refer to Lorentz poles as Regge poles throughout this paper.

[11]M. L. Goldberger, D. Silverman, and C.-I Tan, Phys. Rev. Letters 26, 100 (1971).

[12]As many authors have noted, this kernel strength is too weak to generate a leading $I=0$ pole near 1.

[13]A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I, Sec. (2.12).

[14]Reference 13, Vol. I, Secs. (2.9) and (2.8).

PHYSICAL REVIEW D VOLUME 4, NUMBER 6 15 SEPTEMBER 1971

Wave Equations and Saturation of Current Algebra for $I=\frac{1}{2}$

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(Received 5 October 1970)

The conditions imposed by the saturation of the isospin-factored current algebra are studied for a class of currents which includes the canonical currents belonging to second-order wave equations. The aim is to understand in a simple, transparent way the nature and extent of the conditions imposed by current algebra, and to see how, in particular models, these conditions reduce more general wave equations to the special form $(ap^2+2g\cdot p+s)\psi(p)=0$ predicted in earlier work. A recently proposed wave equation which appears at first sight to contradict the prediction is shown to contain no contradiction.

I. INTRODUCTION

In recent papers[1-3] the saturation of current algebra at infinite momentum has been studied in some detail. Particular attention has been paid to the case of the isospin-factored algebra (corresponding to saturation with a tower of particles of fixed isospin, e.g., a tower of $I=\frac{1}{2}$ K mesons or Ξ particles). No physically satisfactory solution has been found for this case, and the nontrivial solutions which have been found can be expressed in terms of the canonical current associated with an infinite-component wave equation of the form

$$(ap^2+2g_\mu p^\mu+b)\psi(p)=0\,, \tag{1.1}$$

where $\psi(p)$ carries the infinite-component representation of $SL(2,C)$ and a, b, and g_μ are $SL(2,C)$ scalars and vectors, respectively. However, these results for the isospin-factored algebras have been derived in most cases by guesswork,[3] and in the one attempt at a systematic analysis,[2] they come as the end result of some algebraic manipulations in the course of which a number of technical assumptions are made (see below). Hence it would seem to be desirable to obtain a better *intuitive* understanding of how the results come about. This desire has recently been accentuated by the appearance of a model[4] which at first sight seems to contradict the results of Ref. 2.

Accordingly, the broader purpose of the present paper is to try to achieve the better intuitive understanding just mentioned of the isospin-factored case. A secondary purpose is to use this better understanding to show that the contradiction between the model of Ref. 4 and the results of Ref. 2 is only apparent – the reason being that, in the model, not all the physical requirements have been imposed.

To fix our ideas, let the matrix elements of the local current (at time zero) $J_\mu(x,0)$, between one-particle states $|np\rangle$ where p is the 3-momentum and n denotes mass $m(n)$, spin, helicity, and internal quantum numbers, be

$$\begin{aligned}\langle n'p'|J_\mu(x,0)|np\rangle &= e^{i(p'-p)\cdot x}\psi^\dagger_{n'}(p')\beta I_\mu(p_0'p'pp_0)\psi_n(p)\\ &= e^{i(p'-p)\cdot x}\psi^\dagger_{n'}(0)\beta L_{n'}{}^{-1}(p')\\ &\quad\times I_\mu(p_0'p'pp_0)L_n(p)\psi_n(0)\,,\end{aligned} \tag{1.2}$$

where

$$L_n(p)=\left[\frac{m(n)}{p_0}\right]^{1/2}\exp\left[-i\sinh^{-1}\left(\frac{p}{m(n)}\right)\frac{\vec{p}\cdot\vec{K}}{p}\right].$$

Here, $\psi_n(p)$ belongs to any representation (finite-dimensional, infinite sum of finite-dimensional, or strictly infinite-dimensional representation) of the spin group $SL(2,C)$, β is a metric in spinor space, I_μ is a Lorentz 4-vector, and K is the boost vector in spinor space.

It should be emphasized that the current $J_\mu(x,0)$ and the states $|np\rangle$ are the conventional current and states of second-quantized field theory. In particular, the current $J_\mu(x,0)$ is self-adjoint and the states $|np\rangle$ have positive norm. This will be assumed throughout. Only for the matrix *elements* $\langle n'p'|J_\mu(x,0)|np\rangle$ do we introduce spinorial notation, and the metric β in spinor space is not assumed to be positive.

In conventional theories the "dynamics" of the form factors is contained in I_μ, and the "kinematics" in the boost term $L_n(p)$. However, the saturation of the isospin-factored current apparently reverses the situation. It apparently forces I_μ to take the simple linear form

$$I_\mu(p_0'p'pp_0) = a(p_\mu' + p_\mu) + 2g_\mu \tag{1.3}$$

associated with the wave equation (1.1), and hence forces the momentum-transfer dependence of the form factors to reside in the factors $L_n(p)$, in which case it can be nontrivial only if $\psi(p)$ carries a strictly infinite-dimensional representation of SL(2, C). The problem is to understand how this reversal of the roles of $I_\mu(p'p)$ and $L_n(p)$ comes about.

One can separate the question into two parts.

(1) Why does the saturation force $I_\mu(p'p)$ to be a polynomial [and hence remove from $I_\mu(p'p)$ the "dynamical" content of the theory]?

(2) Given that $I_\mu(p'p)$ is a polynomial, why does the saturation force it to be linear?

In this paper we shall attempt to answer only the second question, and that only for a special class of currents. However, we feel that the answer for this special case clarifies the general situation, since it is probable that the mechanism that reduces polynomials to linear functions for the special class of currents is the mechanism which is operating also for general nonpolynomial currents.

Before proceeding with the discussion of the present paper there are two points which we should perhaps discuss, as they have been treated only implicitly in previous articles. The first concerns the general philosophy of saturation at $p_3 = \infty$. More precisely, since the transition to $p_3 = \infty$ is a kinematical exercise, one may legitimately ask: Where does the physics lie? The answer is that the saturation assumption at $p_3 = \infty$ is not meant to provide a complete physical theory – "there is no Hamiltonian" – but is meant only to provide restrictions on possible theories, or rather to provide a restrictive framework within which one still has to choose a specific theory. The situation is analogous to Lagrangian theory, which provides a framework within which one still has to choose a specific Lagrangian. Indeed, one can say that free parameters allowed by the choice of representation of SL(2, C) in Eq. (1.1) correspond roughly to the free choice of coupling constants in Lagrangian theory.

The actual restrictions imposed by current algebra, in order of strength, are as follows:

(1) Saturation with all physical states at $p_3 = \infty$. The only assumption here is that there is no leakage to finite-momentum states, i.e.,

$$\lim_{p_3''=p_3=\infty;\, p_3'\neq\infty} \sum_{n'p'} \epsilon^{abc} \langle n''p'' | J_0^a(\vec{x},0) | n'p' \rangle \times \langle n'p' | J_0^b(\vec{y},0) | np \rangle = 0. \tag{1.4}$$

(2) Saturation with genuine one-particle states at $p_3 = \infty$, i.e., with states of discrete mass, which are finitely degenerate in spin and isospin at each mass level.

(3) Saturation with states of uniformly bounded isospin, in particular with states all of which have isospin $I = \frac{1}{2}$. From the Wigner-Eckart theorem this assumption is equivalent to the factorization

$$J_0^a = \tfrac{1}{2}\tau^a J_0 \tag{1.5}$$

of the isospin current. One of the main results of the present paper will be to see these restrictions manifesting themselves explicitly for simple models.

The second point concerning previous results that we should like to mention here concerns the technical assumptions used in Ref. 2. We should like to summarize them here.

First, throughout Ref. 2, it is assumed that the "angular condition" has an analytic expansion $\sum_{n,m} c_{nm} k_1^n k_2^m$ for sufficiently small $\underline{k} = (k_1, k_2)$, where $\underline{k}$ is the transverse momentum transfer. This is probably not too restrictive an assumption. A possibly more restrictive assumption is that the expansion terminates in the sense that it actually implies only a finite number of conditions, i.e.,

$$c_{mn} = 0 \quad \text{for } n, m \leq N < \infty \Rightarrow c_{nm} = 0 \quad \text{for all } n, m. \tag{1.6}$$

The basis for this assumption is that it is self-consistent and that it can actually be proved in the case that one saturates also the time-space current algebra at $p_3 = \infty$ (with symmetric Schwinger terms).

The set of k-independent equations displayed in the table of Ref. 2 were derived under these two assumptions.[5]

The question of the mass spectrum and the manifest covariance of the solutions was discussed within the context of this table. Regarding the mass spectrum, it was shown that if the symmetric operator K of the table was self-adjoint then all nontrivial solutions had a spacelike ($p^2 < 0$) part. A motivation for thinking that K should be self-adjoint was that, in the case that the solutions of the table could be expressed as wave equations, K could be identified as the generator of a finite unitary Lorentz transformation in spinor space. Given the existence of spacelike solutions, it was then made plausible that the only model in which the spacelike and timelike parts of the solution were not coupled by the current was the free-quark model. (Coupling by the current has already been proved in a number of specific models.[6]) It should be emphasized that this second result on coupling is a plausibility argument and no more.

With respect to manifest covariance, it was shown without any assumptions that the table of k-independent equations allowed three primitive classes of solutions corresponding to the three allowed eigenvalues $(0, \pm 1)$ of an operator B defined in that table, and that these primitive solutions could be expressed in terms of the canonical currents belonging to the manifestly covariant wave equations

$$(p^2 - 2g_\mu p^\mu - b)\psi(p) = 0\,,$$
$$(\gamma_\mu p^\mu - b)\psi(b) = 0\,, \tag{1.7}$$
$$\gamma_5\psi(p) = \pm\,\psi(p)\,,$$

respectively. From this result, it was conjectured, but not proved, that the most general solution to the equations of the table could be obtained by coupling Eqs. (1.7) by means of a p-independent coupling. This conjecture leads immediately to Eq. (1.1).

The number of loopholes left by the above set of assumptions and conjectures, together with the complicated algebra which was necessary to obtain the results, are what make it desirable that the problem be understood in a simpler and more intuitive way, even at the expense of generality – hence our restriction to question (2) above and a relatively simple class of currents.

We shall begin by considering the problem of saturating the isospin-factored current at all momenta for the class of currents considered (Sec. II). (In this connection, since it is well known that saturation at finite momentum leads to physical difficulties, we should perhaps make it clear that we shall be allowing the possibility of spacelike solutions for wave equations, although ultimately, of course, one is aiming at a solution which does not have a spacelike part.) In Sec. III we derive the condition imposed by current algebra at infinite momentum, and in Secs. IV and V we apply this condition to two simple models. In Sec. VI we consider the more complicated model of Ref. 4, and show that the wave equations for this model can always be derived from an equation of the form (1.1). In Sec. VII we apply the current-algebra condition to the model, with interesting results. The technical details of this model are contained in an appendix. We conclude by discussing in Sec. VIII the relationship between our work and that of other authors, particularly that of Ref. 4.

II. SATURATION AT FINITE MOMENTUM

Let the matrix elements of the current be as in (1.2), where $I_0(p_0'p'pp_0)$ is a polynomial and β is a p-independent metric in spinor space (not necessarily positive) introduced so that $\bar\psi\psi = \psi^\dagger\beta\psi$ is Lorentz-invariant, where $\psi^\dagger\psi \geq 0$. It is convenient to define a Hamiltonian operator $H(p)$ in spinor space by the relation

$$H(p)\psi_n(p) = \pm[p^2 + m^2(n)]^{1/2}\psi_n(p)\,. \tag{2.1}$$

Equation (1.2) can be written in the form

$$\langle n'p'|J_0(x,0)|np\rangle = e^{i(p'-p)\cdot x}\psi_{n'}^\dagger(p')I(p',p)\psi_n(p)$$
$$= e^{i(p'-p)\cdot x}J(n'ppn)\,, \tag{2.2}$$

where

$$I(p'p) = \beta\hat I_0(\beta^{-1}H(p')\beta, p', p, H(p)) \tag{2.3}$$

depends only on the 3-momenta p' and p, the caret on $\hat I_0$ indicating that p_0' and p_0 are to be moved to the left and right of all spinor operators in I_0 before substitution. The Hermiticity of the current $J_\mu(x,0)$ in physical Hilbert space implies

$$I(p'p) = I^\dagger(pp')\,. \tag{2.4}$$

The normalization of the states $\psi_n(p)$ is fixed by the current algebra and the normalization of the physical states as follows: The factorization of the isospin current implies that

$$J_\mu^a(x,t) = \tfrac{1}{2}\tau^a J_\mu(x,t)\,, \tag{2.5}$$

where the τ^a are Pauli matrices, and in particular, implies that

$$I^a = \tfrac{1}{2}\tau^a Q = \tfrac{1}{2}\tau^a\int d^3x\,J_0(x,t)\,,\quad I^3 = \pm\tfrac{1}{2} \tag{2.6}$$

where I^a is the isospin. Hence

$$Q^2 = Q \neq 0 \quad\text{or}\quad Q = 1\,. \tag{2.7}$$

In other words, for the factored algebra, the reduced charge is not only conserved but is unity. Now integrating (2.2) over x and inserting (2.7), we obtain

$$\psi_{n'}^\dagger(p)I(pp)\psi_n(p) = [(2\pi)^3\delta(p'-p)]^{-1}\langle n'p'|np\rangle$$
$$= \delta_{n'n}\,, \tag{2.8}$$

which fixes the normalization of the $\psi_n(p)$ as required. Note that from the completeness of the states $\psi_n(p)$, Eq. (2.8) implies also

$$I^{-1}(pp) = \sum_n \psi_n(p)\psi_n^\dagger(p) > 0\,. \tag{2.9}$$

Note that the positivity of $I(pp)$ implied by (2.9) is a consequence of the factored current algebra and the positivity of the norm for the physical states $|np\rangle$. The positivity of the metric β in spinor space is not assumed, and since $I(pp) = \beta I_0(pp)$, it is not necessarily implied by the positivity of $I(pp)$.

For the nonintegrated current (2.5), the equal-time current algebra

$$[J_0^a(x,0), J_0^b(y,0)] = i\epsilon^{abc} J_0^c(x,0)\delta(x-y) \tag{2.10}$$

implies

$$J_0(x,0)J_0(y,0) = J_0(x,0)\delta(x-y)\,; \tag{2.11}$$

inserting this result into (2.2), we obtain

$$\sum_{n''}\int d^3p''\, e^{i(p'-p'')\cdot x} J(n'p'p''n'') e^{i(p''-p)\cdot y} J(n''p''pn) = (2\pi)^3 e^{i(p'-p)\cdot x} J(n'p'pn)\delta(x-y)\,, \tag{2.12}$$

whence, on multiplication by $\exp(iq\cdot y)$ and integration over y, we have

$$\sum_{n''}\int d^3p''\, e^{i(p'-p'')\cdot x} J(n'p'p''n'') J(n''p''pn)\delta(p''-p+q) = J(n'p'pn) e^{i(p'-p+q)\cdot x} \tag{2.13}$$

or

$$\sum_{n''} J(n'p'p''n'')J(n''p''pn) = J(n'p'pn)\,, \tag{2.14}$$

whence, using the definition of $J(n'p'pn)$ in (2.2) and the completeness relation (2.9), we obtain, finally,

$$I(p'p'')I^{-1}(p''p'')I(p''p) = I(p'p) \tag{2.15}$$

as our current-algebra condition. Note that there is no integration over p''.

So far, apart from the polynomial condition, the current is completely general. We now specialize to the class of *separable* currents, i.e., currents of the form

$$I_0(p_0'p'pp_0) = I_0'(p_0'p') + I_0(pp_0)\,. \tag{2.16}$$

This class of currents includes those in which we shall be most interested, namely, the canonical currents belonging to second-order wave equations:

$$I_\mu(p_0'p'pp_0) = T_{\mu\nu}p^\nu + T_{\nu\mu}p'^{\,\nu} + \Gamma_\mu\,, \tag{2.17}$$

where

$$(T_{\mu\nu}p^\mu p^\nu + \Gamma_\mu p^\mu + S)\psi(p) = 0\,,$$
$$\Gamma_\mu = \beta^{-1}\Gamma_\mu^\dagger\beta, \quad T_{\mu\nu} = \beta^{-1}T_{\nu\mu}^\dagger\beta\,. \tag{2.18}$$

For the separable currents (2.16), the separability condition and the Hermiticity condition (2.4) imply that

$$I(p'p) = I^\dagger(p') + I(p)\,. \tag{2.19}$$

Inserting this result into the current-algebra condition (2.18), we obtain

$$[I^\dagger(p') + I(p'')]I^{-1}(p''p'')[I^\dagger(p'') + I(p)] = I^\dagger(p') + I(p)\,, \tag{2.20}$$

whence, using (2.19), we have

$$[I^\dagger(p') - I^\dagger(p'')]I^{-1}(p''p'')[I(p) - I(p'')] = 0\,. \tag{2.21}$$

Setting $p' = p$ and recalling that $I(p''p'')$ is positive, we obtain

$$I(p) - I(p'') = 0 \quad \text{or} \quad I(p) = I(0)\,, \tag{2.22}$$

whence, from (2.19),

$$I(p'p) = I(0,0) = I\,, \tag{2.23}$$

where I is independent of p' and p.

Inserting this result into the expression (2.2) for the matrix elements of the current, we obtain

$$\langle n'p'|J_0(x,0)|np\rangle = e^{i(p'-p)\cdot x}\,\bar\psi_{n'}(p')(\beta^{-1}I)\psi_n(p)\,, \tag{2.24}$$

whence, using Lorentz invariance, we have

$$\langle n'p'|J_\mu(x,0)|np\rangle = e^{i(p'-p)\cdot x}\,\bar\psi_{n'}(p')I_\mu\psi_n(p)\,, \tag{2.25}$$

where I_μ is a vector operator which is independent of p' and p and such that $\beta I_0 = I$. From Eq. (2.25) the strength of the current algebra is evident: It eliminates the p', p dependence of $I_\mu(p'p)$.

Furthermore, for a conserved current, it follows from the p', p independence of I_μ that $\psi(p)$ satisfies a linear wave equation of the form

$$(I_\mu p^\mu - m)\psi(p) = 0\,, \tag{2.26}$$

where m is a p'-, p-independent scalar.[7] Thus current algebra forces a linear wave equation. In particular, if the current is the canonical current belonging to a higher-order wave equation, then the higher-order equation must be compatible with (2.26). For example, comparison of the second-order wave equation given above with (2.26) squared yields

$$T_{\mu\nu} + T_{\nu\mu} = \left\{\frac{1}{m}I_\mu, \frac{1}{m}I_\nu\right\}\,,$$
$$\Gamma_\mu = -\frac{2}{m}I_\mu\,. \tag{2.27}$$

We note in conclusion that the crucial step in going from a p', p-dependent to a p', p-independent $I_\mu(p'p)$, namely, the step (2.21) → (2.22), depends critically on the Hermiticity of the current and the positivity of $I(p''p'')$. These conditions are demanded by the physics of the problem (observable currents and positive norms for the physical states), and if they are not satisfied the conclusions do not follow, as we shall see later in counterexamples.

III. SATURATION AT INFINITE MOMENTUM

Consider again the matrix elements (1.2) of the current and let[8]

$$\begin{aligned}\langle n'|F(\underline{k})|n\rangle &= \lim_{\kappa\to\infty}\langle n'p_1'p_2'\kappa|J_0(0)|np_1p_2\kappa\rangle \\ &= \lim_{\kappa\to\infty}\bar\psi_{n'}(p_1'p_2'\kappa)I_0(p_0'p'pp_0)\psi_n(p_1p_2\kappa) \\ &= \lim_{\kappa\to\infty}\left[\frac{m(n')m(n)}{p_0'p_0}\right]^{1/2}\bar\psi_{n'}(0)e^{-i[\ln m(n')]K_3}e^{i\underline{p}'\cdot\underline{E}}(e^{i\sigma K_3}I_0(p_0'p'pp_0)e^{-i\sigma K_3})e^{-i\underline{p}\cdot\underline{E}}e^{i[\ln m(n)]K_3}\psi_n(0) \\ &= \bar\psi_{n'}(0)e^{-i[\ln m(n')]K_3}[m(n')]^{1/2}e^{i\underline{p}\cdot\underline{E}}\{I_0(p_0'p'pp_0)+I_3(p_0'p'pp_0)\}e^{-i\underline{p}\cdot\underline{E}}e^{i[\ln m(n)]K_3}[m(n)]^{1/2}\psi_n(0) \\ &= \bar\phi_{n'}(0)e^{i\underline{p}'\cdot\underline{E}}I_\alpha(p_0'p'pp_0)e^{-i\underline{p}\cdot\underline{E}}\phi_n(0) \\ &= \phi_{n'}^\dagger(0)e^{i\underline{p}'\cdot\underline{E}^\dagger}I(p'p)e^{-i\underline{p}\cdot\underline{E}}\phi_n(0)\,,\end{aligned}\tag{3.1}$$

where $I(p'p)$ is defined as in (2.3) with $0\to\alpha=0+3$, $\sigma=\sinh^{-1}\kappa$, $\langle n'|n\rangle=\delta_{n'n}$, $\underline{p}=(p_1,p_2)$, $\underline{k}=\underline{p}'-\underline{p}$, $\underline{E}=(K_1+R_2,K_2-R_1)$, and $\vec{R},\vec{K}$ are the generators of SL$(2,C)$ in spinor space. [Note that in the final expression in (3.1), $p_0'+p_3'=p_0+p_3=1$, so that by using the standard deceleration $\exp(i\sigma K_3)$, we have effectively pulled back the infinite-momentum hypersurface $p_3'=p_3=\infty$ to this position.]

Saturation at infinite momentum for the isospin-factored current means applying the current-algebra condition (2.11) to the expression (3.1). With the normalization $\langle n'|n\rangle=\delta_{n'n}$, this amounts to setting

$$F(\underline{k}')F(\underline{k})=F(\underline{k}'+\underline{k})\,.\tag{3.2}$$

Before inserting the right-hand side of (3.1) into (3.2), however, we specialize for simplicity to the class of currents whose α component is separable on the hypersurface $p_0'+p_3'=p_0+p_3=1$, i.e., for which

$$I(p'p)=I^\dagger(p')+I(p)\quad\text{at}\quad p_0'+p_3'=p_0+p_3=1\,.\tag{3.3}$$

This class of currents includes, of course, those which are separable at all momenta, but it is a strictly larger class as we shall see below. Inserting (3.3) in (3.1), we obtain the simplification

$$\begin{aligned}\langle n'|F(\underline{k})|n\rangle &= \phi_{n'}^\dagger(0)e^{i\underline{p}'\cdot\underline{E}^\dagger}[I^\dagger(p')+I(p)]e^{-i\underline{p}\cdot\underline{E}}\phi_n(0) \\ &= \phi_{n'}^\dagger(0)\{[e^{i\underline{p}'\cdot\underline{E}^\dagger}I^\dagger(p')e^{-i\underline{p}'\cdot\underline{E}}]e^{i\underline{k}\cdot\underline{E}}+e^{i\underline{k}\cdot\underline{E}^\dagger}[e^{i\underline{p}\cdot\underline{E}^\dagger}I(p)e^{-i\underline{p}\cdot\underline{E}}]\}\phi_n(0) \\ &= \phi_{n'}^\dagger(0)(I^\dagger e^{i\underline{k}\cdot\underline{E}}+e^{i\underline{k}\cdot\underline{E}^\dagger}I)\phi_n(0)\,,\end{aligned}\tag{3.4}$$

where $I=I(p)$ at $p_0+p_3=1$, $\underline{p}=0$, $p_0-p_3=M^2=(\text{mass operator})^2$ on $\phi_n(0)$, and we have used the fact that the conjugation with $\exp(-i\underline{p}\cdot\underline{E})$ is in spinor space only and $\underline{E}$(Poincaré) $=\underline{E}+\underline{E}$(orbital) commutes with $I_\alpha(p)$. Note that the right-hand side of (3.4) depends only on $\underline{p}'-\underline{p}$ and not on $\underline{p}'+\underline{p}$, a fact which follows from general considerations (Ref. 1) and which we have anticipated in (3.1) by writing $F(\underline{k})$.

Let us now apply the (isospin-factored) current-algebra condition (3.2) to (3.4). First, for $\underline{k}=0$, we obtain the condition

$$\langle n'|n\rangle=\phi_{n'}^\dagger(0)(I^\dagger+I)\phi_n(0)\,,\tag{3.5}$$

which fixes the relative normalization of $|n\rangle$ and $\phi_n(0)$; i.e.,

$$\phi_n(0)=(I^\dagger+I)^{-1/2}|n\rangle\,.\tag{3.6}$$

Note that Eq. (3.6) makes physical sense, since current conservation yields

$$0=(p_\mu'-p_\mu)\psi_{n'}^\dagger(p')\hat{I}_\mu(p_0'p'pp_0)\psi_n(p)=\tfrac{1}{2}[m^2(n')-m^2(n)]\,\phi_{n'}^\dagger(0)(I^\dagger+I)\phi_n(0)\tag{3.7}$$

when evaluated for $\underline{p}'=\underline{p}=0$. Inserting (3.6) into (3.4), we obtain the identification

$$F(\underline{k})=(I^\dagger+I)^{-1/2}(I^\dagger e^{i\underline{k}\cdot\underline{E}}+e^{i\underline{k}\cdot\underline{E}^\dagger}I)(I^\dagger+I)^{-1/2}\,.\tag{3.8}$$

We are now free to apply (3.2) to (3.8) for all $\underline{k}',\underline{k}$. However, only $\underline{k}=-\underline{k}'$ will be necessary. In that case we obtain

$$(I^\dagger e^{i\underline{k}\cdot\underline{E}}+e^{i\underline{k}\cdot\underline{E}^\dagger}I)(I^\dagger+I)^{-1}(I^\dagger e^{-i\underline{k}\cdot\underline{E}}+e^{-i\underline{k}\cdot\underline{E}^\dagger}I)=I^\dagger+I\,,\tag{3.9}$$

or, letting

$$I^\dagger e^{-i\underline{k}\cdot\underline{E}}+e^{-i\underline{k}\cdot\underline{E}^\dagger}I=[I^\dagger+I+\Delta(\underline{k})]e^{-i\underline{k}\cdot\underline{E}},\tag{3.10}$$

$$[I^\dagger + I + \Delta^\dagger(\underline{k})](I^\dagger + I)[I^\dagger + I + \Delta(\underline{k})] = I^\dagger + I + \Delta^\dagger(\underline{k}) + \Delta(\underline{k}) . \tag{3.11}$$

From this equation we see at once that

$$\Delta^\dagger(\underline{k})(I^\dagger + I)\Delta(\underline{k}) = 0 \quad \text{or} \quad \Delta(\underline{k}) = \Delta^\dagger(\underline{k}) = 0 .$$

Hence

$$\underline{E}^\dagger I = I\underline{E} \quad \text{or} \quad [\underline{E}, \beta^{-1}I] = 0 . \tag{3.12}$$

Conversely, if (3.12) is satisfied, the current-algebra condition (3.2) is satisfied for all $\underline{k}', \underline{k}$. Hence we have the result that *for currents whose α component is separable on the hypersurface $p_0' + p_3' = p_0 + p_3 = 1$, a necessary and sufficient condition for current algebra at infinite momentum is Eq. (3.12), where $\beta^{-1}I = I_\alpha(p)$ calculated at $p_0 + p_3 = 1$, $\underline{p} = 0$.* Thus once again we see that current algebra imposes nontrivial conditions on the current. To make the nature of the conditions more understandable we turn now to particular models.

IV. MODEL 1

We shall consider in this section a special case of the second-order wave equations already discussed for finite momenta, namely,

$$(T_{\mu\nu} p^\mu p^\nu - 1)\psi(p) = 0, \quad T_{\mu\nu} = T_{\nu\mu}^\dagger \tag{4.1}$$

$$I_\mu(p_0' p' p p_0) = T_{\mu\nu} p^\nu + T_{\nu\mu} p'^\nu . \tag{4.2}$$

For expository reasons we shall derive the condition imposed by current algebra at $p_3 = \infty$ in two different ways, namely, by using the $p_3 = \infty$ method of Sec. III, and then by using the finite-momentum method of Sec. II, and taking the limit. The result is, of course, the same. Afterwards we shall analyze the result.

1. The method of Sec. III. From (4.2) we have

$$I(p'p) = T_{\alpha\nu} p^\nu + T_{\nu\alpha} p^\nu , \tag{4.3}$$

where

$$I(p) = T_{\alpha\nu} p^\nu \tag{4.4}$$

and

$$I = \tfrac{1}{2} T_{\alpha\alpha} M^2 + \tfrac{1}{2} T_{\alpha\beta} , \quad \beta = 0-3 . \tag{4.5}$$

Hence the current-algebra condition is simply

$$[\underline{E}, I] = \tfrac{1}{2}[\underline{E}, T_{\alpha\alpha} M^2 + T_{\alpha\beta}] = 0 . \tag{4.6}$$

2. The method of Sec. II. At finite momentum we obtained in Eq. (2.22) the result

$$I_0(p)\psi_n(p) = I_0(0)\psi_n(p) \tag{4.7}$$

using only current algebra and completeness. *Afterwards,* we used the fact that (4.7) was valid for all momenta to deduce that

$$I_\mu(p)\psi_n(p) = I_\mu(0)\psi_n(p) . \tag{4.8}$$

Hence in going to the limit it is legitimate to use (4.7) but not (4.8). From (4.7) we have

$$e^{i\sigma K_3} I_0(p) e^{-i\sigma K_3} e^{i\sigma K_3} \psi_n(p) = e^{i\sigma K_3} I_0(0) e^{-i\sigma K_3} e^{i\sigma K_3} \psi_n(p) , \tag{4.9}$$

whence, on taking the limit $\sigma \to \infty$, we obtain

$$I_\alpha(p) e^{-i\underline{p}\cdot\underline{E}} \phi_n(0) = I_\alpha(0) e^{-i\underline{p}\cdot\underline{E}} \phi_n(0) , \tag{4.10}$$

where $p_0 + p_3 = 1$, the zero in $I_\alpha(0)$ referring to p_1 and p_2. Equation (4.10) is the infinite-momentum analog of (4.7). In contradistinction to the finite-momentum case, however, we are now assuming saturation at $p_3 = \infty$; i.e., saturation with the states $p_0 + p_3 = 1$. Hence we can assume (4.10) to hold only for $p_0 + p_3 = 1$. Hence we are not allowed to use the full Lorentz group $L_+^\uparrow$ on $[\exp(-i\underline{p}\cdot\underline{E})]\phi_n(0)$, but only the subgroup which respects the condition $p_0 + p_3 = 1$. The subgroup $[E(2)]$, however, respects also the index α, and hence, using $E(2)$ we cannot deduce (4.10) for any other index. Thus (4.10) is already the full extent of the current-algebra condition at $p_3 = \infty$. It remains only to express (4.10) in infinitesimal form. We have from (4.10)

$$I_\alpha(0) e^{-i\underline{p}\cdot\underline{E}} \phi_n(0) = I_\alpha(p) e^{-i\underline{p}\cdot\underline{E}} \phi_n(0) = e^{-i\underline{p}\cdot\underline{E}} I_\alpha(0) \phi_n(0) , \tag{4.11}$$

whence, since $\underline{p}$ is arbitrary,

$$[\underline{E}, I_\alpha(0)] = 0 \quad \text{on } \phi_n(0) , \tag{4.12}$$

and since $I = I_\alpha(0)$ on $\phi_n(0)$, this equation is identical with (4.6).

We now turn to the question of analyzing (4.6).

First we note that in the special case $T_{\mu\nu} = g_{\mu\nu}$, the wave equation and current of the model become

$$(p^2 - 1)\psi(p) = 0 \tag{4.13}$$

and

$$I_\mu(p'p) = p_\mu' + p_\mu , \tag{4.14}$$

respectively, and that since $T_{\alpha\alpha} = 0$ and $T_{\alpha\beta} = 2$, the $p_3 = \infty$ current-algebra condition (4.6) is automatically satisfied. Thus the $p_3 = \infty$ current algebra allows the solution (4.13), (4.14), a solution which was not allowed by current algebra at finite momenta. The reason for the difference is clear. At general momenta, $I_\mu(p'p)$ cannot possibly satisfy a condition of the form

$$I(p'p'')I^{-1}(p''p'')I(p''p) \sim I(p'p),$$

but at $p_3 = \infty$,

$$I_\alpha(p'p) \to p'_\alpha + p_\alpha = 2,$$

which can satisfy such a condition. This circumstance is clearly the origin of the ap^2 term in Eqs. (1.1). On the other hand, as we see from (1.1) and as we shall see from our models, the extra term ap^2 appears to be the *only* extra freedom allowed by the retreat to $p_3 = \infty$.

We now return to (4.6), and to analyze it, we find it convenient to first apply the wave equation (4.1) to the states $\phi_n(0)$. We obtain

$$T_{\alpha\alpha}M^4 + (T_{\alpha\beta} + T_{\beta\alpha})M^2 + T_{\beta\beta} = 4, \quad \alpha, \beta = 0 \pm 3. \tag{4.15}$$

We can then write this as an equation for I. From (4.5) we obtain

$$(2I + T_{\beta\alpha})(T_{\alpha\alpha})^{-1}(2I - T_{\alpha\beta}) + T_{\beta\beta} = 4, \tag{4.16}$$

where we have assumed $(T_{\alpha\alpha})^{-1} \neq \infty$, in particular, $T_{\mu\nu} \neq g_{\mu\nu}T$.

So far, this is just a reformulation of the wave equation using the definition (4.5). We now apply current algebra by inserting in (4.16) the necessary and sufficient condition (4.6). Using

$$i[E_1, V_\beta] = 2V_1, \quad i[E_1, V_1] = V_\alpha, \quad i[E_1, V_\alpha] = 0, \tag{4.17}$$

and commuting (4.14) once and twice with iE_1, we obtain

$$(4I + 2T_{\beta\alpha})(T_{\alpha\alpha})^{-1}(-T_{\alpha 1}) + (T_{1\alpha})(T_{\alpha\alpha})^{-1}(4I - 2T_{\alpha\beta}) + 2(T_{1\beta} + T_{\beta 1}) = 0 \tag{4.18}$$

and

$$T_{1\alpha}(T_{\alpha\alpha})^{-1}T_{\alpha 1} = T_{11}, \tag{4.19}$$

respectively. Thus current algebra implies (4.18) and (4.19). The important point about (4.19) is that it is an *intrinsic* condition on $T_{\mu\nu}$ and therefore shows at once that not every $T_{\mu\nu}$ in (4.1) can satisfy current algebra. In fact, Eq. (4.19) imposes severe restrictions on $T_{\mu\nu}$, but as they are rather complicated we shall not discuss them further in this paper.

V. MODEL 2

Model 2 is the simplest nontrivial model we could find, namely,

$$(p^4 + 4T_{\mu\nu}p^\mu p^\nu + m)\psi(p) = 0, \quad T_{\mu\nu} = T_{\nu\mu} = T^\dagger_{\mu\nu}, \tag{5.1}$$

$$I_\mu(p'_0p'pp_0) = \tfrac{1}{2}(p'^2 + p^2)(p'_\mu + p_\mu) + 2T_{\mu\nu}(p'^\nu + p^\nu), \tag{5.2}$$

where m is a constant and $\psi(p)$ carries a unitary representation of SL(2, C). I_μ is not separable for all p, but on the hyperplane $p_0 + p_3 = 1$ we have

$$I_\alpha(p'_0p'pp_0) = p'^2 + p^2 + 2T_{\alpha\nu}(p'^\nu + p^\nu), \tag{5.3}$$

and hence

$$I_\alpha(p) = p^2 + 2T_{\alpha\nu}p^\nu, \tag{5.4}$$

$$I \equiv I_\alpha(0) = M^2 + (T_{\alpha\alpha}M^2 + T_{\alpha\beta}) = (1 + T_{\alpha\alpha})M^2 + T_{\alpha\beta}, \text{ on } \phi_n(0), \tag{5.5}$$

where $\beta = 0 - 3$. The current-algebra condition is

$$[\underline{E}, I] = 0, \tag{5.6}$$

and to apply it, it is again convenient to write the wave equation (5.1) on $\phi_n(0)$, i.e.,

$$M^4 + T_{\alpha\alpha}M^4 + (T_{\alpha\beta} + T_{\beta\alpha})M^2 + T_{\beta\beta} + m = 0, \tag{5.7}$$

and then to regard it as an equation for I by substitution from (5.5). We obtain

$$(I + T_{\beta\alpha})(1 + T_{\alpha\alpha})^{-1}(I - T_{\alpha\beta}) + T_{\beta\beta} + m = 0 \tag{5.8}$$

[assuming the existence of $(1 + T_{\alpha\alpha})^{-1}$]. We now commute (5.8) four times with iE_1 and obtain, using the current-algebra condition (5.6),

$$-24T_{\alpha\alpha}(1 + T_{\alpha\alpha})^{-1}T_{\alpha\alpha} + 24T_{\alpha\alpha} = 0 \text{ or } T_{\alpha\alpha} = 0. \tag{5.9}$$

But then

$$T_{\mu\nu} = \lambda g_{\mu\nu} \tag{5.10}$$

and the original combination (5.1), (5.2) reduces to

$$(p^4 + 4\lambda p^2 + m)\psi(p) = 0, \quad I_\mu = \tfrac{1}{2}(p'^2 + p^2 + 4\lambda)(p'_\mu + p_\mu). \tag{5.11}$$

These equations can be solved to yield

$$(p^2 - S)\psi(p) = 0, \quad I_\mu = \pm[(2\lambda)^2 - m]^{1/2}(p'_\mu + p_\mu), \tag{5.12}$$

where $S = 2[\pm(\lambda^2 - \frac{1}{4}m^2)^{1/2} - \lambda]$ is a p-independent scalar, and this combination is manifestly of the form (1.1). Thus in this model we see in a very transparent way how the current algebra reduces the combination (5.1), (5.2) to the form found in Ref. 2.

Note that what reduces is not the wave equation alone but the *combination* of wave equation and current; i.e., the current reduces in such a way that it becomes (up to the factor $\pm[(2\lambda)^2 - m]^{1/2}$) the canonical current for the reduced wave equation. That this reduction is a nontrivial consequence of current algebra will be seen in the next model.

VI. MODEL 3: FACTORIZATION

Model 3 was introduced in Ref. 4 as an apparent counterexample to the results of Ref. 2. It can be written in the form

$$[-P^AP^Bg^{CD}L_{AC}L_{BD}-\delta P^2-\tfrac{1}{4}e^4(p^2-m_-{}^2)^2]\psi=0\,, \tag{6.1}$$

$$I_\mu=(P'^A+P^A)[2m_2\delta_\mu^B+(p'_\mu+p_\mu)\delta_4^B][\tfrac{1}{2}g^{CD}\{L_{AC},L_{BD}\}+\delta g_{AB}]+\tfrac{1}{4}e^4(p'^2+p^2-2m_-{}^2)(p'_\mu+p_\mu)\,, \tag{6.2}$$

where $m_\pm=m_1\pm m_2$, δ, and e^4 are arbitrary numerical parameters, ψ carries a unitary representation of SO(4, 1) generated by L_{AB}, $A=0,1,2,3,4$; $\mu=0,1,2,3$, and

$$P^A=(2m_2p_\mu,m_+m_--p^2)\,. \tag{6.3}$$

The representations of SO(4, 1) used in Ref. 4 are described in the Appendix and are characterized by $-3<N<-\frac{3}{2}$, where $N(N+3)$ is the second-order Casimir operator. The representations for $-2\leqslant N<-\frac{3}{2}$ are somewhat simpler than those for $-3<N<-2$ and we shall limit the discussion of this section to this range of N.

To discuss this model from our point of view, it is convenient to reduce it to SO(3, 1) notation. In that notation (6.1) and (6.2) take the form

$$(Ap^4+2p^2B_\mu p^\mu+S_{\mu\nu}p^\mu p^\nu+2G_\mu p^\mu+F)\psi(p)=0\,, \tag{6.4}$$

$$I_\mu(p_0'p'pp_0)=A(p'^2+p^2)(p'_\mu+p_\mu)+(p'^2+p^2)B_\mu+B_\nu(p'^\nu+p^\nu)(p'_\mu+p_\mu)+S_{\mu\nu}(p'^\nu+p^\nu)+2G_\mu\,, \tag{6.5}$$

respectively, where, with $g_{00}=1$ and $\pi_\mu=L_{\mu4}$, we have

$$\begin{aligned}&G_\mu=m_+m_-m_2\Gamma_\mu,\quad B_\mu=-m_2\Gamma_\mu,\quad \Gamma_\mu=\{L_{\mu\nu},\pi^\nu\}\,,\\&S_{\mu\nu}=4m_2{}^2T_{\mu\nu}-g_{\mu\nu}S,\quad T_{\mu\nu}=(N+3)g_{\mu\nu}-\tfrac{1}{2}\{\pi_\mu,\pi_\nu\}+\tfrac{1}{2}g^{\lambda\sigma}\{L_{\mu\lambda},L_{\nu\sigma}\}\,,\\&S=\tfrac{1}{2}e^4m_-(m_--m_+)+2m_+m_-A+4m_2{}^2(N+3-\delta),\quad A=\pi^2+\tfrac{1}{4}e^4-\delta\,,\\&F=m_+{}^2m_-{}^2A-\tfrac{1}{4}e^4m_-{}^2(m_+{}^2-m_-{}^2)\,.\end{aligned} \tag{6.6}$$

In Eqs. (6.1)–(6.5), p^2 denotes 4-momentum squared.

Thus the wave-equation in model 3 is a generalization of that used in model 2. However, the restrictions on model 3 are more severe, as we are working here within a (special) class of unitary representations of SO(4, 1) and within this class, A, B_μ, $S_{\mu\nu}$, G_μ, and F are restricted by (6.6).

It is pointed out in Ref. 4 that Eq. (6.1) factorizes into

$$[(P^AV_A)^2-P^2-\tfrac{1}{4}e^4(p^2-m_-{}^2)]\psi=0\,, \tag{6.7}$$

where V_A is an SO(4, 1) 5-vector, if and only if $N=-2$. This is correct. But the conclusion, that Eq. (6.1) cannot be derived from an equation of the form (1.1) (and hence contradicts the results of Ref. 2) for $N\neq-2$ is not correct. The point is that (6.7) is a five-dimensional factorization, whereas for (6.1) to be derivable from an equation of the form (1.1) only four-dimensional factorization is necessary. To prove this we shall exhibit explicitly a wave equation of the form (1.1) from which (6.1) can be derived for all $-2\leqslant N<-\frac{3}{2}$. A remarkable feature of (6.1) is that it has this property independently of current algebra. This does not mean that the system (6.4), (6.5) automatically satisfies current algebra because we have no guarantee that the current (6.5) belonging to the wave equation (6.4) will agree with the canonical current belonging to the wave-equation of the form (1.1). We leave the investigation of the current-algebra condition to Sec. VII.

The crucial point in the proof that (6.4) is derivable from an equation of the form (1.1) is that for the representations of SO(4, 1) under consideration, one can prove the identity

$$8T_{\mu\nu}+W_\mu W_\nu^\dagger+W_\nu W_\mu^\dagger=0\,, \tag{6.8}$$

where $T_{\mu\nu}$ is defined in (6.6) and

$$W_\mu=[\Gamma_\mu-i(2N+5)\pi_\mu]b^{-1}\,, \tag{6.9}$$

$$b^2=-\nu^2+(N+2)^2,\quad \text{where } \nu^2-1=\tfrac{1}{2}L_{\mu\sigma}L^{\mu\sigma}\,. \tag{6.10}$$

The operator b^2 is non-negative for all N, $-3<N<-\frac{3}{2}$, and is singular, nonsingular, and positive definite for $-3<N<-2$, $N=-2$, $-2<N<-\frac{3}{2}$, respectively (see Appendix). Hence for the range we are considering it is non-negative and nonsingular.

The proof of (6.8) depends only on the properties of the representations of SO(4, 1) used [note that, by definition, $T_{\mu\nu}$ is in the enveloping algebra of SO(4, 1)] and hence is given in the Appendix. Now let

$$(Dp^2 + D_\mu p^\mu + M)\Psi(p) = 0\,,$$

$$D=\begin{pmatrix}-\alpha & b\\ b & -1\end{pmatrix},\quad D_\mu = m_2\begin{pmatrix}0 & W_\mu\\ W_\mu^\dagger & 0\end{pmatrix},\quad M = -m_-^{\,2}\begin{pmatrix}-\gamma & bm_+/m_-\\ bm_+/m_- & -1\end{pmatrix},\quad \Psi=\begin{pmatrix}\psi\\ \phi\end{pmatrix}, \tag{6.11}$$

where

$$\alpha = \tfrac{1}{4}e^4 + (N+3-\delta)\,,\qquad \gamma = \tfrac{1}{4}e^4 + (m_+^{\,2}/m_-^{\,2})(N+3-\delta)\,. \tag{6.12}$$

By inspection,

$$(bp^2 + m_2 W_\mu p^\mu - m_+ m_- b)\phi = (\alpha p^2 - \gamma m_-^{\,2})\psi\,,$$
$$(bp^2 + m_2 W_\mu^\dagger p^\mu - m_+ m_- b)\psi = (p^2 - m_-^{\,2})\phi\,. \tag{6.13}$$

Hence, by eliminating ϕ,

$$\{(\alpha - b^2)p^4 - 2m_2 p^2 \Gamma_\mu p^\mu - m_2^{\,2}(W_\nu p^\nu)(W_\lambda^\dagger p^\lambda) + [2m_+ m_- b^2 - m_-^{\,2}(\alpha+\gamma)]p^2$$
$$+ 2m_+ m_- m_2 \Gamma_\mu p^\mu + m_-^{\,2}(\gamma m_-^{\,2} - b^2 m_+^{\,2})\}\psi = 0\,. \tag{6.14}$$

But

$$\alpha - b^2 = \pi^2 + \tfrac{1}{4}e^4 - \delta = A\,,$$
$$2m_+ m_- b^2 - m_-^{\,2}(\alpha+\gamma) = -2m_+ m_- A + 2m_+ m_- \alpha - m_-^{\,2}(\alpha+\gamma) = -S\,, \tag{6.15}$$
$$m_-^{\,2}(\gamma m_-^{\,2} - b^2 m_+^{\,2}) = m_-^{\,2}\tfrac{1}{4}e^4(m_-^{\,2} - m_+^{\,2}) + m_-^{\,2}m_+^{\,2}A = F\,.$$

Hence, using (6.8) and (6.6), we obtain from (6.14) the equation

$$\{Ap^4 + 2p^2 B_\mu p^\mu + 4m_2^{\,2}T_{\mu\nu}p^\mu p^\nu - Sg_{\mu\nu}p^\mu p^\nu + 2G_\mu p^\mu + F\}\psi(p) = 0\,, \tag{6.16}$$

which is exactly Eq. (6.4), as required.

The derivation of (6.14) from (6.11) shows not only that Eq. (6.1) can be derived from an equation of the form (1.1) for all $-2 \le N < -\frac{3}{2}$, but also that the problems of factorizing the wave equation and satisfying current algebra are not directly related. They may be indirectly related through the current, because although the canonical current $D(p_\mu' + p_\mu) + D_\mu$ for (6.11) will automatically satisfy current algebra with the metric $(D + D_\alpha)^{-1}$ (at $p_3 = \infty$), we do not know whether the canonical current (6.5) agrees with this current, and whether it satisfies current algebra. This is the question which will be considered in Sec. VII.

VII. MODEL 3: CURRENT-ALGEBRA CONDITION

To investigate the condition imposed on the current of model 3 by current algebra at infinite momentum, we note that on the hypersurface $p_0 + p_3 = 1$, $I_\alpha(p_0' p' p p_0)$ of (6.5) reduces to

$$I_\alpha(p_0' p' p p_0) = 2A(p'^2 + p^2) + B_\alpha(p'^2 + p^2) + 2B_\mu(p'^\mu + p^\mu) + S_{\alpha\nu}(p'^\nu + p^\nu) + 2G_\alpha\,, \tag{7.1}$$

which is separable, with

$$I_\alpha(p) = (2A + B_\alpha)p^2 + 2B_\mu p^\mu + S_{\alpha\nu}p^\nu + G_\alpha; \tag{7.2}$$

hence

$$I \equiv I_\alpha(0) = (2A + B_\alpha)M^2 + (B_\alpha M^2 + B_\beta) + \tfrac{1}{2}(S_{\alpha\alpha}M^2 + S_{\alpha\beta}) + G_\alpha$$
$$= (2A + 2B_\alpha + \tfrac{1}{2}S_{\alpha\alpha})M^2 + (B_\beta + \tfrac{1}{2}S_{\alpha\beta} + G_\alpha)$$
$$= 2XM^2 + Y\,, \tag{7.3}$$

where M^2 is the mass operator on the states

$$\phi_n(0) = \sqrt{m(n)}\,\exp[i\ln m(n) K_3]\psi_n(0).$$

To apply the current-algebra condition which, as we have seen in Sec. III, is

$$[\underline{E}, I] = 0 \tag{7.4}$$

for all currents which are separable on the hyperplane, we proceed as in the simple model 2. First, we write the wave equation on $\phi_n(0)$, where it reduces to

$$AM^4+(B_\alpha M^2+B_\beta)M^2+\tfrac{1}{4}[S_{\alpha\alpha}M^4+2S_{\alpha\beta}M^2+S_{\beta\beta}]+(G_\alpha M^2+G_\beta)+F=0 \tag{7.5}$$

or

$$XM^4+YM^2+Z=0\,, \tag{7.6}$$

where X, Y are as defined in (7.3), and

$$Z=\tfrac{1}{4}S_{\beta\beta}+G_\beta+F\,. \tag{7.7}$$

We now assume that A is nonsingular. In that case (see Appendix), X is nonsingular, and if we substitute I for M^2 in (7.6), using (7.3), we obtain

$$(I+Y)X^{-1}(I-Y)+4Z=0\,. \tag{7.8}$$

We now apply the current-algebra condition (7.4). Commuting (7.8) four times with iE_1, and using (7.4) and the definitions of X, Y, Z, we obtain as a *necessary* condition for current algebra

$$6(2B_\alpha+S_{\alpha\alpha})X^{-1}(-2B_\alpha-S_{\alpha\alpha})+24S_{\alpha\alpha}=0 \tag{7.9}$$

or

$$B_\alpha S_{\alpha\alpha}{}^{-1}B_\alpha=A\,, \tag{7.10}$$

where, by definition (6.6), $S_{\alpha\alpha}{}^{-1}$ exists unless $m_2=0$. (If $m_2=0$, Eqs. (6.4) and (6.5) collapse to the model of Sec. V for the trivial case in which the current actually vanishes $[(2\lambda)^2=m]$.) With $m_2\neq 0$, Eq. (7.10) can be analyzed using SO(4, 1) techniques (see Appendix) and the result is that a necessary and sufficient condition for (7.10) is

$$\tfrac{1}{4}e^4-\delta=-\tfrac{1}{2},\quad N=-\tfrac{5}{2}\,, \tag{7.11}$$

but in that case A is singular. It follows that the current-algebra condition can be satisfied for at most those cases for which A is singular, and since for the representation $-3<N<-2$, A can be singular only for special values of the parameters, and for the representations $-2\leq N<-\frac{3}{2}$, it is strictly nonsingular, we see that, in general, the current of Eqs. (6.4) and (6.5) does not satisfy current algebra.

At first sight this result seems to contradict that of Ref. 4. It will be shown in Sec. VIII that there is no real contradiction. The basic reason is that in Ref. 4 the conventional requirement of positive norm for the *physical* states has been relaxed, thus allowing a wider but, in this respect, unphysical class of solution.

VIII. COMPARISON WITH OTHER RESULTS

We first wish to compare the results of Sec. VII with those of Ref. 4. For this purpose, it turns out to be instructive to first consider an earlier paper[9] by one of the same authors, in which a similar result is obtained for finite-dimensional representations of SL(2, C), since this case is simpler and is also in apparent contradiction with Sec. II above. To see that there is no real contradiction, we take the simplest possible case, namely the scalar Klein-Gordon equation of mass m, with both positive- and negative-frequency solutions. The current postulated for this case is

$$\langle n'p'|J_0(x,0)|np\rangle=\psi^\dagger_{n'}(p')(p'_0+p_0)\psi_n(p)e^{i(p'-p)\cdot x}\,, \tag{8.1}$$

where $n, n'=\pm$, or, in the notation of our Sec. II, where $H(p)$ is now 2×2,

$$I(p'p)=\begin{pmatrix}\omega'+\omega & \omega'-\omega\\ \omega-\omega' & -(\omega+\omega')\end{pmatrix},\quad \omega=+(p^2+m^2)^{1/2}\,. \tag{8.2}$$

The current-algebra condition (2.15) of Sec. II above now reads

$$\begin{aligned}I(p'p'')I^{-1}(p''p'')I(p''p)-I(p'p)&=I(pp'')\left\{\sum_n\frac{\psi_n(p'')\psi_n^\dagger(p'')}{\eta(p'')}\right\}I(p''p)-I(p'p)\\ &=\begin{pmatrix}\omega'+\omega'' & \omega'-\omega''\\ -\omega'+\omega'' & -\omega'-\omega''\end{pmatrix}\begin{pmatrix}(2\omega'')^{-1} & 0\\ 0 & -(2\omega'')^{-1}\end{pmatrix}\begin{pmatrix}\omega''+\omega & \omega''-\omega\\ \omega-\omega'' & -\omega''-\omega\end{pmatrix}-\begin{pmatrix}\omega'+\omega & \omega'-\omega\\ \omega-\omega' & -\omega'-\omega\end{pmatrix}\\ &=0\,,\end{aligned} \tag{8.3}$$

where $\eta_n(p)=2n\,\omega(p)$, and this equation turns out to be an identity. Thus current algebra is automatically satisfied for the model, as claimed, and the conclusion seems to contradict the results of our Sec. II since $I(p'p)$ is manifestly p',p-dependent.

The resolution of the paradox is that with the choice (8.1) of the current, the charge cannot be positive as demanded by current algebra [Eq. (2.7)] unless the norm of the physical states is not definite. Indeed, integrating (8.1) over x and using $Q=1$ for $n'=n$, we obtain

$$(2\pi)^{-3}\langle np'|np\rangle=2\psi_n^\dagger(p)p_0\psi_n(p)\delta(p'-p)=\eta_n(p)\psi_n^\dagger(p)\psi_n(p)\delta(p'-p)\,, \tag{8.4}$$

which is not definite since $\eta_n(p)=\pm 2\,\omega(p)$ for $n=\pm$, by definition. It should perhaps be emphasized that the indefinite norm of the physical states is known to the authors of Ref. 9 and is stated explicitly. However, what the comparison of that paper with our Sec. II shows is that the indefinite norm is of critical importance in evading the conditions imposed by current algebra. With positive norm we get the strong condition (2.23) leading to a linear wave equation. With indefinite norm we get no condition.

We are now in a better position to understand why the model of Ref. 4 does not contradict the results of Ref. 2. In the model, we have an equation similar to (8.3), namely,

$$I(p'p'')\left[\sum_{\pm}\left(\sum_n+\int dn\right)\frac{\psi_n(p)\psi_n^\dagger(p)}{\eta_n(p)}\right]I(p''p)=I(p'p)\,, \tag{8.5}$$

where the integral represents the contribution from continuum states, so that, just as in (8.3), current algebra is formally satisfied. But just as in (8.3) formal satisfaction is not enough. The twin requirements of positive (unit) charge and positive norm for all physical states [i.e., all states used in the summation and integration in (8.5)] have still to be imposed, and until they are imposed and shown to be satisfied by I there is, as in the case of (8.3), no contradiction with the results of Ref. 2. In point of fact all that is actually true in Ref. 4 is that $p_0\eta_n(p)>0$ for the discrete states, and this result has already been seen to be insufficient for the simple case (8.3) where there is no continuum contribution.

In the above connection two remarks might not be out of place. First, it has already been remarked by Leutwyler[10] that if one relaxes the positivity of the norm, solutions of the isospin-factored algebra which are physically satisfactory in every other way, can easily be constructed. Secondly, one might ask why, if we are so insistent on the positivity of the physical norm, do we accept the wave equations (1.1) which have spacelike parts that are equally unphysical. The answer is that we have not been able to prove conclusively that the spacelike parts are coupled to the timelike parts by the current in all nontrivial cases. If, as we suspect, the two parts are coupled by the current in all but trivial cases, then indeed the wave equations (1.1) would not be acceptable, and there would be *no* physically satisfactory solution to the problem of saturating the isospin-factored current algebra at $p_3=\infty$. The conclusion would then be either that many-particle states are necessary for saturation, or that in the physical world the isospin explodes; i.e., we should expect K's and Ξ's with $I\geq\frac{3}{2}$ from future experiments. And indeed there are presently some strong indications of the existence of resonances with exotic values of isospin.[11]

We turn now to other authors. In Ref. 12 it is shown (theorem 1) that a current of the form (1.3) is *sufficient* to satisfy the isospin-factored algebra at $p_3=\infty$. The results of Ref. 2 indicate, as we have said, that this form of the current is *necessary* as well as sufficient, and in Sec. IV we have partially verified this result in the sense that we have shown that not every current of the form $T_{\mu\nu}p^\nu+p^\nu T_{\nu\mu}$ can be a solution. This shows in particular, that the restriction of the authors of Ref. 12 from the general current with which they start to currents of the form (1.3) is a necessary prerequisite for their results. References 3 and 13 represent, as far as we know, the only attacks made on the nonfactored problem and so are beyond the scope of the present paper. We note only that the relation $Q=1$ which is deduced in the factored case, is not necessarily true in the nonfactored case, but has been imposed and plays an important role in Ref. 3.

APPENDIX: TECHNICAL DETAILS FOR MODEL 3

The class of unitary irreducible representations of SO(4, 1) underlying model 3 has been described by Fronsdal[14] and is characterized by the values of the two Casimir operators

$$Q\equiv\tfrac{1}{2}L_{AB}L^{AB}=N(N+3),\quad -3<N<-\tfrac{3}{2} \tag{A1}$$

$$R\equiv N_AN^A=0,\quad N^A=\tfrac{1}{8}\epsilon^{ABCDE}L_{BC}L_{DE}\,, \tag{A2}$$

where L_{AB} are the SO(4, 1) generators and ϵ^{ABCDE} is the Levi-Civita symbol ($A=0,1,2,3,4;\ g_{00}=1$). It can be shown that actually

$$N_A = 0 . \quad \text{(A3)}$$

For $-2 \leqslant N < -\frac{3}{2}$, the two conventional Casimir operators (j_0, ν) of $SO(3,1) \subset SO(4,1)$ take the values $j_0 = 0, -\infty < \nu^2 \leqslant 0$, and for $-3 < N < -2$ we may have in addition the eigenvalue $\nu^2 = (N+2)^2$. It is to avoid the complications arising from this discrete eigenvalue that we restrict ourselves to the range $-2 \leqslant N < -\frac{3}{2}$ in Sec. VI.

The two SO(4, 1) vectors which appear in Secs. VI and VII are $\pi_\mu = L_{\mu 4}$ and

$$\Gamma_\mu = \{L_{\mu\nu}, \pi^\nu\} = i[\pi^2, \pi_\mu] , \quad \text{(A4)}$$

and in the sequel we shall need the inner products

$$\pi^2 = \nu^2 - 1 - Q , \quad \text{(A5)}$$

$$\Gamma^2 = 9\pi^2 + 4(\pi^2 + 1)(\pi^2 + Q) , \quad \text{(A6)}$$

$$\pi \cdot \Gamma = -\Gamma \cdot \pi = i[3\pi^2 + 2(\pi^2 + Q)] , \quad \text{(A7)}$$

which can be calculated directly from (A1)–(A4). It will be convenient to use an $E(2)$ basis for SO(3, 1),[15] i.e., to let SO(3, 1) act on $L_2(\underline{\epsilon})$ where $\underline{\epsilon}$ is a two-dimensional vector. In this basis π_μ takes the form

$$\pi_\alpha = g\epsilon, \quad \alpha = 0+3, \quad \epsilon^2 = \epsilon_1^2 + \epsilon_2^2 , \quad \text{(A8)}$$

where g is an operator in the space of the Casimir operator ν^2 only, and satisfies

$$[\nu^2, [\nu^2, g]] = 2\{\nu^2, g\} - g . \quad \text{(A9)}$$

Using the relation $[\pi_\mu, \pi_\nu] = i L_{\mu\nu}$ we have also the condition

$$[g, [g, \nu^2]] = 2(1 + g^2) , \quad \text{(A10)}$$

and using (A5) and (A8)–(A10) we can write Q as

$$Q = -\tfrac{5}{4} + \nu^2 + g\nu^2 g - \tfrac{1}{4}(1 + g^2) - \tfrac{1}{4}[\nu^2, g]^2 . \quad \text{(A11)}$$

We now turn to the calculations of Secs. VI and VII.

Section VII. In our present notation

$$\begin{aligned} A &= \nu^2 - 1 - Q + \tfrac{1}{4}e^4 - \delta , \\ B_\alpha &= -im_2[\nu^2, g]\epsilon , \\ S_{\alpha\alpha} &= -4m_2^2(1+g^2)\epsilon^2 , \\ X &= e^{im_2 g\epsilon} A e^{-im_2 g\epsilon} . \end{aligned} \quad \text{(A12)}$$

In particular, X is nonsingular if A is nonsingular. We next analyze the condition $B_\alpha S_{\alpha\alpha}^{-1} B_\alpha = A$, which then reduces to

$$[\nu^2, g](1+g^2)^{-1}[\nu^2, g] = 4(\nu^2 - Q - 1 + \tfrac{1}{4}e^4 - \delta) , \quad \text{(A13)}$$

where $1 + g^2 > 0$, since $g = g^\dagger$. Now from (A10) we have

$$[[\nu^2, g], (1+g^2)^{-1}] = -4g(1+g^2)^{-1} . \quad \text{(A14)}$$

Hence the denominator in (A13) can be pulled to the left, and we obtain

$$[\nu^2, g]^2 = 4g\nu^2 g - 4g^2\nu^2 + 4(1+g^2)(\nu^2 - Q - 1 + \tfrac{1}{4}e^4 - \delta) . \quad \text{(A15)}$$

Comparing this equation with (A11) we have

$$g^2(e^4 - 4\delta - 4Q - 3) + (e^4 - 4\delta + 2) = 0 . \quad \text{(A16)}$$

But from (A10) we see that g cannot be a c number; hence (A16) splits into

$$\tfrac{1}{4}e^4 - \delta = -\tfrac{1}{2} \text{ and } Q = -\tfrac{5}{4}, \quad \text{(A17)}$$

as required.

Section VI. The relation to be proved is

$$8T_{\mu\nu} + W_\mu W_\nu^\dagger + W_\nu W_\mu^\dagger = 0 , \quad \text{(A18)}$$

where $T_{\mu\nu}$ and W_μ are defined in (6.6) and (6.9), respectively. Since $T_{\mu\nu}$ is symmetric (A18) has only the traceless and scalar components

$$4T_{\alpha\alpha} + W_\alpha W_\alpha^\dagger = 0 , \quad \alpha = 0+3 \quad \text{(A19)}$$

and

$$4T_\mu{}^\mu + W_\mu W^{\dagger\mu} = 0 , \quad \text{(A20)}$$

respectively, and it will be convenient to treat these separately.

In our present notation the traceless component (A19) can be written in the form

$$4(1+g^2) - \{[\nu^2, g] - (2N+5)g\} b^{-2} \{[\nu^2, g] + (2N+5)g\} = 0 \quad \text{(A21)}$$

or, equivalently,

$$\{[\nu^2, g] + (2N+5)g\}(1+g^2)^{-1}\{[\nu^2, g] - (2N+5)g\} = -4b^2 = 4[\nu^2 - (N+2)^2] . \quad \text{(A22)}$$

Using (A14) to pull the factor $(1+g^2)^{-1}$ to the left, we obtain

$$[\nu^2, g]^2 = 4(\nu^2 - Q - \tfrac{3}{2}) + 4g\nu^2 g - g^2 , \quad \text{(A23)}$$

and since this equation is just a modified form of (A11), Eq. (A19) is established.

The scalar component (A20) can be written in the form

$$4(\nu^2+N^2+7N+11)+\{\Gamma_\mu - i(2N+5)\pi_\mu\}b^{-2}\{\Gamma^\mu + i(2N+5)\pi^\mu\}=0\,, \tag{A24}$$

and to extract the b^{-2} factor we proceed as follows: From the definitions of b^2 and Γ_μ and Eqs. (A8) and (A9) we have

$$[b^2,\pi_\mu]=i\Gamma_\mu,\quad [b^2,\Gamma_\mu]=-i(4\nu^2-1)\pi_\mu+2\Gamma_\mu\,, \tag{A25}$$

whence

$$\pi_\mu\frac{1}{b^2}=\frac{1}{b^2}\pi_\mu+\frac{i}{b^2}\Gamma_\mu\frac{1}{b^2}\,,\quad \Gamma_\mu\frac{1}{b^2}=\frac{1}{b^2}\Gamma_\mu-\frac{i(4\nu^2-1)}{b^2}\pi_\mu\frac{1}{b^2}+\frac{2}{b^2}\Gamma_\mu\frac{1}{b^2} \tag{A26}$$

and

$$\begin{aligned}
&\pi_\mu\frac{1}{b^2}\pi^\mu=\frac{1}{b^2}\pi^2+\frac{i}{b^2}\Gamma_\mu\frac{1}{b^2}\pi^\mu\,,\quad \Gamma_\mu\frac{1}{b^2}\pi^\mu=\frac{1}{b^2}\Gamma\cdot\pi-\frac{i(4\nu^2-1)}{b^2}\pi_\mu\frac{1}{b^2}\pi^\mu+\frac{2}{b^2}\Gamma_\mu\frac{1}{b^2}\pi^\mu\,,\\
&\pi_\mu\frac{1}{b^2}\Gamma^\mu=\frac{1}{b^2}\pi\cdot\Gamma+\frac{i}{b^2}\Gamma_\mu\frac{1}{b^2}\Gamma^\mu,\quad \Gamma_\mu\frac{1}{b^2}\Gamma^\mu=\frac{1}{b^2}\Gamma^2-\frac{i(4\nu^2-1)}{b^2}\pi_\mu\frac{1}{b^2}\Gamma^\mu+\frac{2}{b^2}\Gamma_\mu\frac{1}{b^2}\Gamma^\mu\,.
\end{aligned} \tag{A27}$$

Solving these four equations we obtain, using (A6) and (A7),

$$\begin{aligned}
\begin{Bmatrix}\pi_\mu b^{-2}\pi^\mu\\ \pi_\mu b^{-2}\Gamma^\mu\\ \Gamma_\mu b^{-2}\pi^\mu\\ \Gamma_\mu b^{-2}\Gamma^\mu\end{Bmatrix}
&=\frac{1}{(b^2-1)^2-4\nu^2}\begin{Bmatrix}(b^2-2)\pi^2+i\Gamma\cdot\pi\\ (b^2-2)\pi\cdot\Gamma+i\Gamma^2\\ b^2\Gamma\cdot\pi-i(4\nu^2-1)\pi^2\\ b^2\Gamma^2-i(4\nu^2-1)\pi\cdot\Gamma\end{Bmatrix}\\
&=\frac{1}{(b^2-1)^2-4\nu^2}\begin{Bmatrix}(b^2+1)\pi^2+2(\nu^2-1)\\ i[(3b^2+4\nu^2-1)\pi^2+2b^2(\nu^2-1)]\\ -i[(3b^2+4\nu^2-1)\pi^2+2b^2(\nu^2-1)]\\ (5b^2+4b^2\nu^2+12\nu^2-3)\pi^2+2(2b^2+4\nu^2-1)(\nu^2-1)\end{Bmatrix}.
\end{aligned} \tag{A28}$$

Inserting these results into (A24) and using the definition of b^2 and (A5), we obtain an identity. This establishes Eq. (A20) and hence Eq. (A18).

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[1]R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966).

[2]S.-J. Chang, R. Dashen, and L. O'Raifeartaigh, Phys. Rev. Letters 21, 1026 (1968); Phys. Rev. 182, 1819 (1969).

[3]A. Barut and G. Komen, Phys. Rev. D 1, 418 (1970).

[4]C. Fronsdal and A. M. Harun-Ar Rashid, ICTP Trieste report (unpublished).

[5]A further assumption, that solutions for which a spacelike part existed and was definitely coupled to the timelike part by the current be rejected, has since been shown to be unnecessary by T. Garavaglia, Phys. Rev. D 3, 627 (1971).

[6]S.-J. Chang and L. O'Raifeartaigh, Phys. Rev. 171, 1587 (1968); H. Bebié, F. Ghielmetti, V. Gorgé, and H. Leutwyler, *ibid.* 177, 2133 (1969); 177, 2146 (1969).

[7]S.-J. Chang and L. O'Raifeartaigh, Phys. Rev. 170, 1316 (1968).

[8]For the kinematics of the infinite-momentum limit, see H. Leutwyler, Acta Phys. Austriaca Suppl. V, 320 (1968).

[9]G. Cocho, C. Fronsdal, and R. White, Phys. Rev. 180, 1547 (1969).

[10]H. Leutwyler, Phys. Letters 31B, 214 (1970).

[11]S. Meshkov, International Conference on Duality and Symmetry in Hadron Physics, Tel-Aviv, Israel, 1971 (unpublished).

[12]B. Hamprecht and H. Kleinert, Phys. Rev. 180, 1410 (1969).

[13]I. Khan, U. Niederer, and L. O'Raifeartaigh, Proc. Royal Irish Acad. 70A, 47 (1970).

[14]C. Fronsdal, Phys. Rev. 156, 1665 (1967).

[15]S.-J. Chang and L. O'Raifeartaigh, J. Math. Phys. 10, 21 (1969).