/4}\exp(\tfrac{1}{2}\lambda^2x^2+\sqrt{2}\,\lambda\alpha^*x-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}\alpha^{*2}). \qquad (4)$$

Using this result, and a similar one for $x\to xe^r$ in (3), we get a simple Gaussian integral which yields

$$\langle\alpha|U_r|\beta\rangle=C_r^{-1/2}\exp\left[-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\frac{\alpha^*\beta}{C_r}+\frac{S_r}{2C_r}(\beta^2-\alpha^{*2})\right], \qquad (5)$$

where C_r and S_r are $\cosh r$ and $\sinh r$, respectively. Equation (5) contains the factor $C_r^{-1/2}$ which is missing in Eq. (15) of Ref. 1. We can gain the result for complex z as well by noting the following useful fact: If we define the unitary operator $V_\phi=\exp(\frac{1}{2}i\phi a^\dagger a)$, then we can quite easily show that

$$U_z=V_\phi U_r V_\phi^\dagger, \qquad (6)$$

where $z=re^{i\phi}$.

Since $V_\phi|\beta\rangle = |\beta e^{i\phi/2}\rangle$, the effect of V_ϕ in Eq. (5) would be to multiply β by $e^{i\phi/2}$ and multiply α by $e^{-i\phi/2}$. This would then yield the result of Eq. (15) in Ref. 1 with the additional factor of $C_r^{-1/2}$. Equation (6) is quite useful when dealing with the general (nonminimal) states $|z;\alpha\rangle = U_z|\alpha\rangle$, as we have just seen.

The presence of the factor $C_r^{-1/2}$ has an important effect, namely, it results in $\langle 0|U_z|0\rangle$ going to zero as r goes to infinity. Hence in the limit $r\to\infty$ we get an inequivalent representation.

By an inequivalent representation of the canonical commutation relation $[a, a^\dagger]=1$ we mean a pair of operators $b, b^\dagger$ which satisfies the relation $[b, b^\dagger]=1$ and is related to the pair $a, a^\dagger$ by means of $b = SaS^\dagger$ and $b^\dagger = Sa^\dagger S^\dagger$. The operator S has the property that it maps the space in which a and $a^\dagger$ act into a space which is orthogonal to it. This means that any vector in the domain of a is orthogonal to any vector in the domain of b. Since the transformation S does preserve the commutation relation it is sometimes called an improper unitary transformation and the representation $b, b^\dagger$ an improperly equivalent representation.

In Ref. 1 we treat r as a positive number, i.e., the modulus of z. The minimal states are then seen to be a special case of the states $|z;\alpha\rangle = U_z|\alpha\rangle$ for which z is real. This is quite true and in order to get all of the minimal states this way we simply take the *two* values of ϕ which make z real, namely, $\phi = 0, \pi$. When dealing with the general state $|z;\alpha\rangle$ this is a very convenient parametrization. For the case of the minimal states it is perhaps more convenient to simply consider r as a real number which can be of either sign. In Ref. 1 this point was not stressed and we mention it here to prevent any possible ambiguity from arising.

It is interesting to consider the transformation induced by U_r in terms of position and momentum. The operator U_r is given by

$$U_r = e^{ir(xp+px)/2}. \tag{7}$$

A simple calculation then reveals that

$$\begin{aligned} U_r x U_r^\dagger &= xe^r , \\ U_r p U_r^\dagger &= pe^{-r} , \end{aligned} \tag{8}$$

where now r is any real number. So U_r is just a scale transformation of x and p by reciprocal scale factors. This might have been expected since if one starts out with a minimal state the only changes one can make and still retain minimality are translations in position and momentum and reciprocal scale changes. The translations only involve changing the value of α in $|r;\alpha\rangle$ and the scale changes, as we have just seen, are generated by U_r.

The minimum-uncertainty states have an old and large literature. The first occurrence of them seems to be in a paper by Schrödinger[3] in 1926. A number of their properties may be found in a wide range of papers, some of which are given below.[4] An interesting paper on states which minimize the uncertainty product of operators other than position and momentum has been written by Jackiw.[5] The equivalence-class structure and the unitary equivalence to the coherent states as well as the stated connection with coherence was first pointed out in Ref. 1.

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