residues of all its poles are curls, we conclude that $B_{\mu\nu}(N+1, \Sigma)$ differs from a curl by at most a bilinear expression in any N momenta other than Σ,

$$B_{\mu\nu}(N+1, \Sigma) = \sum \alpha_{ij}(p_\mu^i p_\nu^j - p_\mu^j p_\nu^i) + \text{ curl}, \qquad (8)$$

where the summation extends over all i and j such that

$$1 \leqslant i < j \leqslant N. \qquad (9)$$

Put equal to zero all momenta excepting p_i, p_j, and p_{N+1}. Provided that $N \geqslant 3$, the momentum of at least one WW line has been put equal to zero, so that the left-hand side of (8) vanishes by construction. The curl vanishes since $\Sigma = 0$. Thus, $\alpha_{ij} = 0$. This argument is applied to all i and j satisfying (9), thereby proving that $B_{\mu\nu}(N+1, \Sigma)$ is a curl. This completes the induction.

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Unitary Models of the Pomeranchuk Singularity*

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Two models of production amplitudes are presented which do not lead to a violation of the Froissart bound when the leading Regge singularity reaches $J=1$. Both models have a bootstrap solution with the two-body amplitude having the form $[(J-1)^2 - R_0^2 t]^{-3/2}$.

It is well known that the multi-Regge model in its simplest form leads to a violation of the Froissart bound for total cross sections if the leading Regge pole reaches 1 at $t=0$.[1] This difficulty can of course be avoided by assuming that the Pomeranchuk pole has an intercept, $\alpha(0)$, slightly less than 1.[2] Alternatively one can attempt to construct models of production amplitudes which will not lead to a violation of the unitarity bound when the leading J-plane singularity reaches 1. In this note we present two such models. One is based on the unitarity model of Dash, Fulco, and Pignotti[3] (DFP model) and the other on a generalization of the relativistic eikonal model.[4] Both of these models reduce to the multi-Regge model if the leading Regge singularity is well below 1, but they give strikingly different results when it reaches 1.

In particular, both models present a self-consistent solution with the two-body amplitude having the J-plane structure

$$M_{22}(J, t) = K/[(J-1)^2 - R_0^2 t]^{3/2} \qquad (1)$$

as suggested by the eikonal model.[4,5] In these models the two-body amplitude satisfies unitarity exactly, so there is no violation of the Froissart bound. Interestingly enough a simple pole with $\alpha(0)=1$ does not bootstrap itself in our models, although it does not lead to a violation of the unitarity bound.

Let us first consider the DFP model. Following Ref. 3, we work in impact-parameter space and write

$$M_{2n}(s, \vec{b}) = B_{2n}(1 - iB_{22}/4s)^{-1}, \quad n = 2, 3, \ldots . \qquad (2)$$

The form of Eq. (2) guarantees that the unitarity condition will be satisfied at high energies, if all multiparticle effects are included in the B_{2n}. In the spirit of the multi-Regge model we write[6]

$$B_{2n} = G^2 g^{n-2} \prod_{i=1}^{n-1} M_{22}(s_i, \vec{b}_i). \qquad (3)$$

s_i is the subenergy of the i and $i+1$ particles and $\vec{b}_i$ is their separation in impact-parameter space.

We have taken the masses of all particles to be unity. If all subenergies are large,

$$\prod_{i=1}^{n-1} s_i \cong s \quad \text{and} \quad \sum_{i=1}^{n-1} \vec{b}_i = \vec{b}\,.$$

For the Pomeranchuk singularity we expect B_{22} to be pure imaginary at high energies. Writing $B_{22}=iC$, we see that the two-particle amplitude will satisfy unitarity exactly provided

$$C(s,\vec{b}) = \sum_{n=3}^{\infty} |B_{2n}|^2 \rho_n \equiv \sum_{n=3}^{\infty} C_n(s,\vec{b})\,, \tag{4}$$

where in the approximation mentioned in Ref. 6 the n-particle phase-space factor ρ_n is given by

$$\rho_n \cong \frac{1}{4}\left(\frac{1}{4\pi}\right)^{n-1} \prod_{i=1}^{n-1} \int d^2 b_i \int_0^1 dy_i\, \delta\left(s\prod_{j=1}^{n-1} y_j - 1\right), \tag{5}$$

with $y_j = 1/s_j$. Now Eq. (1) implies that $M_{22}(s,\vec{b})$ is of the form

$$M_{22}(s,\vec{b}) = 4is\theta(b_0^2 - b^2)\,, \tag{6}$$

with $b_0 = R_0 \ln(s/s_0)$. It is an easy matter to compute the C_n's and we find

$$C_n(s,\vec{b}) = \frac{3\beta s[a(b_0^2-b^2)^{1/2}]^{3n-5}}{(b_0^2-b^2)^{1/2}(3n-5)!}\,\theta(b_0^2-b^2)\,, \tag{7}$$

where a and β are constants depending on R_0 and g. Therefore,[7]

$$\begin{aligned} C(s,\vec{b}) &= \beta s \frac{\exp[a(b_0^2-b^2)^{1/2}]}{(b_0^2-b^2)^{1/2}}\,\theta(b_0^2-b^2), \quad b_0^2 \gg b^2 \\ &= 3\beta s \frac{a^4(b_0^2-b^2)^{3/2}}{4!}\,\theta(b_0^2-b^2), \quad b_0^2 \cong b^2\,. \end{aligned} \tag{8}$$

Substituting Eq. (8) into Eq. (2) for the case $N=2$ gives, to leading order in $\ln(s/s_0)$,

$$M_{22}(s,\vec{b}) = 4is\theta(b_0^2-b^2)\,,$$

or (9)

$$M_{22}(s,t) = 8\pi i s b_0 \frac{J_1(b_0\sqrt{-t})}{\sqrt{-t}}\,;$$

so the bootstrap is complete.[8,9]

It should be noticed that the S matrix is given by

$$S_{22}(s,\vec{b}) = \frac{1+iB_{22}/4s}{1-iB_{22}/4s}\,; \tag{10}$$

so $S_{22}=-1$ for $b^2 \ll b_0^2$ and large s. In other words all scattering takes place inside a disk of radius $b_0 = R_0\ln(s/s_0)$. Except for a small region at the edge of the disk, the scattering is entirely elastic and we have, to leading order in $\ln(s/s_0)$,

$$\sigma_{\text{tot}} = \sigma_{\text{elastic}} = 4\pi R_0^2 \ln(s/s_0)^2\,. \tag{11}$$

The inelastic cross section comes entirely from the edge of the disk and goes to a constant at very high energies. Obviously the average multiplicity goes to 2.

Let us now turn to the eikonal model. We shall consider diagrams of the type shown in Fig. 1. The wavy lines may be thought of as quantum electrodynamics or φ^3 ladders. The loops at which the particles are produced will eventually be approximated by constants. As usual one is to consider not only the diagram of Fig. 1 but all those that can be obtained from it by interchanging the legs of the ladders. The important new feature in this model is that when the leading J-plane singularity reaches 1, diagrams in which the produced particles come from different chains are just as important as those in which they come from a single chain. We believe that this feature will hold for any model based on Feynman diagrams.

We denote the amplitude in which n_1-2 particles are produced from one chain, n_2-2 from a second chain, etc. by $M_{2;n_1\cdots n_N}$. When all subenergies are large, the usual eikonal calculation gives[10]

$$\begin{aligned} M_{2;n_1\cdots n_N} &= e^{i\delta(s,\vec{b})}\left(\frac{i}{2s}\right)^{N-1} \prod_{i=1}^{N} g^{n_i-2} \\ &\quad\times \prod_{j=1}^{n_i-1} M_{22}(s_{jn_i};\vec{b}_{jn_i})\,, \end{aligned} \tag{12}$$

where

$$\prod_{j=1}^{n_i-1} s_{jn_i} = s\,, \quad \sum_{j=1}^{n_i-1} \vec{b}_{jn_i} = \vec{b}$$

and

$$e^{i\delta} = 1 + \frac{i}{2s} M_{22}\,. \tag{13}$$

At high energies we expect M_{22} and therefore δ to be pure imaginary; so we write $\delta = ia$. The unitarity condition for M_{22} then becomes[11]

$$\begin{aligned} \operatorname{Im} M_{22}(s,\vec{b}) &= 2s(1-e^{-a}) \\ &= s(1-e^{-a})^2 + se^{-2a}\sum_{N=1}^{\infty}\frac{[C'(s,\vec{b})/s]^N}{N!}\,, \end{aligned} \tag{14}$$

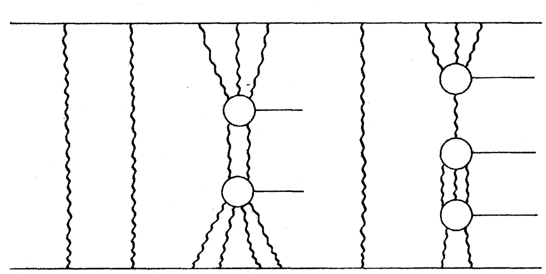

FIG. 1. A typical diagram for the production amplitudes in the eikonal model.

with

$$C'(s,\vec{b})=\sum_{n=3}^{\infty}C_n'(s,b)$$
$$=\sum_{n=3}^{\infty}g^{2n-4}\prod_{i=1}^{n-1}|M_{22}(s_i,\vec{b}_i)|^2\rho_n\,. \qquad (15)$$

If $M_{22}(s,\vec{b})=2is\theta(b_0^2-b^2)$, then we have $C_n'(s,\vec{b})=(\frac{1}{4})^n C_n(s,b)$ and a bootstrap solution is achieved with[8,9]

$$a(s,\vec{b})=C'/2s\,. \qquad (16)$$

Although the elastic scattering amplitude differs from that of the DFP model only by a factor of 2, the physics is completely different. In the present model we have a totally absorbing disk of radius $b_0=R_0\ln(s/s_0)$, and $S_{22}(s,b)=0$ for $b^2\ll b_0^2$. The elastic and inelastic cross sections are equal and we find

$$\sigma_{\text{tot}}=2\sigma_{\text{elastic}}=2\sigma_{\text{inelastic}}=2\pi R_0^2\ln(s/s_0)^2\,. \qquad (17)$$

The cross section for the production of n particles is easily found by expanding the last term in Eq. (14) in powers of g^2. We have

$$\sigma_n=\int d^2b\,e^{-C'/s}\sum_{a_i=0}^{\infty}\sum_{m=0}^{\infty}\delta_{m-\Sigma a_i}\,\delta_{n-2-\Sigma i a_i}\times\prod_{i=1}^{\infty}\frac{1}{a_i!}\left(\frac{C_{i+2}'}{s}\right)^{a_i}, \qquad (18)$$

which leads to an average multiplicity of

$$\bar{n}=K(s/s_0)^{\epsilon}/\ln(s/s_0)^2\,. \qquad (19)$$

It should be noted that unlike the DFP model all production in the eikonal model comes from the interior of the disk.

We realize that the two models presented here are quite crude since the form of the production amplitudes are expected to be realistic only when all subenergies are large. Furthermore the approximations made to compute phase-space integrals are only valid also when all the subenergies are large. Nevertheless, it appears that the main feature of both models is independent of the approximations we have made, namely, that unitarity requires that the leading J-plane singularity be iterated when this singularity is near $J=1$. It also seems quite likely that as suggested by the eikonal model, diagrams corresponding to production of particles from more than one chain should be included in the inelastic amplitudes.

While it is clear that some predictions such as multiplicities and inelastic cross sections are quite model-dependent, others, like the $(\ln s)^2$ behavior of total cross sections and the $(\ln s)^{-2}$ shrinkage of the diffraction peaks, seem to be consequences of the general features of the models.[5]

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[6]Although Eqs. (3) and (12) are at most expected to be valid when all subenergies are large we will use them throughout phase space. We also make the usual approximations to phase space which are only valid for large subenergies. We do not believe these approximations alter the qualitative features of the model.

[7]Fourier transforming Eq. (8) gives the well-known result $C(s,t)\propto s^{1+(a^2+R_0^2t)^{1/2}}$, which shows that the simple multiperipheral model would give a violation of the Froissart bound for the input singularity of Eq. (1).

[8]Notice that the bootstrap condition does not determine the magnitude of R_0.

[9]If we had started by assuming that the leading J-plane singularity entering B_{2n}, $n>3$ was a simple pole with $\alpha(0)=1$, then, of course, $B_{22}=iC$ would have a pole with $t=0$ intercept to the right of $J=1$. Substituting into Eq. (2) leads to an expression for M_{22} having the form of Eq. (1). Thus the bootstrap solution will be achieved by a second iteration. A similar result is true for the eikonal model.

[10]We only consider interactions between nearest neighbors on the same chain and between the two incident particles. The latter give rise to long-range correlations which are crucial if unitarity is to be satisfied. Other long-range correlations arising from non-nearest-neighbor interactions have been neglected for simplicity. However, they may well be important.

[11]To be consistent with our nearest-neighbor approximation we must only retain terms in the inelastic unitarity sum arising when ladders of equal length are joined together.