Further Constraints on the $\pi^0\pi^0$ s-Wave Amplitude*

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On the basis of positivity and crossing symmetry, we derive further constraints on the $\pi^0\pi^0$ s wave. One of these relates the physical region to the region below threshold, where crossing is easy to apply. It is sufficiently strong that none of the models of unitarized $\pi\pi$ partial waves which we have examined satisfies it. We also derive sufficient conditions which indicate that our results cannot be improved very much. Finally, we show how the knowledge of the s wave below threshold can give us useful information about the physical partial waves.

I. INTRODUCTION

In recent years considerable effort has been devoted to the study of rigorous constraints on the $\pi\pi$ partial-wave amplitudes, and many different results have been obtained and have proved to be very useful in constructing low-energy models for $\pi\pi$ scattering.[1] Martin and his collaborators[2] have derived a large number of such constraints on the basis of crossing symmetry, analyticity, and the positivity of the absorptive parts of the amplitudes. Based on the work of Balachandran and Nuyts,[3] one of us[4] has found a general crossing-symmetric parametrization of the $\pi\pi$ partial-wave amplitudes below threshold, and has also obtained some restrictions[5] imposed by positivity on this parametrization. In this paper we present a more powerful technique than that of Ref. 5 for obtaining the positivity constraints on the crossing-symmetric expansion of the $\pi^0\pi^0$ partial-wave amplitudes, and derive stronger results. We also present some sufficient conditions to ensure both positivity and crossing symmetry. We indicate how our constraints can be useful in analyzing models of $\pi^0\pi^0$ scattering.

The philosophy of this paper differs from others[6] which have recently considered the same problem. All the constraints can be reformulated as conditions on the coefficients in the expansion of the s wave in terms of an orthogonal set of polynomials in the unphysical region. The previous authors have tried to find general conditions which can be used to restrict all the coefficients. On the other hand, we are interested here in obtaining stronger conditions on the coefficients of the polynomials of low order, those that govern the grosser features of the behavior of the s wave. We feel that, at the moment, it is more meaningful to compare these grosser features with predictions of a given model of the $\pi^0\pi^0$ s wave. Also, by concentrating on these lower coefficients, we are able to show that our new constraints are almost optimal.

We will follow Ref. 5 closely in formulating the problem. According to Ref. 4, the elastic $\pi^0\pi^0$ partial waves below threshold can be parametrized as

$$f_l(s)=\sum_{\sigma=l}^{\infty} 2(\sigma+1)A_{\sigma l}(1-s)^l P_{\sigma-l}^{(2l+1,0)}(2s-1) \quad (1.1)$$

for $l=0, 2, 4, \ldots,$ where $P_{\sigma-l}^{(2l+1,0)}(2s-1)$ are Jacobi polynomials and

$$A_{\sigma l}=\sum_{p=\{\sigma/3\}}^{[\sigma/2]}(C_p^\sigma)(b_p^\sigma)_l\,. \quad (1.2)$$

Here C_p^σ are arbitrary constants, and $(b_p^\sigma)_l$ are given by

$$(b_p^\sigma)_l=\frac{(\sigma-l)!(\sigma+l+1)!}{(2\sigma+1)!}\times\int_{-1}^{1}dz\,P_l(z)(z^2+3)^{3p-\sigma}(1-z^2)^{\sigma-2p}. \quad (1.3)$$

We have used the notation

$\{\sigma/3\}$ = smallest integer $\geq \sigma/3$,

$[\sigma/2]$ = largest integer $\leq \sigma/2$,

and have chosen units such that the pion mass $=\frac{1}{2}$.

Notice that in this parametrization only the arbitrary constants C_p^σ, but not $A_{\sigma l}$, are independent. Consequently, $A_{\sigma l}$ with the same σ but different l will be related to each other through Eq. (1.2). Explicit calculation for the first few $(b_p^\sigma)_l$ from (1.3) will show, for example, that we must have

$$\begin{aligned}&\text{for } \sigma=1, \quad A_{10}=0;\\&\text{for } \sigma=2, \quad A_{20}=\tfrac{5}{2}A_{22};\\&\text{for } \sigma=3, \quad A_{30}=-A_{32};\\&\text{for } \sigma=4, \quad A_{40}=\tfrac{7}{2}A_{42}=\tfrac{7}{2}A_{44};\\&\text{for } \sigma=5, \quad A_{50}=-2A_{52}=-2A_{54};\end{aligned} \quad (1.4)$$

and so on. In general, for a given σ there will be $\{\sigma/3\}$ such relations, each involving only those $A_{\sigma l}$ with $l\leq\sigma$. It is exactly the whole set of these relations among $A_{\sigma l}$ that makes the parametrization (1.1) crossing symmetric.

Now in the range $0<s<1$ one can also express $f_l(s)$ for $l \geq 2$ by a Froissart-Gribov formula

$$f_l(s)=\frac{4}{\pi(1-s)}\int_1^\infty dt\, A(t,s)Q_l\left(\frac{2t}{1-s}-1\right), \qquad (1.5)$$

where

$$A(t,s)=\sum_{l'\text{even}}(2l'+1)\operatorname{Im}f_{l'}(t)P_{l'}\left(\frac{2s}{t-1}+1\right), \qquad (1.6)$$

with

$$\operatorname{Im}f_{l'}(t)\geq 0. \qquad (1.7)$$

The representation (1.5)–(1.7) satisfies explicitly the positivity condition, but needs further constraints on $\operatorname{Im}f_{l'}(t)$ in order to satisfy crossing symmetry. On the other hand, the representation (1.1)–(1.3) satisfies crossing symmetry by construction, but needs further restrictions on the arbitrary constants C_p^σ to meet the positivity requirement. Therefore, on equating these two representations of $f_l(s)$, one will obtain both the crossing constraints on $\operatorname{Im}f_{l'}(t)$ and the positivity constraints on C_p^σ. In this paper we find some necessary and some sufficient constraints on C_p^σ for $\sigma=2, 3, 4$ for $f_l(s)$ of (1.1) to satisfy positivity [i.e., to be expressible also by the representation (1.5)–(1.7)].

The paper is organized as follows. In Sec. II we analyze some of the crossing conditions (1.4) which can be transformed into crossing sum rules for $\operatorname{Im}f_{l'}(t)$ in the physical region. Using these we describe in Sec. III a method for obtaining constraints on C_p^σ for $\sigma=2, 3, 4$, and then generalize these to higher σ in Sec. IV. In Sec. V we calculate the allowable region of C_p^σ from a Mandelstam representation with the most general positive double-spectral function, and compare it with our corresponding region. The reader interested only in our results can turn immediately to Sec. VI. There we reformulate the results of Sec. III as constraints on the s wave alone, and then examine whether existing models satisfy them. Finally in Sec. VII we give an interpretation of the constraints on C_p^σ and describe how our analysis can be of practical value in analyzing models for the $\pi^0\pi^0$ scattering. Conclusions are presented in Sec. VIII. Appendix A contains useful properties of the functions $B_{\sigma l}^{l'}(t)$ which will appear in Sec. II, while Appendix B contains the detailed numerical descriptions of the allowable region for C_p^σ which we found in Sec. III.

II. CROSSING CONSTRAINTS ON $\operatorname{Im}f_{l'}(t)$

One can invert Eq. (1.1) to solve for $A_{\sigma l}$ in terms of $f_l(s)$:

$$A_{\sigma l}=\int_0^1 ds\,(1-s)^{l+1}P_{\sigma-l}^{(2l+1,0)}(2s-1)f_l(s), \qquad (2.1)$$

where $\sigma \geq l$ and $l=0, 2, 4, \ldots$. Now for $l\geq 2$, one can insert the representation (1.5)–(1.7) into (2.1) and get

$$A_{\sigma l}=\frac{4}{\pi}\sum_{l'=0}^{\infty}(2l'+1)\int_1^\infty dt\operatorname{Im}f_{l'}(t)B_{\sigma l}^{l'}(t), \qquad (2.2)$$

where

$$B_{\sigma l}^{l'}(t)=\int_0^1 ds\,(1-s)^l P_{\sigma-l}^{(2l+1,0)}(2s-1) \times Q_l\left(\frac{2t}{1-s}-1\right)P_{l'}\left(\frac{2s}{t-1}+1\right). \qquad (2.3)$$

Although only $B_{\sigma l}^{l'}(t)$ with even l appear in our problem here, $B_{\sigma l}^{l'}(t)$ with odd l are also well defined by Eq. (2.3), and all of them will be considered for the moment. Several important properties of $B_{\sigma l}^{l'}(t)$ will be noted here. First of all, $B_{\sigma\sigma}^{l'}(t)$ is positive for all $t\geq 1$; for the Jacobi polynomial of the zeroth degree is identically equal to one, and hence the integrand in (2.3) becomes positive for all s in the range of integration.

Next $B_{\sigma l}^{l'}(t)$ can be proved to satisfy the following recurrence relation (see Appendix A):

$$2(2l+1)l'(l'+1)B_{\sigma l}^{l'}(t)=(l+1)(\sigma-l)(\sigma+l+2)[B_{\sigma l+1}^{l'}(t)+B_{\sigma l}^{l'}(t)]+l(\sigma-l+1)(\sigma+l+1)[B_{\sigma l}^{l'}(t)+B_{\sigma l-1}^{l'}(t)], \qquad (2.4)$$

by which one can express each $B_{\sigma l}^{l'}(t)$ as $B_{\sigma\sigma}^{l'}(t)$ times a coefficient that depends on σ, l, and l', but not on t. In particular, we find

$$B_{42}^{l}(t)=[\tfrac{7}{48}l(l+1)(l^2+l-8)+1]B_{44}^{l}(t) \qquad (2.5)$$

which will be useful later on.

Before we go on any further, however, we will invoke a useful theorem[7]: If the fixed-t dispersion relation for $\pi^0\pi^0$ elastic scattering requires two subtractions, then given partial-wave amplitudes $f_l(s)$ consistent with crossing, analyticity, and the positivity condition

$$\operatorname{Im}f_l(s)\geq 0, \quad s\geq 1, \quad l=0, 2, 4, \ldots \qquad (2.6)$$

we can find new amplitudes $\bar{f}_l(s)$ satisfying the same properties for which

$$\operatorname{Im}\bar{f}_l(s)=\operatorname{Im}f_l(s), \quad s\geq 1, \quad l\geq 2$$

but for which $\operatorname{Im}\bar{f}_0(s)$ is arbitrary, except for (2.6).

This theorem means that the imaginary part of the s-wave amplitude is not subject to any crossing constraints. In view of this unique property of the s wave, it will be convenient to isolate the s-wave contributions to $A_{\sigma l}$ in (2.2), and hence, to write, for $l \geq 2$,

$$A_{\sigma l} = A^0_{\sigma l} + \bar{A}_{\sigma l}\,, \tag{2.7}$$

where

$$A^0_{\sigma l} = \frac{4}{\pi}\int_1^\infty dt\,\mathrm{Im}f_0(t)B^0_{\sigma l}(t) \tag{2.8}$$

and

$$\bar{A}_{\sigma l} = \frac{4}{\pi}\sum_{l'=2}^{\infty}(2l'+1)\int_1^\infty dt\,\mathrm{Im}f_{l'}(t)B^{l'}_{\sigma l}(t)\,. \tag{2.9}$$

Since $\mathrm{Im}f_0(t)$ can be arbitrary for the amplitude to remain crossing symmetric, according to the theorem $A^0_{\sigma l}$ automatically satisfies the crossing conditions (1.4), and therefore $\bar{A}_{\sigma l}$ must satisfy them too; e.g.,

$$\begin{aligned} &\text{for } \sigma = 4\,, \quad \bar{A}_{42} = \bar{A}_{44}\,; \\ &\text{for } \sigma = 5\,, \quad \bar{A}_{52} = \bar{A}_{54}\,; \end{aligned} \tag{2.10}$$

and so on. Equations (2.9) and (2.10) will give rise to an infinite set of crossing sum rules for $\mathrm{Im}f_{l'}(t)$ for $l' \geq 2$. For example, the first crossing condition in (2.10) is equivalent to

$$\sum_{l=2}^{\infty}(2l+1)\int_1^\infty dt\,\mathrm{Im}f_l(t)\left[B^l_{42}(t) - B^l_{44}(t)\right] = 0\,. \tag{2.11}$$

This set of crossing sum rules is equivalent to the set derived in Ref. 7 or Ref. 8.

III. POSITIVITY CONSTRAINTS ON C^σ_p FOR σ = 2, 3, 4

Now we turn to our main problem, to find the positivity constraints on C^σ_p for $\sigma = 2, 3, 4$. Here we have only C^2_1, C^3_1, and C^4_2 to consider. However, it will be more convenient to work with $A_{\sigma l}$ instead of C^σ_p directly. The relevant $A_{\sigma l}$ in this case are [see (1.2)]

$$A_{22} = \tfrac{4}{15}C^2_1\,, \quad A_{32} = -\tfrac{4}{105}C^3_1\,,$$

and

$$A_{42} = A_{44} = \tfrac{16}{315}C^4_2\,.$$

The only crossing condition given by (2.10) is $\bar{A}_{44} = \bar{A}_{42}$, or equivalently (2.11).

Since the positivity condition (1.7) does not restrict in any way the over-all scale of $f_l(s)$ in (1.5), the absolute scale of all $A_{\sigma l}$ in (1.1) is left undetermined. Hence we can choose to use $A_{\sigma l}/A_{22}$ instead of $A_{\sigma l}$ in our analysis. Let us call

$$x = -A_{32}/A_{22} \quad \text{and} \quad y = A_{44}/A_{22}\,. \tag{3.1}$$

Since we will force our $\mathrm{Im}f_{l'}(t)$ to obey the crossing sum rule (2.11), the equality $\bar{A}_{44} = \bar{A}_{42}$ and hence $A_{44} = A_{42}$ will always be guaranteed. Therefore, there is no need from now on to consider A_{42} separately in this section.

Ideally one would like to find a region of (x, y) which is necessary and sufficient for A_{22}, A_{32}, and A_{44} to be expressible by the representation (2.2) with $\mathrm{Im}f_{l'}(t)$ satisfying both the positivity condition (1.7) and all the crossing conditions in (2.10). Such a region would be the complete region of (x, y) for the parametrization (1.1) to satisfy both positivity and crossing symmetry. However, this region is difficult to find, and we will be less ambitious and try to find a region of (x, y) which is necessary and sufficient for A_{22}, A_{32}, and A_{44} to be expressible by (2.2) with $\mathrm{Im}f_{l'}(t)$ satisfying (1.7) and only the first condition in (2.10), which is (2.11).

To proceed we observe that from (3.1) and (2.7) we can express x and y as

$$x = ax^0 + (1-a)\bar{x}\,, \quad y = ay^0 + (1-a)\bar{y}\,, \tag{3.2}$$

where x^0, y^0, $\bar{x}$, $\bar{y}$, and a are defined, respectively, by

$$x^0 = -A^0_{32}/A^0_{22}\,, \quad y^0 = A^0_{44}/A^0_{22}\,, \tag{3.3}$$

$$\bar{x} = -\bar{A}_{32}/\bar{A}_{22}\,, \quad \bar{y} = \bar{A}_{44}/\bar{A}_{22}\,, \tag{3.4}$$

and

$$a = A^0_{22}/(A^0_{22} + \bar{A}_{22})\,. \tag{3.5}$$

Since $B^l_{22}(t)$ is positive, it follows from (2.8), (2.9), and (1.7) that both A^0_{22} and $\bar{A}_{22}$ are non-negative. Hence the quantity a defined by (3.5) lies between 0 and 1. Then Eq. (3.2) says that the point (x, y) always lies on the straight-line segment between (x^0, y^0) and $(\bar{x}, \bar{y})$. Further, by adjusting the scale of $\mathrm{Im}f_0(t)$, any point (x, y) on that line segment can be obtained.

Consequently, all we have to do now is to find two things: One is a region R^0 for (x^0, y^0) which will be necessary and sufficient for A^0_{22}, A^0_{32}, and A^0_{44} to be expressible by (2.8) with $\mathrm{Im}f_0(t) \geq 0$; the other is a region $\bar{R}$ for $(\bar{x}, \bar{y})$ which will be necessary and sufficient for $\bar{A}_{22}$, $\bar{A}_{32}$, and $\bar{A}_{44}$ to be expressible by (2.9) with $\mathrm{Im}f_{l'}(t)$ satisfying both the positivity condition (1.7) and the crossing condition (2.11). Then the region R, obtained by connecting with straight-line segments all possible pairs of points inside R^0 and $\bar{R}$, will be the necessary and sufficient region of (x, y) for A_{22}, A_{32}, and A_{44} to be expressible by (2.2) with $\mathrm{Im}f_{l'}(t)$ satisfying (1.7) and (2.11). The constraints thus obtained will be the optimum constraints on C^σ_p for $\sigma = 2, 3, 4$, as long as we ignore all but the first crossing sum rule in (2.10).

The region R^0 is easy to find. We first observe that $\mathrm{Im}f_0(t)$ can be arbitrary for the amplitude to remain crossing symmetric, and hence, can be taken to be

$$\mathrm{Im}f_0(t) = W_1\delta(t - t_1)\,, \quad W_1 > 0\,, \quad t_1 \geq 1\,. \tag{3.6}$$

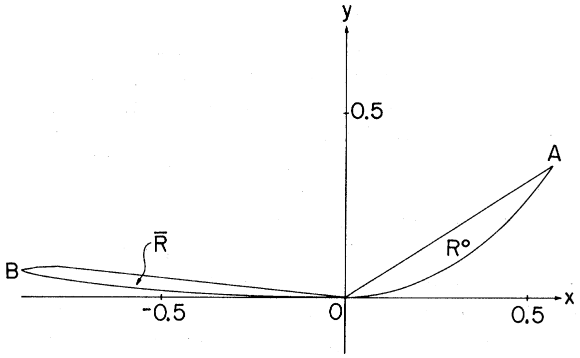

FIG. 1. The regions R^0 and $\bar{R}$ obtained in Sec. III for the case $\sigma=2,3,4$. To find the missing boundary of the region R (the convex hull of R^0 and $\bar{R}$), connect the top end point A of R^0 and the end point B on the upper-left corner of $\bar{R}$ with a straight-line segment, as shown in Fig. 4.

This choice of $\mathrm{Im} f_0(t)$ will lead, through Eqs. (2.8) and (3.3), to a point $P^0(t_1)$ given by

$$x^0(t_1) = -B^0_{32}(t_1)/B^0_{22}(t_1),$$

$$y^0(t_1) = B^0_{44}(t_1)/B^0_{22}(t_1).$$

When we move t_1 along the real axis from 1 to ∞, the corresponding point $P^0(t_1)$ will move around in the x-y plane and describe a certain finite curve, which we shall denote by C^0.

Now consider another choice of $\mathrm{Im} f_0(t)$:

$$\mathrm{Im} f_0(t) = W_1\delta(t-t_1) + W_2\delta(t-t_2), \quad W_i \geq 0, \quad t_i \geq 1.$$

The resulting point (x^0, y^0) in this case will always lie on the line segment between $P^0(t_1)$ and $P^0(t_2)$. Furthermore, every point on this line segment can be realized if the ratio W_1/W_2 is properly adjusted.

It is easy to generalize this consideration to the case of more and more δ functions in $\mathrm{Im} f_0(t)$. Finally, since the most general non-negative function is nothing but a continuous linear superposition of the δ functions at various points with non-negative weights, the point (x^0, y^0) corresponding to the most general $\mathrm{Im} f_0(t) \geq 0$ will fall inside the region formed by connecting with line segments all possible pairs of points on the curve C^0 (the convex hull of C^0). Further, every point inside this region can be realized by an appropriate choice of $\mathrm{Im} f_0(t)$. This is exactly the region R^0 we are looking for. The region R^0 is shown in Fig. 1, and its detailed numerical description will be given in Appendix B.

The region $\bar{R}$ is much more difficult to obtain, because now we have to take the crossing sum rule (2.11) into consideration. We will first use Eq. (2.5) to rewrite this sum rule in a more convenient form:

$$5\int_1^\infty dt\, \mathrm{Im} f_2(t) B^2_{44}(t) = \sum_{l=4}^\infty (2l+1)\alpha_l \int_1^\infty dt\, \mathrm{Im} f_l(t) B^l_{44}(t), \tag{3.7}$$

where

$$\alpha_l = \tfrac{1}{12} l(l+1)(l^2+l-8) > 0, \quad (l \geq 4).$$

We see from (3.7) that not all $\mathrm{Im} f_l(t)$ will be independent of each other. However, since the left-hand side as well as each term on the right-hand side of (3.7) is positive definite for any choice of $\mathrm{Im} f_l(t) \geq 0$, it is always possible, given any $\mathrm{Im} f_l(t)$ for $l \geq 4$, to adjust $\mathrm{Im} f_2(t)$ so that both sides of (3.7) will balance each other. Therefore, in the following analysis we can treat $\mathrm{Im} f_l(t)$ for $l \geq 4$ as independent variables and then force $\mathrm{Im} f_2(t)$ to comply with the crossing condition(3.7).

Now suppose $\mathrm{Im} f_l(t)$ for $l \geq 4$ are already given. Then we can calculate the right-hand side of (3.7), and call the result M,

$$M = \sum_{l=4}^\infty (2l+1)\alpha_l \int_1^\infty dt\, \mathrm{Im} f_l(t) B^l_{44}(t). \tag{3.8}$$

Then our $\mathrm{Im} f_2(t)$ has to satisfy

$$5\int_1^\infty dt\, \mathrm{Im} f_2(t) B^2_{44}(t) = M$$

or, equivalently,

$$\int_1^\infty dt\, \mathrm{Im} f_2(t)/\lambda(t) = 1, \tag{3.9}$$

where

$$\lambda(t) = M/5B^2_{44}(t) > 0. \tag{3.10}$$

Clearly the choice

$$\mathrm{Im} f_2(t) = \lambda(t_1)\delta(t-t_1), \quad t_1 \geq 1 \tag{3.11}$$

satisfies (3.9) and (1.7), and consequently will lead to an allowable point $(\bar{x}(t_1), \bar{y}(t_1))$ through Eqs. (3.4) and (2.9):

$$\bar{x}(t_1) = -\frac{5\lambda(t_1)B^2_{32}(t_1) + g_{32}}{5\lambda(t_1)B^2_{22}(t_1) + g_{22}},$$
$$\bar{y}(t_1) = \frac{5\lambda(t_1)B^2_{44}(t_1) + g_{44}}{5\lambda(t_1)B^2_{22}(t_1) + g_{22}}, \tag{3.12}$$

where

$$g_{\sigma l} = \sum_{l'=4}^\infty (2l'+1)\int_1^\infty dt\, \mathrm{Im} f_{l'}(t) B^{l'}_{\sigma l}(t). \tag{3.13}$$

Again we rewrite $\bar{x}(t_1)$ and $\bar{y}(t_1)$ in the following form:

$$\bar{x}(t_1) = b x_2(t_1) + (1-b)\hat{x},$$
$$\bar{y}(t_1) = b y_2(t_1) + (1-b)\hat{y}, \tag{3.14}$$

where

$$x_l(t) = -B^l_{32}(t)/B^l_{22}(t),$$
$$y_l(t) = B^l_{44}(t)/B^l_{22}(t), \tag{3.15}$$

$$\hat{x} = -g_{32}/g_{22}\,, \quad \hat{y} = g_{44}/g_{22}\,, \tag{3.16}$$

and

$$b = 5\lambda(t_1)B_{22}^2(t_1)/[5\lambda(t_1)B_{22}^2(t_1) + g_{22}]\,. \tag{3.17}$$

We see, therefore, that given any choice of $\mathrm{Im}f_l(t)$ for $l \geq 4$, we can first calculate $(\hat{x}, \hat{y})$ from (3.13) and (3.16) and then choose a value of t_1 and compute $x_2(t_1)$, $y_2(t_1)$, and b from (3.15), (3.17), (3.10), and (3.8), and hence obtain an allowable point $(\bar{x}(t_1), \bar{y}(t_1))$ from Eq. (3.14). If we vary t_1 through all values from 1 to ∞ while maintaining the same $\mathrm{Im}f_l(t)$ for $l \geq 4$, we obtain a certain finite curve $\bar{C}$ in the x-y plane, every point on which is an allowable point for $(\bar{x}, \bar{y})$. We denote the convex hull of $\bar{C}$ by $\bar{R}'$.

It is easy to see that every point inside the region $\bar{R}'$ is an allowable point for $(\bar{x}, \bar{y})$, and further, that the point $(\bar{x}, \bar{y})$ corresponding to the most general $\mathrm{Im}f_2(t)$ consistent with (3.9) will definitely fall inside this region $\bar{R}'$. The first part of this assertion is true, because any point $(\bar{x}, \bar{y})$ inside the region $\bar{R}'$ can always be made to correspond to some $\mathrm{Im}f_2(t)$ of the following form:

$$\mathrm{Im}f_2(t) = \frac{\lambda(t_1)\delta(t-t_1) + C\lambda(t_2)\delta(t-t_2)}{1+C}\,, \quad C \geq 0$$

which satisfies (3.9) and (1.7). The second part is true because the most general $\mathrm{Im}f_2(t)$ satisfying (3.9) can be expressed as a properly-weighted linear superposition of $\lambda(t_1)\delta(t-t_1)$ at various points, i.e.,

$$\mathrm{Im}f_2(t) = \int_1^\infty dt_1 h(t_1)\lambda(t_1)\delta(t-t_1)\,,$$

where

$$h(t_1) = \mathrm{Im}f_2(t_1)/\lambda(t_1) \geq 0$$

and

$$\int_1^\infty dt_1 h(t_1) = 1$$

because of (3.9).

Now we summarize our results so far as follows. Once given a set of $\mathrm{Im}f_l(t)$ for $l \geq 4$, we can first calculate the point $(\hat{x}, \hat{y})$, from which we can immediately generate an allowable region R' for $(\bar{x}, \bar{y})$. This region $\bar{R}'$ is the necessary and sufficient region of $(\bar{x}, \bar{y})$ for $\mathrm{Im}f_2(t)$ to satisfy both (1.7) and (3.9). It is clear that if one can find the complete region for $(\hat{x}, \hat{y})$, then one will be able to obtain all possible allowable regions $\bar{R}'$, and consequently, by taking their convex hull one will obtain the region $\bar{R}$.

Fortunately, the complete region for $(\hat{x}, \hat{y})$ is quite easy to find. Since $\hat{x}$ and $\hat{y}$ are given by (3.13) and (3.16) with $\mathrm{Im}f_l(t)$ all independent and arbitrary, the complete region for $(\hat{x}, \hat{y})$ can be obtained in the same manner as R^0 was obtained earlier in this section. That is to say, we can first take $\mathrm{Im}f_{l'}(t)$ for $l' \geq 4$ to be

$$\mathrm{Im}f_{l'}(t) = \delta_{l'l}\,\delta(t-t_0) \tag{3.18}$$

and calculate the corresponding $(\hat{x}, \hat{y})$. The convex hull of all such points will be the complete region for $(\hat{x}, \hat{y})$.

Thus we have described a method for obtaining the necessary and sufficient region $\bar{R}$ of $(\bar{x}, \bar{y})$ for $\bar{A}_{22}$, $\bar{A}_{32}$, and $\bar{A}_{44}$ to be expressible by (2.9) with $\mathrm{Im}f_{l'}(t)$ satisfying (1.7) and (2.11). To find the region $\bar{R}$ we only need to calculate

$$\begin{aligned}\bar{x}(l, t_0, t_1) &= -\frac{\alpha_l B_{44}^l(t_0)B_{32}^2(t_1) + B_{44}^2(t_1)B_{32}^l(t_0)}{\alpha_l B_{44}^l(t_0)B_{22}^2(t_1) + B_{44}^2(t_1)B_{22}^l(t_0)}\,,\\ \bar{y}(l, t_0, t_1) &= \frac{(\alpha_l+1)B_{44}^l(t_0)B_{44}^2(t_1)}{\alpha_l B_{44}^l(t_0)B_{22}^2(t_1) + B_{44}^2(t_1)B_{22}^l(t_0)}\,.\end{aligned} \tag{3.19}$$

The region $\bar{R}$ is obtained by taking the convex hull of all these points. The region $\bar{R}$ is also shown in Fig. 1, and its numerical details are again given in Appendix B.

IV. GENERALIZATION TO HIGHER σ

Now we can try to apply the same technique described in Sec. III to the case of higher σ. Suppose we wish to find constraints on C_p^σ for $\sigma = 2, 3, \ldots, n$. There are $d(n)+1$ such C_p^σ in total, where

$$d(n)+1 = [n/2](n-[n/2]) - (\{n/3\}-1)(n-\tfrac{3}{2}\{n/3\})\,.$$

Hence among all the relevant $A_{\sigma l}$ $(2 \leq \sigma \leq n)$, only $d(n)+1$ of them will be independent. Moreover, we will actually consider $A_{\sigma l}/A_{22}$ rather than $A_{\sigma l}$ themselves, and thus we are left with $d(n)$-independent variables among $A_{\sigma l}/A_{22}$, which we shall denote by $x_1, x_2, \ldots, x_{d(n)}$. We will also choose $x_1 = x$ and $x_2 = y$, where x and y are given in Sec. III.

To find the constraints on C_p^σ, we will have to find a region in the $d(n)$-dimensional space of $(x_1, x_2, \ldots, x_{d(n)})$, which is necessary and sufficient for $A_{\sigma l}$ $(2 \leq \sigma \leq n)$ to be expressible by (2.2) with $\mathrm{Im}f_{l'}(t)$ satisfying the positivity condition (1.7) and all the crossing conditions in (2.10) for $4 \leq \sigma \leq n$.

To proceed we first separate as before s-wave contributions to $A_{\sigma l}$ from those of the other partial waves, i.e., we write again Eqs. (2.7)–(2.9), and start to find a region R^0 for $(x_1^0, x_2^0, \ldots, x_{d(n)}^0)$ which is necessary and sufficient for $A_{\sigma l}^0$ $(2 \leq \sigma \leq n)$ to be expressible by (2.8) with $\mathrm{Im}f_0(t) \geq 0$. This region can be obtained by taking the convex hull of the curve C^0, which in turn is obtained by inserting $\mathrm{Im}f_0(t) = \delta(t-t_1)$ into (2.8), and varying t_1 through all values from 1 to ∞. This region R^0 can be calculated explicitly without any difficulty, as can be seen in Appendix A.

The region $\bar{R}$, which is the necessary and sufficient region for $\bar{A}_{\sigma l}$ $(2 \leq \sigma \leq n)$ to be expressible by (2.9) with $\mathrm{Im} f_{l'}(t)$ satisfying (1.7) and all the crossing conditions in (2.10) for $\sigma \leq n$, seems however more difficult to calculate, and in fact, we have not been able to find a general method for obtaining such a region for $n \geq 5$. Nevertheless, it is still possible to apply our technique to find another region $\bar{R}'$ which is necessary although not sufficient for $\bar{A}_{\sigma l}$ to have the same properties. Such a region can still provide strong constraints on C_p^σ.

For simplicity we shall describe the procedure to obtain $\bar{R}'$ only for the case $n=5$. In this case we have two crossing conditions to satisfy, namely,

$$\bar{A}_{44} = \bar{A}_{42} \tag{4.1}$$

and

$$\bar{A}_{54} = \bar{A}_{52}\,. \tag{4.2}$$

Since we know how to take care of only one of them at a time, we will first ignore the condition (4.2) and hence treat $\bar{A}_{54}$ and $\bar{A}_{52}$ separately. Suppose we are considering $\bar{A}_{22}$, $\bar{A}_{32}$, $\bar{A}_{44}$, and $\bar{A}_{52}$ only. Then following exactly the same procedure for obtaining $\bar{R}$ in Sec. III, we can find a region $\bar{R}_1$ of $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = -\bar{A}_{52}/\bar{A}_{22}$, which is necessary and sufficient for $\bar{A}_{22}$, $\bar{A}_{32}$, $\bar{A}_{44}$, and $\bar{A}_{52}$ to be expressible by (2.9), (1.7), and (4.1). Similarly if we consider $\bar{A}_{22}$, $\bar{A}_{32}$, $\bar{A}_{44}$, and $\bar{A}_{54}$ only, we can find the corresponding region $\bar{R}_1'$ in the $(\bar{x}, \bar{y}, \bar{z}')$ space, where $\bar{z}' = -\bar{A}_{54}/\bar{A}_{22}$. However, $\bar{z}'$ is supposed to be the same as $\bar{z}$, if we take (4.2) into account. Therefore, the intersection $I_1 = \bar{R}_1 \cap \bar{R}_1'$ will be a necessary region of $(\bar{x}, \bar{y}, \bar{z})$ for $\bar{A}_{\sigma l}$ $(2 \leq \sigma \leq 5)$ to be expressible by (2.9), (1.7), (4.1), and (4.2).

Now we can repeat this procedure all over again having the roles of (4.1) and (4.2) exchanged, and hence find another necessary region I_2 of $(\bar{x}, \bar{y}, \bar{z})$ for $\bar{A}_{\sigma l}$ to have the same properties. Finally we take the intersection of I_1 and I_2, and obtain an even smaller region $\bar{R}' = I_1 \cap I_2$ of $(\bar{x}, \bar{y}, \bar{z})$ which is necessary (but not sufficient) for $\bar{A}_{\sigma l}$ $(2 \leq \sigma \leq 5)$ to be expressible by (2.9), (1.7), (4.1), and (4.2). Clearly this method can be used also in the case of higher n.

It is a relatively easy matter to find a certain sufficient region in the $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{d(n)})$ space for $\bar{A}_{\sigma l}$ $(2 \leq \sigma \leq n)$ to be expressible by (2.9), (1.7), and (2.10), as will be seen in Sec. V. Hence it is possible for one to tell how far apart our region $\bar{R}'$ is from the region $\bar{R}$. However, without doing all this we want to argue that the region $\bar{R}'$ obtained by the above method must be strongly constrained. To show this we use the method just described to obtain the region $\bar{R}'$ for the case $n=4$, and show that $\bar{R}'$ is a large improvement over the previous result in Ref. 5.

To find the region $\bar{R}'$ for $n=4$, we first ignore the crossing condition (2.11) and treat all $\mathrm{Im} f_l(t)$ for $l \geq 2$ as if they were independent of each other. Let us call

$$\bar{y} = \bar{A}_{44}/\bar{A}_{22} \quad \text{and} \quad \bar{y}' = \bar{A}_{42}/\bar{A}_{22}$$

to distinguish $\bar{A}_{44}$ from $\bar{A}_{42}$ for the moment. Then following exactly the same procedure for obtaining the region R^0 in Sec. III, we can find a region $\bar{R}_1$ in the (x, y) plane which is necessary and sufficient for $\bar{A}_{22}$, $\bar{A}_{32}$, and $\bar{A}_{44}$ to be expressible by (2.9) with $\mathrm{Im} f_{l'}(t)$ satisfying (1.7) alone. Similarly we can find the corresponding region $\bar{R}_1'$ in the (x, y') plane. The intersection $\bar{R}' = \bar{R}_1 \cap \bar{R}_1'$ is a necessary (but not sufficient) region of (x, y) for $A_{\sigma l}$ $(2 \leq \sigma \leq 4)$ to be expressible by (2.9) with $\mathrm{Im} f_{l'}(t)$ satisfying (1.7) and (2.11). The region $\bar{R}'$ is shown in Fig. 2 together with $\bar{R}$. This result is indeed much better than the previous result, as can be seen from Fig. 3.

If one could somehow manage to find the region $\bar{R}$ in general case, then one could take the convex hull of R^0 and $\bar{R}$ to obtain the region R, the necessary and sufficient region for $A_{\sigma l}$ $(2 \leq \sigma \leq n)$ to satisfy positivity (1.7) and crossing conditions (2.10) for $4 \leq \sigma \leq n$. Notice that if one projects this $d(n)$-dimensional region R onto the x-y plane, one may find there a region even smaller than the region we previously found for the case $n=4$. This can happen because our previous constraints are not sufficient conditions for $A_{\sigma l}$ to satisfy $\sigma \geq 5$ crossing conditions. However, as will be seen in Sec. V, the improvement one can make over our results in Sec. III by going to higher n is rather

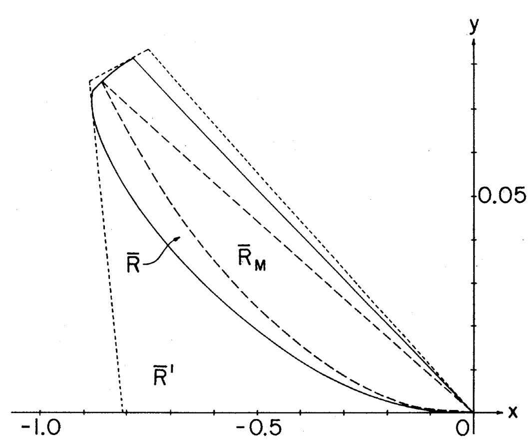

FIG. 2. Comparison of the region $\bar{R}$ with the region $\bar{R}'$ of Sec. IV and with the region $\bar{R}_M$ of Sec. V. $\bar{R}$ is the region bounded by the solid curves, $\bar{R}'$ the one bounded by the dotted lines, and $\bar{R}_M$ by the dashed curves. $\bar{R}_M$ is contained in $\bar{R}$, which in turn is contained in $\bar{R}'$. Notice the difference in scale for the x and y coordinates.

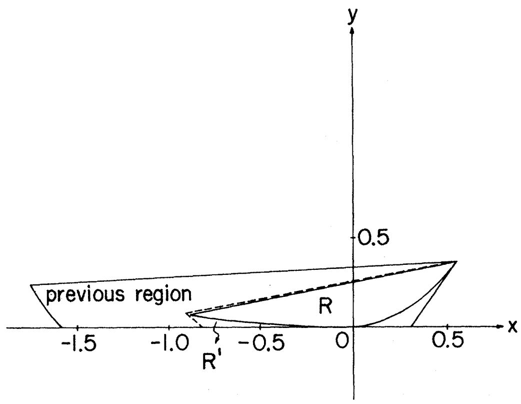

FIG. 3. Comparison of our region R (the innermost) with the previous region of Ref. 5 (the outermost) and with the region R' (in between). The region R' is just the convex hull of R^0 and $\bar{R}'$ (see Figs. 1 and 2). R and R' share the right-hand-side boundary.

limited, as far as constraints on (x, y) are concerned. Hence our region R is not too far apart from the complete region for (x, y).

V. COMPARISON WITH THE RESULTS FROM A MANDELSTAM REPRESENTATION

To see how good our results in Sec. III are, we can take a simple example of a scattering amplitude that satisfies the analyticity, positivity, and crossing properties of the elastic $\pi^0\pi^0$ scattering, and calculate the corresponding allowable region of (x,y). Such a region will be a sufficient region of (x, y) for $A_{\sigma l}$ $(2 \leq \sigma \leq 4)$ to be expressible by (2.2) with $\text{Im}f_{l'}(t)$ satisfying positivity (1.7) and all the crossing conditions in (2.10). However, since the imaginary part of the s-wave amplitude can be chosen arbitrarily, i.e., every point in the region R^0 is already realizable, it is not necessary to consider this part of the allowable region for the purpose of comparison. Hence, we will calculate only the allowable region for $(\bar{x}, \bar{y})$ for a given model.

Here we shall take for such a model the most general twice-subtracted Mandelstam representation with a positive symmetric double-spectral function (the single-spectral-function terms lead to the region R^0, because they involve only the imaginary part of the s wave):

$$F(s, t) = \frac{st}{\pi^2}\int\int_1^\infty ds'dt'\frac{\rho(s', t')}{s't'(s'-s)(t'-t)}$$

$$+\text{cyclic terms},$$

where

$$\rho(s', t') = \rho(t', s') \geq 0 .$$

After removing the imaginary part of the s wave, we find the absorptive part to be

$$\bar{A}(t', s) = \frac{1}{\pi}\int_1^\infty ds'\rho(s', t')\left(\frac{1}{s'-s} + \frac{1}{s'+t'+s-1} - \frac{2}{t'-1}\ln\frac{s'+t'-1}{s'}\right).$$

The d-wave amplitude below threshold can be expressed as

$$\bar{f}_2(s) = \frac{4}{\pi(1-s)}\int_1^\infty dt'\bar{A}(t', s)Q_2\left(\frac{2t'}{1-s} - 1\right)$$

from which we can calculate $\bar{A}_{\sigma 2}$ and hence $(\bar{x}, \bar{y})$ using (2.1) In practical computation we set

$$\rho(s', t') = \delta(s'-s_0)\delta(t'-t_0) + \delta(s'-t_0)\delta(t'-s_0) \tag{5.1}$$

because the most general positive symmetric function is simply a linear superposition of such expressions at various points (s_0, t_0). For each choice of (s_0, t_0) we find an allowable point for $(\bar{x}, \bar{y})$. Then we take the convex hull of all such points and thus obtain a region $\bar{R}_M$, shown in Fig. 2 together with $\bar{R}$. This region $\bar{R}_M$ is a sufficient region of $(\bar{x}, \bar{y})$ for $\bar{A}_{22}$, $\bar{A}_{32}$, and $\bar{A}_{44}$ to be expressible by (2.9) with $\text{Im}f_{l'}(t)$ satisfying (1.7) and (2.11). The boundary of $\bar{R}_M$ consists of two parts, a curved portion and a straight line. It is interesting to note that this curved portion of the boundary corresponds to those $\rho(s', t')$ with $s_0 = t_0$ in Eq. (5.1). Other choices of $\rho(s', t')$ give rise to inside points of $(\bar{x}, \bar{y})$.

Of course one can take the convex hull of $\bar{R}_M$ and R^0 and thus find a sufficient region R_M of (x, y) for $A_{\sigma l}$ $(\sigma = 2, 3, 4)$ to be expressible by (2.2) with $\text{Im}f_{l'}(t)$ satisfying (1.7) and all the crossing sum rules in (2.10). The complete region of (x, y) which is both necessary and sufficient for $A_{\sigma l}$ to have the same properties will contain R_M, but will be contained in R. Since there is not much room in between R_M and R, we see that our region R is quite close to the complete region.

Following a different approach, Pennington[9] has also studied the positivity and crossing constraints on the $\pi^0\pi^0$ partial-wave amplitudes, and has obtained the necessary and sufficient constraints on (x, y) for $f_2(s)$ and $f_4(s)$ to be positive definite below threshold. However, this property of $f_l(s)$ is only a small consequence of the full positivity condition (1.7), and hence it is not surprising that his results are much weaker than ours.

Finally we also compare our results with the original results of Ref. 5, as shown in Fig. 3.

VI. RESULTS REFORMULATED AS CONSTRAINTS ON s WAVE ALONE

All the constraints on C_p^σ for $\sigma=2, 3, 4$ we just obtained can be reformulated as constraints on the s-wave amplitude alone, which will be more useful in analyzing models of $\pi^0\pi^0$ scattering. We first express all the relevant $A_{\sigma 0}$ and $A_{\sigma l}^0$ in terms of the s wave explicitly. We have from Eq. (2.1),

$$A_{10}=\int_0^1 (1-s)(3s-1)f_0(s)ds\,,$$

$$A_{20}=\int_0^1 (1-s)(10s^2-8s+1)f_0(s)ds\,,$$

$$A_{30}=\int_0^1 (1-s)(35s^3-45s^2+15s-1)f_0(s)ds\,,$$

$$A_{40}=\int_0^1 (1-s)(126s^4-224s^3+126s^2-24s+1)f_0(s)ds\,;$$

from Eqs. (2.8) and (A6),

$$A_{22}^0=\frac{2}{3\pi}\int_1^\infty \mathrm{Im}f_0(t)[Q_2(2t-1)-Q_3(2t-1)]dt\,,$$

$$A_{32}^0=\frac{-1}{2\pi}\int_1^\infty \mathrm{Im}f_0(t)[Q_3(2t-1)-Q_4(2t-1)]dt\,,$$

$$A_{44}^0=\frac{2}{5\pi}\int_1^\infty \mathrm{Im}f_0(t)[Q_4(2t-1)-Q_5(2t-1)]dt\,.$$

Then our results can be summarized as follows:

$$A_{10}=0\,, \qquad (6.1)$$

$$A_{20}>\tfrac{5}{2}A_{22}^0>0\,, \qquad (6.2)$$

$$(x,y)\in R\,, \qquad (6.3)$$

where

$$x=5A_{30}/2A_{20}\,,\ y=5A_{40}/7A_{20}'\,;$$

$$(x^0,y^0)\in R^0\,, \qquad (6.4)$$

where

$$x^0=-A_{32}^0/A_{22}^0\,,\ y^0=A_{44}^0/A_{22}^0\,;$$

$$(\bar{x},\bar{y})\in \bar{R}\,, \qquad (6.5)$$

where

$$\bar{x}=\frac{x-ax^0}{1-a}\,,\quad \bar{y}=\frac{y-ay^0}{1-a}$$

with $a=5A_{22}^0/2A_{20}$.

Condition (6.1) follows from crossing symmetry alone [see (1.4)], while condition (6.4) from the positivity of $\mathrm{Im}f_0(t)$ alone. The rest are consequences of both positivity and crossing symmetry. Condition (6.2) can be proved from (1.4), (2.7), (2.9), and (1.7), while (6.3) and (6.5) were derived in Sec. III, and the regions R, R^0, and $\bar{R}$ were shown in Fig. 1. Condition (6.5) will turn out to be the strictest constraint. It is particularly interesting because it relates the s wave above threshold [via (x^0, y^0)] to the s wave below threshold [via (x, y)].

Now we can analyze various models for the s-wave $\pi^0\pi^0$ scattering to see whether they satisfy our constraints (6.1)–(6.5). We have examined all the models of Ref. 1 except the one of Basdevant and Lee which by itself does not satisfy crossing symmetry well enough for us to make a meaningful comparison. The model of Auberson *et al.* does not satisfy the crossing condition (6.1) and the other crossing condition[4]

$$2\int_0^1 (1-s)f_0^0(s)ds=5\int_0^1 (1-s)f_0^2(s)ds \qquad (6.6)$$

very well either. However, we have been able to choose a different set of parameters for their model in such a way that both crossing conditions as well as all the constraints[10] listed in their paper are well satisfied. Our choice of the parameters is, in their notation and units,

$$a_0^2=-0.099\,,\quad \alpha^2=0.24\,,\quad A=3.81\,,$$

$$B=-0.6306\,,\quad C=0.3753\,,\quad D=0.052\,63\,.$$

Then we take this modified version of Auberson's model together with those of Le Guillou, Krinsky, and Kang into our analysis.

They all satisfy, by construction, conditions (6.1) and (6.4), neither of which is a joint consequence of crossing and positivity. We list in Table I the calculated values of (x, y), (x^0, y^0), and $(\bar{x}, \bar{y})$ for each model and indicate whether conditions (6.2), (6.3), and (6.5) are satisfied or not in each case.

Thus we see that none of them satisfies all our constraints, despite the fact that they all satisfy the crossing conditions (6.1) and (6.6) on the s-wave amplitudes below threshold, and the positivity condition on $\mathrm{Im}f_0(t)$ above threshold. The point here is that they violate (implicitly) the positivity of the higher partial waves. This can be easily understood as follows. We have seen that the s wave in the unphysical region is related to the higher partial waves in the same region by crossing [see (1.4) and (2.1)], and the latter in turn are related, by the Froissart-Gribov representation (1.5)–(1.7), to the crossed-channel imaginary part of all partial waves in the physical region. Consequently, the positivity of the higher partial waves puts some constraints on the s wave in the unphysical region, and they have nothing to do with the positivity of the s wave.

TABLE I. Calculated values of (x,y), (x^0,y^0), and $(\bar{x},\bar{y})$ for each model.

Model	(x,y)	(x^0,y^0)	$(\bar{x},\bar{y})$	(6.2) satisfied	(6.3) satisfied	(6.5) satisfied
Auberson	$x=0.157$ $y=0.026$	$x^0=0.098$ $y^0=0.019$	$\bar{x}=-0.130$ $\bar{y}=-0.0084$	No	No	No
Le Guillou	$x=0.183$ $y=0.022$	$x^0=0.125$ $y^0=0.027$	$\bar{x}=\ 4.10$ $\bar{y}=-0.312$	No	No	No
Krinsky	$x=0.089$ $y=0.012$	$x^0=0.097$ $y^0=0.019$	$\bar{x}=\ 0.052$ $\bar{y}=-0.018$	Yes	Yes	No
Kang	$x=0.111$ $y=0.017$	$x^0=0.122$ $y^0=0.026$	$\bar{x}=\ 0.052$ $\bar{y}=-0.031$	Yes	Yes	No

VII. INTERPRETATION OF THE CONSTRAINTS ON C_p^0

We want to see how much information about the physical amplitude one can extract from the knowledge of a given point (x, y) defined by (3.1). We first observe that each point (x, y) has been obtained through Eqs. (3.2)–(3.5). Thus each (x, y) must be associated with a certain straight-line segment whose end points (x^0, y^0) and $(\bar{x}, \bar{y})$ are inside the regions R^0 and $\bar{R}$, respectively. Further, the closer the point (x, y) gets to the end point (x^0, y^0), the more important the s wave must be relative to all the other partial waves, in the sense that A_{22}^0 gets bigger than $\bar{A}_{22}$, and vice versa.

Now given a point $P=(x, y)$ inside R, we shall not, in general, be able to tell which line segment it is associated with. However, we do know something about this line segment, for we can draw a straight line through A and P (see Fig. 4) intersecting with the boundary of $\bar{R}$ at C and D, and also a straight line through B and P intersecting with the boundary of R^0 at E and F, and we know for sure that the point (x^0, y^0) associated with P must come from the region AEF, and similarly the point $(\bar{x}, \bar{y})$ from the region $BGCD$.

The analysis in Appendix B indicates that, in

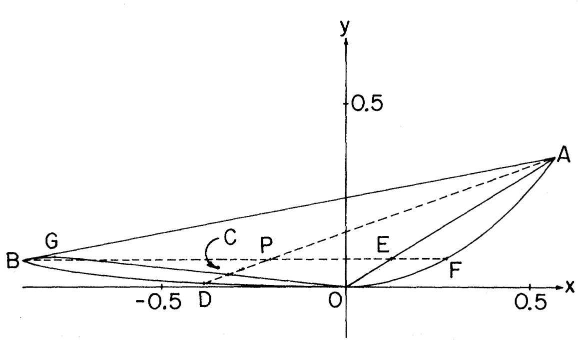

FIG. 4. The point $P=(x,y)$ calculated from any model. The point (x^0,y^0) associated with P must come from the region AEF, and the point $(\bar{x},\bar{y})$ from the region $BGCD$.

general, a point (x^0, y^0) in the upper portion of the region R^0 corresponds to an s wave $\text{Im}f_0(t)$ emphasizing the lower-energy (smaller-t) region and vice versa. Similarly, a point in the upper portion of the region $\bar{R}$ corresponds to $\text{Im}f_l(t)$ for $l \geq 2$ emphasizing the lower-energy region and vice versa. Therefore, the larger y is, the more likely it is for most partial waves to emphasize the low-energy region and vice versa. Also the larger (more positive) x is, the more likely it is for the s wave to dominate the rest of the partial waves in the physical region and vice versa.

Some extreme cases can be easily interpreted. They will happen when the point (x, y) falls somewhere on the boundary of R. The following predictions are all based on the analysis in Appendix B.

(1) If (x, y) falls on the AFO part of the boundary, then there is only the s wave in the physical region, and it will consist of a single sharp resonance at a certain energy uniquely determined by the location of (x, y).

(2) If (x, y) falls on the BDO part of the boundary, then there are only the d wave and the g wave, each of which consists of a single sharp resonance. The location of these two resonances and their relative strength are uniquely determined by the location of (x, y).

(3) If (x, y) falls on the straight line AB of the boundary, then only s, d, and g waves are present, and they are all concentrated at the threshold.

Of course all these extreme cases cannot occur in the $\pi^0\pi^0$ scattering. But if a model happens to produce a point (x, y) extremely close to the boundary of R, we can still make some approximate predictions about the partial waves in the physical region. For example, suppose a model gives rise to a point (x, y) very close to the AFO part of the boundary. Then we know that all higher partial waves than the s wave, if present at all, must be relatively unimportant. Further, it is most likely for the s wave to be concentrated around a small energy region, in which there may be several sharp resonances, or just a broad resonance or something else. In any case, the s wave must be relatively small outside that region.

If the point (x, y) falls somewhere inside the region R, not too close to any part of its boundary, then there are too many possibilities pertaining to such a point for one to predict anything conclusive. However, given additional information about the model under consideration and applying our analysis in a more quantitative manner, one can certainly understand better the structure of the model. To illustrate this point, consider a model for the s-wave elastic $\pi^0\pi^0$ scattering satisfying conditions (6.1)–(6.5). Then it can be shown that there exists at least one set of non-negative functions $\mathrm{Im}f_l(t)$ for $l \geq 2$ in the physical region $t \geq 1$ such that A_{22}, A_{32}, A_{42}, and A_{44} can be expressed by (2.2) and satisfy the crossing conditions (1.4) for $1 \leq \sigma \leq 4$. Furthermore, this set of $\mathrm{Im}f_l(t)$ will satisfy the following equations:

$$\sum_{l=2}^{\infty}(2l+1)\int_1^{\infty} dt\,\mathrm{Im}f_l(t)B_{22}^l(t)=\tfrac{1}{4}\pi(\tfrac{2}{5}A_{20}-A_{22}^0),$$
$$\sum_{l=2}^{\infty}(2l\pm1)\int_1^{\infty} dt\,\mathrm{Im}f_l(t)B_{32}^l(t)=-\tfrac{1}{4}\pi(A_{30}+A_{32}^0),$$
$$\sum_{l=2}^{\infty}(2l+1)\int_1^{\infty} dt\,\mathrm{Im}f_l(t)B_{44}^l(t)=\tfrac{1}{4}\pi(\tfrac{2}{7}A_{40}-A_{44}^0).$$

The right-hand side of each equation above is known for a given model. Such constraints on $\mathrm{Im}f_l(t)$ in the physical region, plus some additional assumptions like f^0 dominance in the d-wave amplitude, and so forth, can lead to some predictions on the behavior of the higher partial waves in the physical region, such as the mass and width of the f^0 resonance. Unfortunately, we have not been able to consider such implications for existing models since they all violate at least one of our conditions.

VIII. CONCLUSIONS

We have found the necessary and sufficient constraints on the parameters C_p^σ for $\sigma=2, 3, 4$ in the crossing-symmetric $\pi^0\pi^0$ partial-wave expansion (1.1)–(1.3), which follow from, and ensure, the existence of a set of amplitudes $\mathrm{Im}f_l(t)$ in the physical region satisfying both the positivity condition (1.7) and the crossing sum rule (2.11). Although they are not quite the complete constraints on C_p^σ to ensure positivity and crossing symmetry, we have nevertheless shown that in fact our results cannot be strengthened very much.

We have also reformulated the results as constraints on the s-wave amplitude alone, and have examined whether they are satisfied in several existing models for the s-wave $\pi\pi$ scattering. It turned out that none of them satisfies all our constraints, which indicates that they have all violated the positivity condition on the higher partial waves. We have also given an interpretation of the constraints on C_p^σ and described how they can be used to extract implications on the higher partial waves in the physical region from a given model of the s-wave amplitude.

APPENDIX A

We wish to establish several properties of $B_{\sigma l}^{l'}(t)$ defined by (2.3). First we shall give a proof of the recurrence relation (2.4). From (2.3) we have

$$l'(l'+1)B_{\sigma l}^{l'}(t)=\int_0^1 ds\,(1-s)^l P_{\sigma-l}^{(2l+1,0)}(2s-1)Q_l\left(\frac{2t}{1-s}-1\right)l'(l'+1)P_{l'}\left(\frac{2s}{t-1}+1\right). \tag{A1}$$

Now we insert the equation

$$l'(l'+1)P_{l'}(x)=\frac{d}{dx}\left[(x^2-1)\frac{d}{dx}P_{l'}(x)\right], \tag{A2}$$

satisfied by the Legendre function $P_{l'}(x)$, into (A1) and then integrate by parts twice to express (A1) in the following form:

$$l'(l'+1)B_{\sigma l}^{l'}(t)=\int_0^1 ds\,F_{\sigma l}(s,t)P_{l'}\left(\frac{2s}{t-1}+1\right), \tag{A3}$$

where $F_{\sigma l}(s, t)$ is a complicated expression involving $Q_l(2t/(1-s)-1)$ and its first and second derivatives, and some polynomials of s and t. Then we use the differential Eq. (A2) satisfied by $Q_l(x)$ and several recurrence relations of Q_l functions to express all the derivatives of Q_l in $F_{\sigma l}(s, t)$ in terms of Q_{l+1}, Q_l, and Q_{l-1} only, so that all t dependence of $F_{\sigma l}(s, t)$ gets absorbed into the three Legendre functions:

$$F_{\sigma l}(s,t)=\frac{l+1}{2(2l+1)}(1-s)^{l+1}Q_{l+1}\left(\frac{2t}{1-s}-1\right)\left(\frac{d}{ds}+s\frac{d^2}{ds^2}\right)P^{(2l+1,0)}_{\sigma-l}(2s-1)$$
$$+\tfrac{1}{2}(1-s)^l Q_l\left(\frac{2t}{1-s}-1\right)\left\{l+[(2l+3)s-1]\frac{d}{ds}-s(1-s)\frac{d^2}{ds^2}\right\}P^{(2l+1,0)}_{\sigma-l}(2s-1)$$
$$+\frac{l}{2(2l+1)}(1-s)^{l-1}Q_{l-1}\left(\frac{2t}{1-s}-1\right)\left\{(2l+1)[(2l+1)s-1]-[(4l+3)s-1](1-s)\frac{d}{ds}+s(1-s)^2\frac{d^2}{ds^2}\right\}P^{(2l+1,0)}_{\sigma-l}(2s-1). \qquad (A4)$$

Finally, we can transform (A4) into

$$F_{\sigma l}(s,t)=\frac{(l+1)(\sigma-l)(\sigma+l+2)}{2(2l+1)}(1-s)^{l+1}Q_{l+1}\left(\frac{2t}{1-s}-1\right)P^{(2l+3,0)}_{\sigma-l-1}(2s-1)$$
$$+\tfrac{1}{2}(\sigma^2-l^2+2\sigma-l)(1-s)^l Q_l\left(\frac{2t}{1-s}-1\right)P^{(2l+1,0)}_{\sigma-l}(2s-1)$$
$$+\frac{l(\sigma-l+1)(\sigma+l+1)}{2(2l+1)}(1-s)^{l-1}Q_{l-1}\left(\frac{2t}{1-s}-1\right)P^{(2l-1,0)}_{\sigma-l+1}(2s-1). \qquad (A5)$$

Here we have made use of the differential equation satisfied by the Jacobi polynomials, and several of their recurrence relations, which can be easily established by using the representation[11]

$$P_n^{(\alpha,0)}(x)=\sum_{m=0}^{n}\frac{(n+\alpha+m)!}{2^m m!(n-m)!(\alpha+m)!}(x-1)^m.$$

Inserting (A5) into (A3) and using the definition (2.3) for $B^{l'}_{\sigma l}(t)$, we obtain the recurrence relation (2.4).

This recurrence relation can be used to find the explicit expression of $B^0_{\sigma l}(t)$, as we show in the following. For $l'=0$, (2.4) becomes

$$(l+1)(\sigma-l)(\sigma+l+2)[B^0_{\sigma,l+1}(t)+B^0_{\sigma l}(t)]+l(\sigma-l+1)(\sigma+l+1)[B^0_{\sigma l}(t)+B^0_{\sigma l-1}(t)]=0$$

from which we can easily establish

$$B^0_{\sigma l}(t)=(-1)^{\sigma-l}B^0_{\sigma\sigma}(t).$$

Hence, all we have to do is to evaluate

$$B^0_{\sigma\sigma}(t)=\int_0^1 ds(1-s)^\sigma Q_\sigma\left(\frac{2t}{1-s}-1\right).$$

The integral can be calculated to be

$$[1/2(\sigma+1)][Q_\sigma(2t-1)-Q_{\sigma+1}(2t-1)]$$

by using the representation[12]

$$Q_\sigma(2x-1)=\frac{1}{2}\sum_{n=\sigma}^{\infty}\frac{n!n!}{(n-\sigma)!(n+\sigma+1)!}\left(\frac{1}{x}\right)^{n+1}.$$

Therefore, we find

$$B^0_{\sigma l}(t)=\frac{(-1)^{\sigma-l}}{2(\sigma+1)}[Q_\sigma(2t-1)-Q_{\sigma+1}(2t-1)] \qquad (A6)$$

which can be used to find the region R^0 of Sec. IV.

Notice that if only $B^0_{\sigma l}(t)$ with even l are considered, as is the case in our problem, $B^0_{\sigma l}(t)$ will be independent of l for a given σ. Hence we have, for example,

$$A^0_{44}=A^0_{42},\quad A^0_{54}=A^0_{52},\quad A^0_{66}=A^0_{64}=A^0_{62},$$

and so on. Thus $A^0_{\sigma l}$ satisfy automatically all the crossing conditions corresponding to those of (2.10).

We can of course use the recurrence relation (2.4) to deduce properties of $B^{l'}_{\sigma l}(t)$ for $l'\neq 0$, but the results will look more complicated in general.

APPENDIX B

In this appendix we shall give a detailed description of the regions R^0 and $\bar{R}$ we obtained in Sec. III, in order for our results to be convenient to use in practice. The boundary of the region R^0 consists of two parts (see Fig. 4), the curved part AFO and the straight-line part AEO. The curve AFO is simply the curve C^0, described by Eq. (3.6), or, more explicitly,

$$x^0(t)=\frac{3[Q_3(2t-1)-Q_4(2t-1)]}{4[Q_2(2t-1)-Q_3(2t-1)]},$$

$$y^0(t)=\frac{3[Q_4(2t-1)-Q_5(2t-1)]}{5[Q_2(2t-1)-Q_3(2t-1)]},$$

with $t\geq 1$. The point A is given by $(x^0(1), y^0(1))$, or $(\frac{9}{16},\frac{9}{25})$, and the portion of the curve near the origin O can be approximated by $y=448x^2/405$.

In Table II we list several calculated values of points on the curve AFO for practical purposes.

The straight line AEO is $y=16x/25$. Generally a point (x, y) in the upper portion of the region R^0

TABLE II. Calculated points on the curve *AFO* (Fig. 4).

t	$x^0(t)$	$y^0(t)$
1.0	0.563	0.360
1.01	0.496	0.277
1.05	0.399	0.178
1.1	0.337	0.126
1.4	0.193	0.0413
2.0	0.110	0.0133
5.0	0.0358	0.0014

corresponds to an s wave $\mathrm{Im} f_0(t)$ emphasizing the lower-energy (i.e., smaller-t) region, and a point in the lower portion, the higher-energy region. This can be seen clearly from the way R^0 was constructed in Sec. III.

The region $\bar{R}$ has a more complicated structure, given by using (3.19) for all $l \geq 4$ and t_0, $t_1 \geq 1$. It appears to us, from taking many different values of l, t_0, t_1, that the boundary is given by 3 pieces, the curved portion *BDO*, the broken-line portion *BG*, and the straight line *GCO*.

The *BG* part of the boundary is obtained by connecting with straight-line segments the following sequence of points successively:

$$\bar{x} = \bar{x}(l, 1, 1) = -\frac{3}{16}\,\frac{155l^3 + 692l^2 + 623l + 150}{37l^3 + 152l^2 + 77l - 50},$$

$$\bar{y} = \bar{y}(l, 1, 1) = \frac{3(l-1)(l+2)(l+3)}{37l^3 + 152l^2 + 77l - 50},$$

$$l = 4, 6, 8, \ldots .$$

The point G is given by

$$(\bar{x}(\infty, 1, 1), \bar{y}(\infty, 1, 1)) = (-\tfrac{465}{592}, \tfrac{3}{37})$$

and the point B by

$$(\bar{x}(4, 1, 1), \bar{y}(4, 1, 1)) = (-\tfrac{3939}{4496}, \tfrac{21}{281}).$$

In Table III we list several points on this part of the boundary.

The straight line *GCO* is given by $y = -16x/155$. The curved portion *BDO* of the boundary is calculated from

$$\bar{x} = \bar{x}(4, t_0, t_1), \quad \bar{y} = \bar{y}(4, t_0, t_1), \quad t_0, t_1 \geq 1$$

where t_0 and t_1 are related to each other through the equation

$$\frac{x_4(t_0)y_4'(t_0) - x_4'(t_0)y_4(t_0)}{y_4'(t_0)} = \frac{x_2(t_1)y_2'(t_1) - x_2'(t_1)y_2(t_1)}{y_2'(t_1)} \tag{B1}$$

when there is a solution for t_0 and $t_1 \geq 1$. [$x_l(t)$ and $y_l(t)$ were defined by (3.15), where $x_l'(t)$ means $dx_l(t)/dt$.] It turns out[13] that (B1) has no solution unless $t_0 > 2.33$. For $t_0 < 2.33$, the boundary is given by

$$\bar{x} = \bar{x}(4, t_0, 1),$$

$$\bar{y} = \bar{y}(4, t_0, 1).$$

We have not been able to find a closed form for the expression of this curve *BDO*, but we know the portion near the origin O can be approximated by

$$\bar{x} \approx \bar{x}(4, t_0, \tfrac{9}{37}t_0) \approx -\frac{111}{56t_0},$$

$$\bar{y} \approx \bar{y}(4, t_0, \tfrac{9}{37}t_0) \approx \frac{4107}{14\,945t_0^2}, \quad \text{as } t_0 \to \infty .$$

Several points on the curve *BDO* are listed in Table IV.

Again in general a point in the upper portion of the region $\bar{R}$ corresponds to $\mathrm{Im} f_l(t)$ for $l \geq 2$ emphasizing the lower-energy region and vice versa. Further, a point closer to the *BDO* part of the boundary tends to emphasize lower partial waves (smaller l).

TABLE III. Calculated points on the segment *BG* (Fig. 4).

l	$\bar{x}(l,1,1)$	$\bar{y}(l,1,1,)$
4	−0.876	0.0747
6	−0.842	0.0778
8	−0.826	0.0790
10	−0.817	0.0796
20	−0.800	0.0805
∞	−0.785	0.0811

TABLE IV. Calculated points on the curve *BDO* (Fig. 4).

t_0	t_1	$\bar{x}$	$\bar{y}$
1.00	1.00	−0.876	0.0747
1.20	1.00	−0.876	0.0707
1.40	1.00	−0.869	0.0666
1.60	1.00	−0.855	0.0623
2.00	1.00	−0.814	0.0541
2.33	1.00	−0.773	0.0479
2.83	1.10	−0.653	0.0332
3.29	1.20	−0.569	0.0247
3.95	1.35	−0.479	0.0173
4.81	1.55	−0.397	0.0117
6.29	1.90	−0.307	0.00689
8.37	2.40	−0.232	0.00389
18.28	4.80	−0.107	0.00082

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[1]See, e.g., G. Auberson *et al.*, Phys. Letters 28B, 41 (1968); J. Le Guillou *et al.*, CERN Report No. CERN-TH-1260, 1970 (unpublished); S. Krinsky, Phys. Rev. D 2, 1168 (1970); J. Kang and B. Lee, *ibid.* 3, 2814 (1971); J. Basdevant and B. Lee, Phys. Letters 29B, 437 (1969).

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[10]In the paper of Auberson *et al.*, condition (*B*) was not completely enforced on $f_0^2(s)$. Here we have followed them in ignoring this condition on $f_0^2(s)$.

[11]See, for example, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1964), p. 775.

[12]See the appendix of Ref. 5.

[13]The authors are grateful to Dr. P. Grassberger for pointing out an error concerning this point in the original preprint.

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Comments on the Scaling Behavior of Currents

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Necessary and sufficient conditions in order for all components of a local operator (such as a current, for example) to have the same dimension are given and discussed. In addition, we then note that exact scale invariance is in an apparent formal contradiction with a definite scaling behavior for the currents. The solution of this contradiction is then traced back to the fact that in scale-invariant theories, the Schwinger term must be quadratically divergent. We finally conclude by briefly discussing Coleman's theorem for scale and conformal transformations.

In a previous paper[1] the equal-time commutator algebra satisfied by the generators of the broken conformal group have been derived. These relations were then used to give necessary and sufficient conditions in order for all components of a local operator (such as a current, for example) to have the same dimension, and to discuss Coleman's theorem for scale and conformal transformations.

In this paper we wish to summarize the discussion given in Ref. 1 and to make some further comments. In addition, we show that exact scale invariance immediately leads to an immediate apparent formal contradiction with definite scaling behavior for the currents. This has already been pointed out in Ref. 8 on the additional assumption that

$$D|\Omega\rangle = 0. \tag{1}$$

In this paper we also schematize that proof and note in addition that Eq. (1) follows immediately from the equal-time commutator algebra of the conformal group and the translation invariance of the vacuum. Moreover, we note that the solution of this apparent contradiction may be traced back to the fact that in scale-invariant theories, the Schwinger term must be quadratically divergent.

We start by writing, for the sake of completeness, the by now well-known relations satisfied by the generators of the broken conformal group. These are

$$[D, P_\mu] = -iP_\mu - \int d^3x\, \partial_\lambda D_\lambda(x)\delta_{\mu 4}, \tag{2}$$

$$[K_\mu, P_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}) - 2\int d^3x\, \delta_{\nu 4}x_\mu\, \partial_\lambda D_\lambda(x), \tag{3}$$

$$[D, M_{\mu\nu}] = \int d^3x(\delta_{\mu 4}x_\nu - \delta_{\nu 4}x_\mu)\partial_\lambda D_\lambda(x), \tag{4}$$

$$[K_\mu, M_{\rho\sigma}] = i(\delta_{\mu\sigma}K_\rho - \delta_{\mu\rho}K_\sigma) + 2\int d^3x\, x_\mu(\delta_{\rho 4}x_\sigma - \delta_{\sigma 4}x_\rho)\partial_\lambda D_\lambda(x), \tag{5}$$

$$[D, K_\mu] = iK_\mu - \int d^3x\, \delta_{\mu 4}x^2\partial_\lambda D_\lambda(x), \tag{6}$$