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Nonlinear Realizations of SU(3)×SU(3) and the Symmetry-Breaking Meson Lagrangian

S. P. Rosen and A. McDonald
Physics Department, Purdue University, Lafayette, Indiana 47907
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We show how an octet of pseudoscalar mesons which transform nonlinearly under SU(3) × SU(3) can be converted to the linear realizations $(3,\bar{3})$ and $(\bar{3},3)$. Our results are manifestly covariant with respect to redefinitions of the meson field. We use them to construct an effective meson Lagrangian in which all the symmetry breaking occurs in the mass term and belongs to a single representation $\{(p,\bar{p})+(\bar{p},p)\}$. From an analysis of π-π scattering lengths, we find it unlikely that the symmetry-breaking interaction belongs either to the self-adjoint sequence (8, 8), (27, 27), ..., or to the triangular one $p=6, 10, 15, \ldots$. There are, however, many other representations, including $\{(3,\bar{3})+(\bar{3},3)\}$, which cannot be ruled out. We also examine weak currents and find that the f_- form factor in K_{l3} decay gives rise to a serious problem in the symmetry breaking.

I. INTRODUCTION

In recent studies of chiral SU(2),[1] we analyzed the relationship between nonlinear and linear realizations, and applied the results to meson dynamics. Our approach bore a strong resemblance to the nonlinear σ model of Gell-Mann and Lévy,[2] for we discovered that if the action of chiral operators upon the pion field is given by[3]

$$[K_a, \pi_b] = -i[\delta_{ab} f(\pi^2) + \pi_a \pi_b g(\pi^2)],$$
$$g = (1+2ff')/(f-2\pi^2 f'), \tag{1.1}$$

then the quantities

$$u = -f/(f^2+\pi^2)^{1/2} \quad \text{and} \quad \hat{\pi}_a = \pi_a/(f^2+\pi^2)^{1/2} \tag{1.2}$$

span the linear representation $(\frac{1}{2},\frac{1}{2})$ of SU(2) ×SU(2),

$$[K_a, u] = -i\hat{\pi}_a, \quad [K_a, \hat{\pi}_b] = i\delta_{ab} u, \tag{1.3}$$

and satisfy the constraint

$$u^2 + \hat{\pi}_a \hat{\pi}_a = 1. \tag{1.4}$$

The space-time derivatives $\partial_\mu u$ and $\partial_\mu \hat{\pi}_a$ transform in exactly the same way as u and $\hat{\pi}_a$ themselves, and so we were able to construct effective Lagrangians, with and without derivative coupling, by the standard methods of linear representation theory. In particular, we studied the effects of broken chiral symmetry upon π-π scattering lengths and weak currents.

We now attempt to carry out the same program in chiral SU(3). This means that, given the general action of chiral operators on the octet of pseudoscalar meson fields,

$$[K_a, \pi_b] = iF_{ab}(\pi) \quad (a, b = 1, \ldots, 8), \tag{1.5}$$

we must construct quantities

$$Z_\alpha = u_\alpha - iv_\alpha \quad \text{and} \quad \bar{Z}_\alpha = u_\alpha + iv_\alpha$$
$$(\alpha = 0, 1, \ldots, 8), \tag{1.6}$$

which transform according to the linear representations[4] $(3,\bar{3})$ and $(\bar{3},3)$ of SU(3)×SU(3):

$$[K_a, Z_b] = d_{abc} Z_c + (\tfrac{2}{3})^{1/2}\delta_{ab} Z_0,$$
$$[K_a, Z_0] = (\tfrac{2}{3})^{1/2} Z_a,$$
$$[K_a, \bar{Z}_b] = -(d_{abc}\bar{Z}_c + (\tfrac{2}{3})^{1/2}\delta_{ab}\bar{Z}_0),$$
$$[K_a, \bar{Z}_0] = -(\tfrac{2}{3})^{1/2}\bar{Z}_a. \tag{1.7}$$

Once we know Z_α and $\bar{Z}_\alpha$, we can use them and their space-time derivatives to form Lagrangians which either preserve chiral symmetry or transform according to a specific representation of SU(3)×SU(3). Thus the symmetry-preserving kinetic part of the meson Lagrangian will be proportional to $(\partial_\mu Z_\alpha)(\partial_\mu \bar{Z}_\alpha)$ and the symmetry-breaking mass term will be a function, $M(Z_0, \bar{Z}_0)$, of Z_0 and $\bar{Z}_0$.

Although we have no *a priori* knowledge of how $M(Z_0,\bar{Z}_0)$ behaves, we shall assume that it belongs to a single representation $\{(p,\bar{p})+(\bar{p},p)\}$ of SU(3) ×SU(3). We also assume that $M(Z_0,\bar{Z}_0)$ is an admixture of SU(3) singlet and octet parts, and we fix this admixture by requiring the Lagrangian to reproduce the empirical masses of the π and K mesons in the tree approximation. The next order of this approximation corresponds to meson-meson scattering, and it enables us to calculate S-wave scattering lengths in terms of m_π, m_K, and the parameters of the SU(3) representation (p). Our formulas agree with those of other writers[5] when (p) represents the triplet, but they are more general because they apply to all types of symmetry breaking.

The problem of constructing Z_α and $\bar{Z}_\alpha$ for SU(3) $\times$SU(3) is much more complicated than the corresponding one of finding u and $\hat{\pi}_a$ for chiral SU(2). Associated with any octet vector π_b ($b=1, \ldots, 8$) there are two SU(3) invariants and one dual vector:

$$X=\pi_b\pi_b, \quad Y=d_{abc}\pi_a\pi_b\pi_c, \quad \Pi_a=d_{abc}\pi_b\pi_c. \tag{1.8}$$

Consequently, the commutation tensor $F_{ab}(\pi)$ of Eq. (1.5) contains more terms than the corresponding SU(2) tensor [Eq. (1.1)], and the invariant functions in it [i.e., the analog of $f(\pi^2)$ and $g(\pi^2)$] depend upon two variables, X and Y, instead of one.

There are, in fact, seven terms in $F_{ab}(\pi)$, but because of the constraints of chiral algebra, not all of them are independent.[6] These algebraic constraints give rise to a set of partial differential equations analogous to the one for g in Eq. (1.1), and as a result, any two of the seven terms can be used to determine the remaining five.

It is not difficult to show that, given the pseudoscalar quantities v_a ($a=1, 2, \ldots, 8$) in Eq. (1.6), we can determine u_0, u_a, and v_0 from the commutators $[K_b, v_a]$. Now for a particular choice of $F_{ab}(\pi)$, v_a will be of the form $\alpha\pi_a+\beta\Pi_a$, where α and β depend upon X, Y, and the invariant functions of the commutation tensor. Our objective, of course, is to find out this dependence; before doing so however, we find it very helpful to consider an example of the inverse procedure. That is, we assume that $v_a=\pi_a$, and then determine the appropriate form of $F_{ab}(\pi)$. Armed with this knowledge, we are able to construct the Z_α and $\bar{Z}_\alpha$ for any choice of commutation tensor, and also to determine the constraints upon these quantities analogous to Eq. (1.4) for the σ model. Our results, which are described in Sec. II below, are all covariant with respect to redefinitions of the meson field.

Since the kinetic Lagrangian is proportional to $(\partial_\mu Z_\alpha)(\partial_\mu \bar{Z}_\alpha)$, our chief problem in constructing the meson Lagrangian is the mass term. If the SU(3) scalar part M_0 belongs to the representation $\{(p,\bar{p})+(\bar{p},p)\}$, then it must be an eigenfunction of the operators K_aK_a and $d_{abc}K_aK_bK_c$ with eigenvalues determined by the parameters of (p). Now M_0 is, in general, a function of X and Y [see Eq. (1.8)], but we can also write it as a function of Z_0 and $\bar{Z}_0$. The latter choice of variables is particularly convenient because it enables us to convert the eigenvalue conditions into relatively simple partial differential equations. We then pick out the appropriate solution by noting that M_0 must be a polynomial in Z_0 and $\bar{Z}_0$ and not an infinite series. In addition, we can derive the octet part, M_8, of the mass term by operating on M_0 with K_8 and $d_{8bc}K_bK_c$. The details are given in Sec. III.

In Sec. IV we analyze meson-meson scattering by expanding the Lagrangian to fourth order in the meson fields. We obtain general formulas for any representation (p) and then consider the special cases when p represents the 3-, 6-, and 8-dimensional representations. In Sec. V we discuss weak currents and the problem of accommodating the K_{l3} form factor f_- within the framework of chiral SU(3). Behavior under redefinitions of the meson field is discussed in an appendix.

II. THE $(3, \bar{3})$ SYSTEM

Macfarlane, Sudbery, and Weisz (MSW)[6] have shown that the most general, nonsingular form of $F_{ab}(\pi)$ [see Eq. (1.5)] is

$$F_{ab}(\pi)=F\delta_{ab}+Bd_{abc}\pi_c+Cd_{abc}\Pi_c+G\pi_a\pi_b + S\pi_a\Pi_b+T\pi_b\Pi_a+J\Pi_a\Pi_b, \tag{2.1}$$

where F, B, C, G, S, T, and J are all functions of X and Y [see Eq. (1.8)]. Since $F_{ab}(\pi)$ has even parity, F, C, G, and J are actually functions of X and Y^2 and they need not vanish when $X=Y=0$; B, S, and T on the other hand are each products of Y times a function of X and Y^2, and they do vanish at the origin.

In order that the chiral algebra be satisfied, in particular, that

$$[K_a, K_b]=if_{abc}F_c,$$

where F_c is a generator of SU(3), the functions in Eq. (2.1) must satisfy a set of conditions enunciated by MSW. Rather than repeat all of them here, we quote only those necessary for our analysis. They are[6]

$$\Delta_1F-FG-\tfrac{1}{3}(BX+CY)S=-\tfrac{2}{3}, \quad \Delta_2F-FT-\tfrac{1}{3}(BX+CY)J=0, \tag{2.2a}$$

$$\Delta_1B+XCS+\tfrac{2}{3}BC-2FS=0, \quad \Delta_2B+XCJ-\tfrac{1}{3}C^2-2FJ=0, \tag{2.2b}$$

$$\Delta_1C+2SB+CG+C^2=0, \quad \Delta_2C+2JB+CT=0, \tag{2.2c}$$

$$B^2+2FC-\tfrac{1}{3}XC^2=-1, \tag{2.3}$$

where

$$\Delta_1\equiv 2(F+\tfrac{1}{3}XC+XG+YS)\frac{\partial}{\partial X} + (XB+2YC+3YG+X^2S)\frac{\partial}{\partial Y}, \tag{2.4}$$

$$\Delta_2\equiv 2(B+XT+YJ)\frac{\partial}{\partial X}+(3F-XC+3YT+X^2J)\frac{\partial}{\partial Y}.$$

Although no more than two of the seven functions in Eq. (2.1) are independent, we shall generally find it convenient to work with three of them, namely, F, B, and C. Equation (2.3) then serves the very useful purpose of expressing any one of the three in terms of the other two.

A. General Form of Z_α and $\overline{Z}_\alpha$

When the commutation tensor has its most general form the pseudoscalar octet is given by

$$v_b = \alpha\pi_b + \beta\Pi_b, \tag{2.5}$$

where α and β are as yet unknown functions of X, Y, F, B, and C. It is easily seen from Eqs. (1.6) and (1.7) that $[K_a, v_b]$ contains no SU(3) representation greater than the octet. Consequently, in order to eliminate decuplets and (27)-plets from the commutator, α and β must satisfy

$$\begin{aligned}&\Delta_1\alpha + \alpha G + \tfrac{2}{3}\beta(B+XS) = \Delta_2\alpha + \alpha T + \tfrac{1}{3}\beta(2XJ - C) = 0,\\ &\Delta_1\beta + \alpha S + \beta(C+2G) = \Delta_2\beta + \alpha J + 2\beta T = 0.\end{aligned} \tag{2.6}$$

We now argue that these conditions are sufficient to give a consistent linear realization of SU(3) ×SU(3).

We begin by using the commutation relations of Eqs. (1.6) and (1.7) together with certain properties of the d coefficients, i.e.,[7]

$$\sum_{b=1}^{8} d_{bbc} = 0, \qquad \sum_{b,c=1}^{8} d_{abc}d_{fbc} = \tfrac{5}{3}\,\delta_{af}, \tag{2.7}$$

to identify the remaining members of the $(3,\overline{3})$ system:

$$u_0 = -\tfrac{1}{8}i(\tfrac{3}{2})^{1/2}[K_a, v_a] = (\tfrac{3}{2})^{1/2}[\alpha F + \beta\times\tfrac{1}{3}(BX+CY)], \tag{2.8a}$$

$$u_a = -\tfrac{3}{5}id_{abc}[K_b, v_c] = \alpha'\pi_a + \beta'\Pi_a\,, \tag{2.8b}$$

$$v_0 = \tfrac{1}{8}i(\tfrac{3}{2})^{1/2}[K_a, u_a] = -(\tfrac{3}{2})^{1/2}[\alpha' F + \beta'\times\tfrac{1}{3}(BX+CY)], \tag{2.8c}$$

where

$$\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix} = \begin{bmatrix}B & 2F - \tfrac{1}{3}XC\\ C & -B\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \mathcal{B}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{2.9}$$

Next we observe that if the commutator

$$[K_b, u_0] = -i(\tfrac{2}{3})^{1/2}v_b \tag{2.10}$$

is to be satisfied, then α and β must obey

$$\begin{aligned}&\Delta_1[\alpha F + \beta\times\tfrac{1}{3}(BX+CY)] = -\tfrac{2}{3}\alpha,\\ &\Delta_2[\alpha F + \beta\times\tfrac{1}{3}(BX+CY)] = -\tfrac{2}{3}\beta.\end{aligned} \tag{2.11}$$

That this is indeed the case follows from the algebraic requirements for $F_{ab}(\pi)$ [see Eqs. (2.2) and (2.3)] together with the conditions of Eq. (2.6). Finally we note that α' and β' obey exactly the same conditions as α and β [see Eqs. (2.6) and (2.11)], and so $[K_a, u_b]$ contains nothing higher than an octet, and the relation

$$[K_b, v_0] = +i(\tfrac{2}{3})^{1/2}u_b \tag{2.12}$$

holds valid.

Having shown that the realization is algebraically consistent, we must now determine the unknown functions α and β. We find it convenient to work not with α and β directly, but rather with u_0 and v_0; these latter quantities are of direct group-theoretical interest, and they are equivalent to α and β on account of Eqs. (2.8) and (2.9). In fact, we can use the identity

$$\frac{1}{D}\left(\mathcal{B}\mathcal{C}\begin{bmatrix}b\\ -a\end{bmatrix}[b, -a] + \mathcal{C}\begin{bmatrix}b\\ -a\end{bmatrix}[b, -a]\mathcal{B}\right) \equiv I, \tag{2.13}$$

where

$$\begin{aligned}&I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\\ &\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}\tfrac{1}{3}(BX+CY)\\ -F\end{bmatrix},\\ &D = [a, b]\,\mathcal{C}\,\mathcal{B}\begin{bmatrix}a\\ b\end{bmatrix},\end{aligned} \tag{2.14}$$

to invert the expressions for u_0 and v_0:

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = I\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{D}(\tfrac{2}{3})^{1/2}(v_0 I - u_0\mathcal{B})\begin{bmatrix}a\\ b\end{bmatrix}. \tag{2.15}$$

From Eqs. (2.3) and (2.9) it follows that

$$\mathcal{B}^2 = -I \tag{2.16}$$

and so the expression for α' and β' is

$$\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix} = \frac{1}{D}(\tfrac{2}{3})^{1/2}(u_0 I + v_0\mathcal{B})\begin{bmatrix}a\\ b\end{bmatrix}. \tag{2.17}$$

For future reference we note that the transformation

$$[\alpha, \beta] \to [\alpha', \beta'] \tag{2.18a}$$

is equivalent to

$$(u_0, v_0) \to (-v_0, u_0). \tag{2.18b}$$

Using Eqs. (2.15) and (2.17), we can write down expressions for Z_k and $\overline{Z}_k$ in terms of u_0, v_0 and F, B, and C:

$$\begin{aligned}&Z_0 = u_0 - iv_0, \quad \overline{Z}_0 = u_0 + iv_0,\\ &Z_k = (u_k - iv_k)\\ &\quad = (\tfrac{2}{3})^{1/2}\left(\frac{Z_0}{D}\right)[\pi_k, \Pi_k](I + i\mathcal{B})\begin{bmatrix}a\\ b\end{bmatrix},\\ &\overline{Z}_k = (u_k + iv_k)\\ &\quad = (\tfrac{2}{3})^{1/2}\left(\frac{\overline{Z}_0}{D}\right)[\pi_k, \Pi_k](I - i\mathcal{B})\begin{bmatrix}a\\ b\end{bmatrix}.\end{aligned} \tag{2.19}$$

Now the quantity $\sum_{\gamma=0}^{8} Z_\gamma\overline{Z}_\gamma$ is chirally invariant:

$$[K_a, \textstyle\sum Z_\gamma\overline{Z}_\gamma] = 0, \tag{2.20}$$

and so it must be equal to a constant. Calling this constant n, we find from Eqs. (2.19) and (2.15) that

$$\sum_{\gamma=0}^{8} Z_\gamma \bar{Z}_\gamma = n = Z_0 \bar{Z}_0\left(1+\frac{2E}{3D}\right), \qquad (2.21)$$

where

$$E=\sum_{k=1}^{8}[\pi_k, \Pi_k]\mathcal{B}\,\mathcal{C}\begin{bmatrix}\pi_k\\ \Pi_k\end{bmatrix}, \quad (\mathcal{B}\,\mathcal{C})^T=\mathcal{B}\,\mathcal{C}. \qquad (2.22)$$

The second part of Eq. (2.21) relates $u_0^2+v_0^2$ to a known combination of F, B, and C; consequently we need only one more relation to be able to express u_0 and v_0 separately in terms of F, B, and C. To get at this relation, we consider a special case of $F_{ab}(\pi)$.

B. A Special Case and Its Uses

Suppose that we choose

$$\alpha=1, \quad \beta=0, \qquad (2.23)$$

and denote the corresponding fields and functions by carets; for example,

$$v_b=\hat{\pi}_b, \quad u_0=(\tfrac{3}{2})^{1/2}\hat{F}. \qquad (2.24)$$

Our problem now is the reverse of what it was before: Instead of having to find α and β in terms of the commutation tensor, we must now find the appropriate form of $F_{ab}(\pi)$ for our choice of α and β.

The general conditions of Eq. (2.6) plus the particular values of α and β in Eq. (2.23) imply that in the commutation tensor of Eq. (2.1),

$$\hat{G}=\hat{S}=\hat{T}=\hat{J}=0. \qquad (2.25)$$

Consequently the remaining invariant functions obey

$$\begin{aligned}&\hat{\Delta}_1\hat{F}=-\tfrac{2}{3}, \quad \hat{\Delta}_2\hat{F}=0,\\ &\hat{\Delta}_1\hat{B}=-\tfrac{2}{3}\hat{B}\hat{C}, \quad \hat{\Delta}_2\hat{B}=\tfrac{1}{3}\hat{C}^2,\\ &\hat{\Delta}_1\hat{C}=-\hat{C}^2, \quad \hat{\Delta}_2\hat{C}=0,\\ &\hat{B}^2+2\hat{F}\hat{C}-\tfrac{1}{3}\hat{X}\hat{C}^2=-1,\end{aligned} \qquad (2.26)$$

as can be seen from Eqs. (2.2) and (2.3). One solution of these equations takes the very simple form

$$\hat{F}+2/3\hat{C}=(\tfrac{2}{3})^{1/2}k, \qquad (2.27)$$

where k is a constant which can take all real values except zero. There is also a second solution, as we shall see below.

To make use of the relation between $\hat{F}$ and $\hat{C}$, we treat the identification of v_b with $\hat{\pi}_b$ as being a redefinition of the meson field:

$$\hat{\pi}_b=\alpha\pi_b+\beta\Pi_b. \qquad (2.28)$$

Then, as shown in Appendix A, we can write $\hat{F}$ and $\hat{C}$ as

$$\begin{aligned}&\hat{F}=\alpha F+\beta\times\tfrac{1}{3}(BX+CY),\\ &\hat{C}=\frac{C\alpha^2-2\alpha\beta B-(2F-\frac{1}{3}XC)\beta^2}{\alpha^3-\alpha\beta^2X-\frac{2}{3}\beta^3Y}.\end{aligned} \qquad (2.29)$$

Substituting these expressions into Eq. (2.27), we obtain a new relation between α and β in terms of F, B, and C.

We can verify this relation by using the conditions on α and β in Eqs. (2.6) and (2.11), together with the algebraic requirements of (2.2) and (2.3), to show that

$$\Delta_1\left(\hat{F}+\frac{2}{3\hat{C}}\right)=\Delta_2\left(\hat{F}+\frac{2}{3\hat{C}}\right)=0 \qquad (2.30)$$

when $\hat{F}$ and $\hat{C}$ are as given in Eq. (2.29). Now we have already observed that α' and β' obey exactly the same equations as α and β [see discussion preceding Eq. (2.12)], and so, in addition to Eq. (2.30), we can also conclude that

$$\Delta_1\left(\hat{F}'+\frac{2}{3\hat{C}'}\right)=\Delta_2\left(\hat{F}'+\frac{2}{3\hat{C}'}\right)=0, \qquad (2.31)$$

where $\hat{F}'$ and $\hat{C}'$ are the same as $\hat{F}$ and $\hat{C}$ except that α' and β' take the place of α and β. In the $\hat{v}_b=\hat{\pi}_b$ system, we find that

$$\begin{aligned}&\hat{F}'=\hat{B}\hat{F}+\hat{C}\times\tfrac{1}{3}(\hat{B}\hat{X}+\hat{C}\hat{Y}),\\ &\hat{C}'=\hat{C}/(\hat{B}^3-\hat{B}\hat{C}^2\hat{X}-\tfrac{2}{3}\hat{C}^3\hat{Y}).\end{aligned} \qquad (2.32)$$

At the very beginning of Sec. II, we remarked that $\hat{B}$ must vanish at the origin $\hat{X}=\hat{Y}=0$, while $\hat{F}$ and $\hat{C}$ need not; in fact, because of the last item in Eq. (2.6), F and C cannot vanish at this point. It then follows from Eq. (2.32) that $(\hat{F}'+2/3\hat{C}')$ vanishes when $\hat{X}$ and $\hat{Y}$ are zero. When combined with Eq. (2.31), this means that

$$\hat{F}'+2/3\hat{C}'=0 \qquad (2.33)$$

for all values of X and Y. Equation (2.33) is therefore the second solution to the set of partial differential equations in Eq. (2.26).

C. Solutions for Z_0 and $\bar{Z}_0$

Now that we have new relations for α and β, we need to translate them into relations for Z_0 and $\bar{Z}_0$. In principle, this is not a difficult task, but in practice it involves some tedious manipulation. We begin by reexpressing $\hat{F}$ and $\hat{C}$ in terms of u_0 and v_0, and then we show how the corresponding results for $\hat{F}'$ and $\hat{C}'$ can be obtained by a simple transformation. From the relations thus obtained, we are able to express Z_0 and $\bar{Z}_0$ as functions of F, B, and C.

Comparing Eq. (2.29) with (2.8a), (2.9), and (2.14), we see that

$$\hat{F} = (\tfrac{2}{3})^{1/2} u_0 \tag{2.34}$$

and that the numerator of $\hat{C}$ is

$$L \equiv C\alpha^2 - 2\alpha\beta B - (2F - \tfrac{1}{3}XC)\beta^2 = -[\alpha, \beta]\, \mathcal{C}\mathcal{B} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{2.35}$$

From the expression for $[\alpha, \beta]$ in Eq. (2.15), the definition of D in Eq. (2.14), and the properties

$$\mathcal{B}^2 = -I, \quad \mathcal{C}^T = -\mathcal{C}, \quad \mathcal{B}^T\mathcal{C}\mathcal{B} = \mathcal{C}, \tag{2.36}$$

we find that

$$L = -2(u_0^2 + v_0^2)/3D. \tag{2.37}$$

To evaluate the denominator of $\hat{C}$, we need the result

$$\begin{bmatrix} \frac{2}{3}\beta(\alpha X + \beta Y) \\ \alpha^2 - \frac{1}{3}X\beta^2 \end{bmatrix} = \frac{-4u_0^2}{3D}\begin{bmatrix} a \\ b \end{bmatrix} + \frac{2}{3D^2}[(v_0^2 - u_0^2)I + 2u_0 v_0 \mathcal{B}]\begin{bmatrix} \frac{2}{3}b(aX + bY) \\ a^2 - \frac{1}{3}Xb^2 \end{bmatrix}, \tag{2.38}$$

which can be derived by straightforward manipulation; we then obtain

$$\begin{aligned}\Delta &\equiv (\alpha^3 - \alpha\beta^2 X - \tfrac{2}{3}\beta^3 Y) = -[\alpha, \beta]\,\mathcal{C}\mathcal{B}\begin{bmatrix} \frac{2}{3}\beta(\alpha X + \beta Y) \\ \alpha^2 - \frac{1}{3}X\beta^2 \end{bmatrix} \\ &= \frac{2\sqrt{2}}{3\sqrt{3}\,D}\left[2u_0^3 + v_0(v_0^2 - 3u_0^2)\frac{K}{D^2} + u_0(3v_0^2 - u_0^2)\frac{H}{D^2}\right],\end{aligned} \tag{2.39}$$

where

$$\begin{aligned} K &= -[a, b]\,\mathcal{C}\begin{bmatrix} \frac{2}{3}b(aX + bY) \\ a^2 - \frac{1}{3}Xb^2 \end{bmatrix}, \\ H &= -[a, b]\,\mathcal{C}\mathcal{B}\begin{bmatrix} \frac{2}{3}b(aX + bY) \\ a^2 - \frac{1}{3}Xb^2 \end{bmatrix}. \end{aligned} \tag{2.40}$$

When applied to Eqs. (2.27) and (2.29), these results enable us to rewrite the relation between $\hat{F}$ and $\hat{C}$ in the form

$$u_0(3v_0^2 - u_0^2)(D^2 - 2H) - v_0(v_0^2 - 3u_0^2)(2K) = 3(u_0^2 + v_0^2)D^2 k. \tag{2.41}$$

Another relation of this kind comes from Eq. (2.33) for $\hat{F}'$ and $\hat{C}'$. Since the functions $\hat{F}'$ and $\hat{C}'$ are derived from $\hat{F}$ and $\hat{C}$ by substituting (α', β') for (α, β), and since this substitution is equivalent to the interchange of u_0 and v_0 in Eq. (2.18b), it follows that the second relation can be obtained by applying Eq. (2.18b) directly to Eq. (2.41). Thus we have

$$-v_0(3u_0^2 - v_0^2)(D^2 - 2H) - u_0(u_0^2 - 3v_0^2)(2K) = 0. \tag{2.42}$$

The right-hand side vanishes by virtue of Eq. (2.33).

The coefficients of $(D^2 - 2H)$ and $(2K)$ in Eqs. (2.41) and (2.42) are the real and imaginary parts of $(u_0 \pm iv_0)^3$. Therefore, by taking one relation plus or minus i times the other, and using the results

$$\begin{aligned} &(2H - D^2)^2 + 4K^2 = -\tfrac{2}{3}D^3\Phi, \\ &Z_0\overline{Z}_0 \equiv u_0^2 + v_0^2 = -\tfrac{3}{2}nD/\Phi, \\ &\Phi = -(\tfrac{3}{2}D + E) \end{aligned} \tag{2.43}$$

[see Eq. (2.21)], we find that

$$\begin{aligned} \overline{Z}_0^3 &= (u_0 + iv_0)^3 = \frac{9n^{3/2}}{4\Phi^2}[(2H - D^2) + 2Ki], \\ Z_0^3 &= (u_0 - iv_0)^3 = \frac{9n^{3/2}}{4\Phi^2}[(2H - D^2) - 2Ki], \end{aligned} \tag{2.44}$$

$$\sqrt{n} = 3k.$$

Thus we have found expressions, albeit rather complicated ones, for Z_0 and $\overline{Z}_0$ in terms of F, B, C, and the constant n.

D. Properties of Z_k and $\overline{Z}_k$

As we mentioned in the Introduction, there are two independent SU(3) scalars in this theory, and all other scalars must be functions of them. Now, in the $(3, \overline{3})$ system we are given two scalars, Z_0 and $\overline{Z}_0$, and we can manufacture seven more, for example, $(Z_k Z_k)$ and $(d_{ijk}Z_i Z_j Z_k)$. Therefore, if we treat Z_0 and $\overline{Z}_0$ as the independent scalars, the manufactured ones will be functions of Z_0 and $\overline{Z}_0$. This means that there must be several nonlinear relations among the elements of the $(3, \overline{3})$ system.

We have already encountered one such relation in Eq. (2.21):

$$\sum_{k=1}^{8} Z_k \overline{Z}_k = n - Z_0\overline{Z}_0, \tag{2.45}$$

where n is a constant. The other relations express Z_k and $\overline{Z}_k$ in terms of Z_0 and $\overline{Z}_0$, and relate quantities like $d_{ijk}Z_i Z_j$ to Z_k and $\overline{Z}_k$. To derive them, we shall need the identity

$$\mathcal{B}^T\begin{bmatrix} l & m \\ m & n \end{bmatrix}\mathcal{B} + \begin{bmatrix} l & m \\ m & n \end{bmatrix} \equiv [l(2F - XC/3) - 2mB - nC]\,\mathcal{C}\mathcal{B}, \tag{2.46}$$

which holds for any symmetric matrix, and also

$$\begin{bmatrix} \frac{2}{3}b(aX+bY) \\ a^2-\frac{1}{3}Xb^2 \end{bmatrix} = (-\tfrac{1}{6}EI+\tfrac{1}{3}\,\mathcal{C}\mathfrak{X}\mathcal{B})\begin{bmatrix} a \\ b \end{bmatrix},$$

$$\mathfrak{X} = \begin{bmatrix} \pi_k \\ \Pi_k \end{bmatrix}[\pi_k, \Pi_k] = \begin{bmatrix} X & Y \\ Y & \frac{1}{3}X^2 \end{bmatrix}, \tag{2.47}$$

where E is given in Eq. (2.22).

Let us begin with $Z_k Z_k$. From Eqs. (2.19) and (2.46), we have

$$Z_k Z_k = \frac{2Z_0^{\;2}}{3D^2}[a, b](-2\mathcal{B}^T\mathfrak{X}\mathcal{B} + E\,\mathcal{C}\mathcal{B} + 2i\,\mathfrak{X}\mathcal{B})\begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.48}$$

Using the properties of $\mathcal{B}$ and $\mathcal{C}$ in Eq. (2.36), we can write the term in round parentheses as

$$-(\mathcal{C}\mathcal{B}+i\,\mathcal{C})(-EI+2\,\mathcal{C}\mathfrak{X}\mathcal{B}) - iE\,\mathcal{C}. \tag{2.49}$$

The last term does not contribute to $Z_k Z_k$ because it is antisymmetric, and so from Eq. (2.47) and the definitions of H and K in Eq. (2.40) we can write

$$Z_k Z_k = \frac{2Z_0^{\;2}}{D^2}(2H+2iK). \tag{2.50}$$

With the aid of Eqs. (2.43) and (2.44), this becomes

$$Z_k Z_k = 2(\sqrt{n}\,\overline{Z}_0 + Z_0^{\;2}). \tag{2.51}$$

In a similar way, we can also show that

$$\overline{Z}_k \overline{Z}_k = 2(\sqrt{n}\,Z_0 + \overline{Z}_0^{\;2}). \tag{2.52}$$

We now consider products involving d coefficients. From Eq. (2.19) we have

$$d_{ijk}Z_i\overline{Z}_j = \frac{2Z_0\overline{Z}_0}{3D^2}[a, b]\;(I-i\,\mathcal{B}^T)\begin{bmatrix} \Pi_k & \frac{1}{3}X\pi_k \\ \frac{1}{3}X\pi_k & \frac{2}{3}Y\pi_k - \frac{1}{3}X\Pi_k \end{bmatrix}(I+i\,\mathcal{B})\begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.53}$$

Using the identity of Eq. (2.47) and dropping the antisymmetric part of the matrix sandwiched between the $[a, b]$ vectors, we find that

$$d_{ijk}Z_i\overline{Z}_j = \frac{-4}{3D}Z_0\overline{Z}_0[\pi_k, \Pi_k]\begin{bmatrix} a \\ b \end{bmatrix}, \tag{2.54}$$

and hence that

$$d_{ijk}Z_i\overline{Z}_j = -(\tfrac{2}{3})^{1/2}(Z_0\overline{Z}_k + \overline{Z}_0 Z_k). \tag{2.55}$$

In the case of $d_{ijk}Z_iZ_j$, we require some lengthy algebraic manipulations to show that

$$d_{ijk}Z_iZ_j = \frac{2Z_0^{\;2}}{3D^2}\left(2[\pi_k, \Pi_k](I-i\,\mathcal{B})\begin{bmatrix} \frac{2}{3}b(aX+bY) \\ a^2-\frac{1}{3}Xb^2 \end{bmatrix} + 2D[\pi_k, \Pi_k]\begin{bmatrix} a \\ b \end{bmatrix}\right). \tag{2.56}$$

Inserting the identity of Eq. (2.13) between $[\pi_k, \Pi_k]$ and the rest of the first term, and using the definitions of Eq. (2.40) together with the results of (2.43) and (2.44), we find that

$$d_{ijk}Z_iZ_j = (\tfrac{2}{3})^{1/2}(Z_0Z_k - \sqrt{n}\,\overline{Z}_k). \tag{2.57}$$

Similarly,

$$d_{ijk}\overline{Z}_i\overline{Z}_j = (\tfrac{2}{3})^{1/2}(\overline{Z}_0\overline{Z}_k - \sqrt{n}\,Z_k). \tag{2.58}$$

Combining these results with those for Z_kZ_k, $\overline{Z}_k\overline{Z}_k$, and $Z_k\overline{Z}_k$, we obtain the following expressions for trilinear scalar quantities:

$$\begin{aligned} d_{ijk}Z_iZ_jZ_k &= (\tfrac{2}{3})^{1/2}[2Z_0^{\;3} + 3\sqrt{n}\,Z_0\overline{Z}_0 - (\sqrt{n})^3], \\ d_{ijk}Z_iZ_j\overline{Z}_k &= -(\tfrac{2}{3})^{1/2}(nZ_0 + Z_0^{\;2}\overline{Z}_0 + 2\sqrt{n}\,\overline{Z}_0^{\;2}), \\ d_{ijk}Z_i\overline{Z}_j\overline{Z}_k &= -(\tfrac{2}{3})^{1/2}(n\overline{Z}_0 + \overline{Z}_0^{\;2}Z_0 + 2\sqrt{n}\,Z_0^{\;2}), \\ d_{ijk}\overline{Z}_i\overline{Z}_j\overline{Z}_k &= (\tfrac{2}{3})^{1/2}[2\overline{Z}_0^{\;3} + 3\sqrt{n}\,Z_0\overline{Z}_0 - (\sqrt{n})^3]. \end{aligned} \tag{2.59}$$

We also note that because[8]

$$if_{ijk}\pi_i\pi_j = if_{ijk}\pi_i\Pi_j = 0, \tag{2.60}$$

all products involving the antisymmetric coefficients vanish:

$$if_{ijk}Z_iZ_j = if_{ijk}Z_i\overline{Z}_j = if_{ijk}\overline{Z}_i\overline{Z}_j = 0. \tag{2.61}$$

Besides our argument about the number of independent SU(3) scalars, there is another way of understanding the relations described above. If we consider the Kronecker product of the representations $(3, \overline{3})$ and $(\overline{3}, 3)$ we can, in general, pick out a term corresponding to the (8, 1). For Z_k and $\overline{Z}_k$ this term is

$$\sum_{i,j=1}^{8}(if_{ijk}+d_{ijk})Z_i\overline{Z}_j + (\tfrac{2}{3})^{1/2}(\delta_{i0}\delta_{jk}+\delta_{j0}\delta_{ik})Z_i\overline{Z}_j \tag{2.62}$$

and it corresponds to converting the nonlinear meson field to the linear realization (8, 1). There is, however, a general theorem[9] which states that the only linear realizations to which the meson field can be converted are of the form $(m, \overline{m})$; therefore the expression in Eq. (2.62) must vanish. A check of Eqs. (2.55) and (2.61) reveals that it does indeed vanish. In other words the above relations ensure that we can never construct representations from Z_k and $\overline{Z}_k$ which violate the general theorem of Coleman, Wess, and Zumino.[9]

E. Expansion of Z_γ and $\overline{Z}_\gamma$

When we construct effective Lagrangians and apply them to physical processes, we have to expand them in powers of the meson field. In particular, for meson-meson scattering we must go as far as the fourth power. Since the Lagrangians are functions of Z_γ and $\bar{Z}_\gamma$, we expand these quantities to the appropriate order.

We begin by considering the functions F, C, G, B, and S. On grounds of parity, the first three are functions of X and Y^2, and so, to fourth order in π_b, they can be expanded as

$$F=F_0+XF_x+\frac{X^2}{2!}F_{xx}+\cdots,$$
$$C=C_0+XC_x+\frac{X^2}{2!}C_{xx}+\cdots, \tag{2.63}$$
$$G=G_0+XG_x+\frac{X^2}{2!}G_{xx}+\cdots,$$

where quantities with subscripts are constants. For the same reason the functions B and S are given by

$$B=YB_0+\cdots,\quad S=YS_0+\cdots. \tag{2.64}$$

Since B^2 and $S(BX+CY)$ are of sixth order in the meson field, they can be neglected in Eqs. (2.3) and (2.2c), and the resulting equations, namely,

$$2FC-\tfrac{1}{3}XC^2=-1,$$
$$2(F+\tfrac{1}{3}XC+XG)\frac{\partial F}{\partial X}-FG=-\tfrac{2}{3}, \tag{2.65}$$

enable us to express C and G in terms of the expansion coefficients of F:

$$C_0=\frac{-1}{2F_0},\quad C_x=\frac{1}{2F_0^3}(F_xF_0+\tfrac{1}{12}),$$
$$C_{xx}=\frac{F_{xx}}{2F_0^2}-\frac{1}{2F_0^5}(2F_0F_x+\tfrac{1}{3})(F_0F_x+\tfrac{1}{12}), \tag{2.66}$$
$$G_0=\frac{2}{F_0}(F_0F_x+\tfrac{1}{3}),\quad G_x=2F_{xx}+\frac{F_x}{F_0^2}(4F_0F_x+\tfrac{1}{3}).$$

With the aid of Eq. (2.66), we can expand the formula for Z_0^3 in Eq. (2.44):

$$Z_0^3=-n^{3/2}\left[1-\frac{X}{F_0^2}+\frac{iY}{3F_0^3}+\frac{X^2}{F_0^4}(2F_0F_x+\tfrac{3}{4})+\cdots\right]. \tag{2.67}$$

From this expression we find that, when $X=Y=0$,

$$[Z_0]_0=-\sqrt{n},\quad \left[\frac{\partial Z_0}{\partial X}\right]_0=-\frac{[Z_0]_0}{3F_0^2},$$
$$\left[\frac{\partial^2 Z_0}{\partial X^2}\right]_0=\frac{2[Z_0]_0}{3F_0^4}(2F_0F_x+\tfrac{5}{12}), \tag{2.68}$$
$$\left[\frac{\partial Z_0}{\partial Y}\right]_0=\frac{i[Z_0]_0}{9F_0^3}.$$

Here, and hereafter, square brackets with a subscript zero around a single term are used to indicate the value of the enclosed quantity at the origin. We now use Taylor's theorem to write

$$Z_0=[Z_0]_0\left(1-\frac{X}{3F_0^2}+\frac{iY}{9F_0^3}+\frac{X^2}{3F_0^4}(2F_0F_x+\tfrac{5}{12})+\cdots\right). \tag{2.69}$$

In a similar way, we have

$$\bar{Z}_0=[\bar{Z}_0]_0\left(1-\frac{X}{3F_0^2}-\frac{iY}{9F_0^3}+\frac{X^2}{3F_0^4}(2F_0F_x+\tfrac{5}{12})+\cdots\right), \tag{2.70}$$

where

$$[\bar{Z}_0]_0=-\sqrt{n}=[Z_0]_0\,. \tag{2.71}$$

To calculate the formulas for Z_k and $\bar{Z}_k$, we note that the matrix factor in Eq. (2.19) has the form

$$\frac{1}{D}[\pi_k,\Pi_k](I\pm i\mathfrak{B})\begin{bmatrix}a\\ b\end{bmatrix}=-\frac{\Pi_k}{2F_0^2}\left(1-\frac{X}{F_0^2}(2F_0F_x+\tfrac{1}{12})\right)-\frac{\pi_kY}{12F_0^4}\mp i\frac{\pi_k}{F_0}\left(1-\frac{X}{F_0}F_x\right)+\cdots. \tag{2.72}$$

Now using the expansion of Z_0 and $\bar{Z}_0$, we find that

$$Z_k=(\tfrac{2}{3}n)^{1/2}\left[\frac{\Pi_k}{2F_0^2}\left(1-\frac{X}{F_0^2}(2F_0F_x+\tfrac{5}{12})\right)-\frac{\pi_kY}{36F_0^4}+i\frac{\pi_k}{F_0}\left(1-\frac{X}{F_0^2}(F_0F_x+\tfrac{1}{3})\right)\right]+\cdots \tag{2.73}$$

and

$$\bar{Z}_k=(\tfrac{2}{3}n)^{1/2}\left[\frac{\Pi_k}{2F_0^2}\left(1-\frac{X}{F_0^2}(2F_0F_x+\tfrac{5}{12})\right)-\frac{\pi_kY}{36F_0^4}-\frac{i\pi_k}{F_0}\left(1-\frac{X}{F_0^2}(F_0F_x+\tfrac{1}{3})\right)\right]+\cdots. \tag{2.74}$$

These formulas and the ones for Z_0 and $\bar{Z}_0$ hold for all choices of F_0 and F_x (except $F_0=0$), but as a check we apply them to the special case of $\hat{F}$ in Sec. II B above.

In the system for which $v_b=\hat{\pi}_b$, the functions $\hat{F}$, $\hat{B}$, and $\hat{C}$ can be expanded as in Eq. (2.63), and they satisfy Eq. (2.66). In addition, they also obey Eq. (2.27), which when expanded to first order in X becomes

$$\hat{F}_0+\frac{2}{3\hat{C}_0}+\hat{X}\left(\hat{F}_x-\frac{2\hat{C}_x}{3\hat{C}_0^2}\right)-(\tfrac{2}{3})^{1/2}k=0. \tag{2.75}$$

Setting the coefficient of $\hat{X}$ and the constant term to zero, and using Eq. (2.66), we find that

$$\hat{F}_0=-3(\tfrac{2}{3})^{1/2}k=-(\tfrac{2}{3}n)^{1/2},\quad \hat{F}_0\hat{F}_x+\tfrac{1}{3}=0. \tag{2.76}$$

Now, by definition [see Eq. (1.6)]

$$\bar{Z}_k-Z_k=2iv_k,$$

but from Eqs. (2.73), (2.74), and (2.76),

$$\bar{Z}_k-Z_k=2i\hat{\pi}_k$$

in the caret system. Thus our expansions for Z_k and $\bar{Z}_k$ are consistent with the system in which we start by taking $v_b = \hat{\pi}_b$.

III. HIGHER REPRESENTATIONS

As a result of the Coleman-Wess-Zumino theorem[9] and the relations in Sec. II D, the only SU(3) $\times$SU(3) representations we can construct from products of Z_k and $\bar{Z}_k$ are those of the class $(p, \bar{p})$. The distinctive feature of this class is that the Kronecker product of (p) and its conjugate representation $(\bar{p})$ always contains an SU(3) singlet in addition to octets and other multiplets. This singlet is important because we can generate all the other members of $(p, \bar{p})$ from it, and because we can compute it for all choices of (p).

Let us denote the singlet by $\mathcal{S}$. It commutes with the SU(3) generators F_i,

$$[F_i, \mathcal{S}] = 0, \tag{3.1}$$

and so from

$$[F_i, K_j] = if_{ijk}K_k$$

and the Jacobi identity

$$[F_i, [K_j, \mathcal{S}]] + [K_j, [\mathcal{S}, F_i]] + [\mathcal{S}, [F_i, K_j]] \equiv 0, \tag{3.2}$$

we find that

$$[F_i, [K_j, \mathcal{S}]] = if_{ijk}[K_k, \mathcal{S}]. \tag{3.3}$$

This means that the commutator of $\mathcal{S}$ with the chiral operator K_j behaves as an SU(3) octet. By a similar argument, the double commutator $[K_i, [K_j, \mathcal{S}]]$ behaves as a second-rank tensor, the triple commutator as a third-rank tensor, and so on. Thus by commuting $\mathcal{S}$ with chiral operators enough times, we can generate a series of SU(3) representations with zero triality.

If $\mathcal{S}$ belongs to $(p, \bar{p})$, then the members of this series must also belong to $(p, \bar{p})$. To show this we note that the algebra of SU(3)$\times$SU(3) contains four Casimir operators L_a $(a = 1-4)$ which commute with the generators,

$$[F_i, L_a] = [K_i, L_a] = 0, \tag{3.4}$$

and whose eigenvalues serve to define an irreducible representation. The singlet $\mathcal{S}$ is an eigenstate of L_a,

$$[L_a, \mathcal{S}] = l_a \mathcal{S}, \tag{3.5}$$

and so, from the Jacobi identity for L_a, K_i, and $\mathcal{S}$, it follows that $[K_i, \mathcal{S}]$ is also an eigenstate with the same eigenvalue as $\mathcal{S}$:

$$[L_a, [K_i, \mathcal{S}]] = l_a[K_i, \mathcal{S}]. \tag{3.6}$$

Similarly, the double and higher commutators of $\mathcal{S}$ with chiral operators are also eigenstates of L_a with l_a as their eigenvalue. Thus commuting $\mathcal{S}$ with the K_i never takes us out of the $(p, \bar{p})$ representation.

For a fixed (p), the Clebsch-Gordan series of $(p\times\bar{p})$ runs from the singlet up to some maximal representation (k) and then stops. Consequently, when we commute $\mathcal{S}$ with a succession of chiral operators, we generate new representations until we reach (k); then we either annihilate $\mathcal{S}$ or begin to repeat ourselves. For example, the operator

$$K_+ = K_1 + iK_2 \tag{3.7}$$

raises the isospin by one unit every time it is applied to $\mathcal{S}$, and so there will be some number N, corresponding to the maximum isospin contained in (k), such that the $(N+1)$th commutator of K_+ on $\mathcal{S}$ will vanish:

$$[K_+, [K_+, \ldots, [K_+, \mathcal{S}]\cdots] = 0, \quad (N+1 \text{ factors}). \tag{3.8}$$

When we convert the eigenvalue equations (3.5) to differential equations, this result will serve as a boundary condition upon the solution.

To proceed further, we need to know the exact forms of the operators L_a and the eigenvalues l_a. We describe the representation (p) by two integers (μ_1, μ_2) which represent the number of quark and antiquark indices, respectively. For $(\bar{p})$ the role of quarks and antiquarks are interchanged:

$$(p) \equiv (\mu_1, \mu_2), \quad (\bar{p}) \equiv (\mu_2, \mu_1). \tag{3.9}$$

The dimension of (p) is

$$D(\mu_1, \mu_2) = \tfrac{1}{2}(\mu_1 + 1)(\mu_2 + 1)(\mu_1 + \mu_2 + 2) \tag{3.10}$$

and the eigenvalues of the quadratic and cubic Casimir operators are[10]

$$\begin{aligned} m_2(\mu_1, \mu_2) &= \tfrac{1}{3}[\mu_1{}^2 + \mu_2{}^2 + (\mu_1 + \mu_2)^2 + 6(\mu_1 + \mu_2)], \\ m_3(\mu_1, \mu_2) &= \tfrac{1}{9}(\mu_2 - \mu_1)[(\mu_1 + 2\mu_2)(\mu_2 + 2\mu_1) \\ &\quad + 9(\mu_1 + \mu_2 + 1)], \end{aligned} \tag{3.11}$$

respectively. The statement that $\mathcal{S}$ belongs to the representation $(p, \bar{p})$ of SU(3)$\times$SU(3) then means that

$$\tfrac{1}{4}[F_i \pm K_i, [F_i \pm K_i, \mathcal{S}]] = \tfrac{1}{2}m_2(\mu_1, \mu_2)\mathcal{S}, \tag{3.12a}$$

$$\tfrac{1}{8}d_{ijk}[F_i \pm K_i, [F_j \pm K_j, [F_k \pm K_k, \mathcal{S}]]] = \mp\tfrac{1}{2}m_3(\mu_1, \mu_2)\mathcal{S}, \tag{3.12b}$$

where we have made use of

$$\begin{aligned} m_2(\mu_1, \mu_2) &= +m_2(\mu_2, \mu_1), \\ m_3(\mu_1, \mu_2) &= -m_3(\mu_2, \mu_1). \end{aligned} \tag{3.13}$$

Since $\mathcal{S}$ is a singlet, these equations reduce to

$$[K_i, [K_i, \mathcal{S}]] = 2m_2(\mu_1, \mu_2)\mathcal{S}, \tag{3.14a}$$

$$d_{ijk}[K_i, [K_j, [K_k, \mathcal{S}]]] = -4m_3(\mu_1, \mu_2)\mathcal{S}. \tag{3.14b}$$

We can convert these conditions to partial differential equations by observing that $\mathcal{S}$ must be a function of Z_0 and $\bar{Z}_0$. Then, with the aid of the relations in Sec. II D, we obtain

$$(Z_0^2+\sqrt{n}\,\bar{Z}_0)\frac{\partial^2 \mathcal{S}}{\partial Z_0^2}+(Z_0\bar{Z}_0-n)\frac{\partial^2 \mathcal{S}}{\partial Z_0\partial \bar{Z}_0}+(\bar{Z}_0^2+\sqrt{n}\,Z_0)\frac{\partial^2 \mathcal{S}}{\partial \bar{Z}_0^2}+4\left(Z_0\frac{\partial \mathcal{S}}{\partial Z_0}+\bar{Z}_0\frac{\partial \mathcal{S}}{\partial \bar{Z}_0}\right)=\tfrac{3}{2}m_2(\mu_1,\mu_2)\mathcal{S} \tag{3.15}$$

and

$$\begin{aligned}&[2Z_0^3+3\sqrt{n}\,Z_0\bar{Z}_0-(\sqrt{n})^3]\frac{\partial^3 \mathcal{S}}{\partial Z_0^3}+(3Z_0^2\bar{Z}_0+6\sqrt{n}\,\bar{Z}_0^2+3nZ_0)\frac{\partial^3 \mathcal{S}}{\partial Z_0^2\partial \bar{Z}_0}-(3Z_0\bar{Z}_0^2+6\sqrt{n}\,Z_0^2+3n\bar{Z}_0)\frac{\partial^3 \mathcal{S}}{\partial Z_0\partial \bar{Z}_0^2}\\&\quad-[2\bar{Z}_0^2+3\sqrt{n}\,Z_0\bar{Z}_0-(\sqrt{n})^3]\frac{\partial^3 \mathcal{S}}{\partial \bar{Z}_0^3}+15(Z_0^2+\sqrt{n}\,\bar{Z}_0)\frac{\partial^2 \mathcal{S}}{\partial Z_0^2}-15(\bar{Z}_0^2+\sqrt{n}\,Z_0)\frac{\partial^2 \mathcal{S}}{\partial \bar{Z}_0^2}+20\left(Z_0\frac{\partial \mathcal{S}}{\partial Z_0}-\bar{Z}_0\frac{\partial \mathcal{S}}{\partial \bar{Z}_0}\right)\\&\qquad=-9m_3(\mu_1,\mu_2)\mathcal{S}\,.\end{aligned} \tag{3.16}$$

Since quarks have isospin zero or $\frac{1}{2}$, the largest isospin contained in $(p,\bar{p})$ is

$$N=(\mu_1+\mu_2). \tag{3.17}$$

Accordingly, the $(\mu_1+\mu_2+1)$th commutator of K_+ on $\mathcal{S}$ must vanish as in Eq. (3.8). This means that all the $(\mu_1+\mu_2+1)$th partial derivatives of $\mathcal{S}$ must vanish,

$$\frac{\partial^{N+1}\mathcal{S}}{\partial Z_0^{N+1}}=\frac{\partial^{N+1}\mathcal{S}}{\partial Z_0^{N}\partial \bar{Z}_0}=\cdots=\frac{\partial^{N+1}\mathcal{S}}{\partial \bar{Z}_0^{N+1}}=0, \tag{3.18}$$

where N is given by Eq. (3.17). Therefore the required solution of Eqs. (3.15) and (3.16) is a polynomial of degree N in Z_0 and $\bar{Z}_0$.

If we write this polynomial as

$$\mathcal{S}=\sum_{m,n}a_{m,n}(Z_0)^m(\bar{Z}_0)^n \tag{3.19}$$

and substitute it in Eqs. (3.15) and (3.16), we obtain two recursion relations for the coefficients $a_{m,n}$. The general solution is too cumbersome to quote, and so we shall, instead, cite a few illustrative examples. In the $(3,\bar{3})$ case, $\mu_1=1$, $\mu_2=0$, and $\mathcal{S}=Z_0$ as expected; in representations for which $\mu_1+\mu_2=2, 3$, we have

$$\begin{aligned}&(6,\bar{6}):\quad \mu_1=2,\quad \mu_2=0,\quad \mathcal{S}=\bar{Z}_0+3(Z_0^2/\sqrt{n}),\\&(8,8):\quad \mu_1=\mu_2=1,\quad \mathcal{S}=1-9(Z_0\bar{Z}_0/n),\\&(10,\overline{10}):\quad \mu_1=3,\quad \mu_2=0,\quad \mathcal{S}=1-18(Z_0\bar{Z}_0/n)-27(Z_0/\sqrt{n})^3,\\&(15,\overline{15}):\quad \mu_1=2,\quad \mu_2=1,\quad \mathcal{S}=Z_0-3(\bar{Z}_0^2/\sqrt{n})-9(Z_0^2\bar{Z}_0/n).\end{aligned} \tag{3.20}$$

For higher representations we recommend that each case be handled on its own.

Although the general formula for $\mathcal{S}$ is unwieldy, it does reduce to manageable proportions when it is expanded as a power series in the meson field, and only the first few terms are kept. Up to fourth order $\mathcal{S}$ is given by

$$\mathcal{S}=[\mathcal{S}]_0+\left[\frac{\partial \mathcal{S}}{\partial X}\right]_0X+\left[\frac{\partial \mathcal{S}}{\partial Y}\right]_0Y+\frac{1}{2!}\left[\frac{\partial^2 \mathcal{S}}{\partial X^2}\right]_0X^2+\cdots, \tag{3.21}$$

where the square brackets with subscript zero denote values at the point $\pi_b=X=Y=0$. The coefficients of X and Y can be written as products of partial derivatives of $\mathcal{S}$ with respect to Z_0 and $\bar{Z}_0$ times partial derivatives of Z_0 and $\bar{Z}_0$ with respect to X and Y; for example,

$$\left[\frac{\partial \mathcal{S}}{\partial X}\right]_0=\left[\frac{\partial \mathcal{S}}{\partial Z_0}\frac{\partial Z_0}{\partial X}+\frac{\partial \mathcal{S}}{\partial \bar{Z}_0}\frac{\partial \bar{Z}_0}{\partial X}\right]_0. \tag{3.22}$$

The derivatives of Z_0 and $\bar{Z}_0$ at the origin are already contained in Eqs. (2.68)–(2.71), and those of $\mathcal{S}$ can be computed from Eqs. (3.15) and (3.16). In this way, $\mathcal{S}$ is found to be

$$\mathcal{S}=[\mathcal{S}]_0[1-\tfrac{1}{8}m_2X/F_0^2-\tfrac{1}{20}im_3Y/F_0^3+\tfrac{1}{4}m_2(\tfrac{1}{40}m_2+F_0F_x+\tfrac{17}{120})X^2/F_0^4+\cdots]\,. \tag{3.23}$$

We have already pointed out that once we know $\mathcal{S}$ we can determine the remaining members of the $(p,\bar{p})$ representation by commuting $\mathcal{S}$ with K_i. Of particular interest are the two octets

$$M_i=[K_i,\mathcal{S}]=(\tfrac{2}{3})^{1/2}\left(Z_i\frac{\partial\mathcal{S}}{\partial Z_0}-\bar{Z}_i\frac{\partial\mathcal{S}}{\partial\bar{Z}_0}\right) \tag{3.24}$$

and

$$\begin{aligned}\tilde{M}_i&=d_{ijk}[K_j,[K_k,\mathcal{S}]]\\&=\tfrac{2}{3}d_{ijk}\left(Z_jZ_k\frac{\partial^2\mathcal{S}}{\partial Z_0^{\,2}}-2Z_j\bar{Z}_k\frac{\partial^2\mathcal{S}}{\partial Z_0\partial\bar{Z}_0}+\bar{Z}_j\bar{Z}_k\frac{\partial^2\mathcal{S}}{\partial\bar{Z}_0^{\,2}}\right)+\tfrac{5}{3}(\tfrac{2}{3})^{1/2}\left(Z_i\frac{\partial\mathcal{S}}{\partial Z_0}+\bar{Z}_i\frac{\partial\mathcal{S}}{\partial\bar{Z}_0}\right).\end{aligned} \tag{3.25}$$

For the $(3,\bar{3})$ representation $\mathcal{S}$ is equal to Z_0, and M_i and $\tilde{M}_i$ are both proportional to Z_i. For the special cases of Eq. (3.20), we find that

$$\begin{aligned}&(6,\bar{6}):\quad M_i=(\tfrac{2}{3})^{1/2}[(6/\sqrt{n})Z_0Z_i-\bar{Z}_i],\qquad \tilde{M}_i=\tfrac{7}{3}M_i,\\&(8,8):\quad M_i=(-9/n)(\tfrac{2}{3})^{1/2}(Z_i\bar{Z}_0-\bar{Z}_iZ_0),\qquad \tilde{M}_i=-(\sqrt{6}/n)(\bar{Z}_0Z_i+Z_0\bar{Z}_i),\\&(10,\overline{10}):\quad M_i=(3\sqrt{6}/n)(-2Z_i\bar{Z}_0-(9/\sqrt{n})Z_0^{\,2}Z_i+2Z_0\bar{Z}_i),\qquad \tilde{M}_i=3M_i,\\&(15,\overline{15}):\quad M_i=(\tfrac{2}{3})^{1/2}\left[Z_i\left(1-\frac{18Z_0\bar{Z}_0}{n}\right)+3\bar{Z}_i\left(\frac{2\bar{Z}_0}{\sqrt{n}}+\frac{3Z_0^{\,2}}{n}\right)\right],\quad \tilde{M}_i=(\tfrac{2}{3})^{1/2}\left[Z_i\left(\frac{17}{3}-\frac{66}{n}Z_0\bar{Z}_0\right)-\bar{Z}_i\left(\frac{2\bar{Z}_0}{\sqrt{n}}+\frac{39Z_0^{\,2}}{n}\right)\right].\end{aligned} \tag{3.26}$$

If we expand the general formulas for M_i and $\tilde{M}_i$ by the same techniques as we used for $\mathcal{S}$, then we obtain

$$M_i=[\mathcal{S}]_0\frac{m_3}{10}\left(\frac{3}{2}\frac{\Pi_i}{F_0^{\,2}}-(3F_0F_x+\tfrac{3}{28}m_2+\tfrac{19}{56})\frac{X\Pi_i}{F_0^{\,4}}-\frac{2m_2-3}{28}\frac{Y\pi_k}{F_0^{\,4}}\right)-i[\mathcal{S}]_0\frac{m_2}{4}\frac{\pi_i}{F_0}\left(1-\frac{X}{F_0^{\,2}}(F_0F_x+\tfrac{1}{10}m_2+\tfrac{1}{15})\right)+\cdots \tag{3.27}$$

and

$$\begin{aligned}\tilde{M}_i=&[\mathcal{S}]_0m_2(2m_2+3)\left[\frac{-\Pi_i}{40F_0^{\,2}}-\frac{2m_2+9}{9\times560}\frac{\pi_iY}{F_0^{\,4}}+\left(F_0F_x+\frac{22m_2+57}{21\times24}\right)\frac{X\Pi_i}{20F_0^{\,4}}\right]\\&+[\mathcal{S}]_0\frac{m_3^{\,2}}{105}\left(\frac{2Y\pi_i}{F_0^{\,4}}-\frac{X\Pi_i}{2F_0^{\,4}}\right)+i[\mathcal{S}]_0\frac{m_3}{2}\frac{\pi_i}{F_0}\left(1-\frac{X}{F_0^{\,2}}(F_0F_x+\tfrac{1}{10}m_2+\tfrac{1}{15})\right)+\cdots.\end{aligned} \tag{3.28}$$

When $\mu_1=\mu_2$ the representation $(p,\bar{p})$ is self-conjugate and m_3 vanishes [see Eqs. (3.9)–(3.11)]; the octet M_i is then of purely odd parity and $\tilde{M}_i$ is even. When either μ_1 or μ_2 is zero, the SU(3) representation (p) is triangular and the product $(p\times\bar{p})$ contains one octet instead of two. In this case we find that M_i and $\tilde{M}_i$ are proportional to one another:

$$\begin{aligned}\tilde{M}_i&=\tfrac{1}{3}(2\mu_1+3)M_i,\qquad \mu_2=0,\\&=-\tfrac{1}{3}(2\mu_2+3)M_i,\quad \mu_1=0.\end{aligned} \tag{3.29}$$

These simplifications will be very useful when we come to consider meson-meason scattering.

IV. MESON-MESON SCATTERING

The meson Lagrangian consists of two parts, a kinetic term which involves derivatives of the meson field, and a mass term which contains no derivatives. The kinetic term need not break the chiral symmetry, but the mass term always does. Accordingly, we consider a model in which all the symmetry breaking occurs in the mass term, and none in the kinetic one.

We shall assume that the symmetry-breaking term belongs to a single representation, $\{(p,\bar{p})+(\bar{p},p)\}$, of SU(3)×SU(3), and is an admixture of its unitary singlet and octet components. We determine this admixture by fitting the Lagrangian to the observed masses of the pion and K meson, and then we can calculate general formulas for meson-meson scattering in terms of the representation (p). Our results for S-wave scattering lengths agree with earlier calculations for the case $p=3$,[5] and they indicate that certain higher representations are not likely to occur in the real world.

In terms of Z_k and $\bar{Z}_k$ the chiral-invariant kinetic-energy term takes the simple form

$$\mathcal{L}_{\mathrm{K.E.}}=-G\sum_{\alpha=0}^{8}\partial_\mu Z_\alpha\partial_\mu\bar{Z}_\alpha. \tag{4.1}$$

The constant G is chosen so that when $\mathcal{L}_{\mathrm{K.E.}}$ is expanded in powers of the meson field, the coefficient of the leading term is $(-\frac{1}{2})$. From Eqs. (2.67)–(2.74) we find that up to fourth order in the meson field,

$$\mathcal{L}_{\text{K.E.}} = -\tfrac{1}{2}(\partial_\mu \pi_k)(\partial_\mu \pi_k)\left[1 - \frac{2X}{F_0^2}(F_0 F_x + \tfrac{1}{3})\right] + \frac{1}{4F_0^2}(\partial_\mu X \partial_\mu X)(2F_0 F_x + \tfrac{1}{3}) - \frac{1}{8F_0^2}(\partial_\mu \Pi_k \partial_\mu \Pi_k). \quad (4.2)$$

The mass term of the Lagrangian can be written as

$$\mathcal{L}_m = -\tfrac{1}{2}(d\mathcal{L}_0 + b\mathcal{L}_8 + \tilde{b}\tilde{\mathcal{L}}_8), \quad (4.3)$$

where $\mathcal{L}_0$ is proportional to the unitary singlet function $\mathcal{S}$ of Eq. (3.23), and $\mathcal{L}_8$ and $\tilde{\mathcal{L}}_8$ are proportional to the octets M_8 and $\tilde{M}_8$, respectively [see Eqs. (3.27) and (3.28)]. These expressions are normalized so that the quadratic terms of $\mathcal{L}_m$ are

$$-\tfrac{1}{2}[dX + (b+\tilde{b})\Pi_8]. \quad (4.4)$$

To fit the pion and K-meson masses, the constants d, b, and $\tilde{b}$ must obey

$$d = \tfrac{1}{3}(m_\pi^2 + 2m_K^2),$$
$$b + \tilde{b} = (\tfrac{4}{3})^{1/2}(m_\pi^2 - m_K^2). \quad (4.5)$$

Notice that in general we cannot determine all three constants by fitting masses; we shall return to this point below.

To calculate S-wave scattering lengths from the fourth-order terms in the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{K.E.}} + \mathcal{L}_m, \quad (4.6)$$

we use the formula[11]

$$a = \frac{-iM(12 \to 1'2')}{8\pi(\mathfrak{M}_1 + \mathfrak{M}_2)}, \quad (4.7)$$

where $M(12 \to 1'2')$ is the invariant amplitude taken with respect to normalized and properly symmetrized two-meson states at threshold, and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are the meson masses. We then find that for π-π scattering,

$$a_0 = \frac{1}{24\pi m_\pi F_0^2}(-m_\pi^2 + 5A),$$
$$a_2 = \frac{1}{24\pi m_\pi F_0^2}(-4m_\pi^2 + 2A),$$
$$A = \frac{17 + 3m_2}{20}d + \frac{19 + 6m_2}{28\sqrt{3}}b + \left(57 + 22m_2 - \frac{48m_3^2}{m_2(2m_2+3)}\right)\frac{\tilde{b}}{84\sqrt{3}}, \quad (4.8)$$

where m_2 and m_3 are the Casimir eigenvalues of Eq. (3.11). For K-K scattering we obtain

$$a_0(KK) = a_0(\overline{KK}) = 0,$$
$$a_1(KK) = \frac{1}{24\pi m_K F_0^2}\left[-4m_K^2 + \frac{17+3m_2}{10}d - \frac{19+6m_2}{28\sqrt{3}}b - \left(57 + 22m_2 - \frac{48m_3^2}{m_2(2m_2+3)}\right)\frac{\tilde{b}}{84\sqrt{3}}\right]. \quad (4.9)$$

Finally, the π-K scattering lengths are

$$a_{3/2} = \frac{1}{8\pi F_0^2(m_\pi + m_K)}[-\tfrac{5}{6}(m_\pi^2 + m_K^2) - m_\pi m_K + A'],$$
$$a_{1/2} = \frac{1}{8\pi F_0^2(m_\pi + m_K)}[-\tfrac{5}{6}(m_\pi^2 + m_K^2) + 2m_\pi m_K + A'], \quad (4.10)$$

where

$$A' = \frac{17+3m_2}{15}d + \frac{19+6m_2}{84\sqrt{3}}b + \left(57 + 22m_2 - \frac{48m_3^2}{m_2(2m_2+3)}\right)\frac{\tilde{b}}{252\sqrt{3}}. \quad (4.11)$$

In all of the above equations, subscripts on scattering lengths correspond to the isospin of the two-meson state.

When $p = 3$, the Casimir eigenvalues are

$$(\mu_1 = 1, \mu_2 = 0):\quad m_2 = \tfrac{8}{3},\quad m_3 = -\tfrac{20}{9}, \quad (4.12)$$

and the scattering lengths become

$$\underline{\pi\text{-}\pi}:\quad a_0 = \frac{7m_\pi}{32\pi F_0^2},\quad a_2 = \frac{-2m_\pi}{32\pi F_0^2}, \quad (4.13\text{a})$$

$$\underline{KK}:\quad a_1 = \frac{-2m_K}{32\pi F_0^2}, \quad (4.13\text{b})$$

$$\underline{\pi K}:\quad a_{1/2} = \frac{m_\pi m_K}{4\pi F_0^2(m_\pi + m_K)},\quad a_{3/2} = -\frac{m_\pi m_K}{8\pi F_0^2(m_K + m_\pi)}. \quad (4.13\text{c})$$

These results agree with those of Cronin[5] if his parameter f is given by $2f^2 = 1/F_0^2$; they are also consistent with the work of Turner.[5] It is interesting to note that the π-π scattering lengths in Eq. (4.13a) are identical with the predictions of chiral SU(2) with the mass term in the $(\tfrac{1}{2}, \tfrac{1}{2})$ representation.[1,3] This is not surprising because the relevant representations of SU(2)×SU(2) contained in $\{(3,\bar{3}) + (\bar{3},3)\}$ are (0, 0) and $(\tfrac{1}{2}, \tfrac{1}{2})$.

Suppose now that we consider the properties of π-π scattering lengths for larger representations (p). From Eq. (4.8) we see that the combination $2a_0 - 5a_2$ is independent of the parameters of (p); moreover, the actual value of the combination, namely,

$$2a_0 - 5a_2 = 3m_\pi/4\pi F_0^2 \quad (4.14)$$

is exactly the same as in the chiral SU(2) case. Another combination is proportional to the quantity A:

$$4a_0 - a_2 = 3A/4\pi m_\pi F_0^2. \tag{4.15}$$

In general, we cannot predict the magnitude of A because we cannot calculate both constants b and $\tilde{b}$ from observed masses [see Eq. (4.5)]. There are, however, certain types of representation for which one of the constants is effectively zero.

If (p) is a self-adjoint representation, then $\mu_1 = \mu_2 = \mu$, and the eigenvalue m_3 vanishes [see Eq. (3.11)]. The even-parity part of M_8 [Eq. (3.27)] is then zero, and we can drop the term $b\mathfrak{L}_8$ from $\mathfrak{L}_m$ [Eq. (4.3)]. With $b=0$, we can determine $\tilde{b}$ from Eq. (4.5), and hence the quantity A:

$$A = m_\pi^2[\tfrac{103}{140} + \tfrac{283}{630}\mu(\mu+2)] + 2m_K^2[\tfrac{2}{35} - \tfrac{47}{630}\mu(\mu+2)], \quad \mu_1 = \mu_2 = \mu. \tag{4.16}$$

Because the K-meson mass is so much larger than the pion mass ($m_K^2 \approx 12 m_\pi^2$), the value of A is always negative and fairly large. For example, in the simple case of (p) an octet with $\mu = 1$, we find that

$$A \approx -2m_\pi^2$$

and that a_0 and a_2 are both negative and of comparable magnitudes. As we increase μ, A becomes more and more negative and the ratio a_2/a_0 tends to 0.4. Thus if we always want a_0 to be positive and much larger than $|a_2|$,[12] we must rule out all symmetry breaking in which (p) is self-adjoint, i.e., (8), (27), (64), ...

If (p) is a triangular representation then $\mu_2 = 0$, and the octets M_8 and $\tilde{M}_8$ of Eqs. (3.27), (3.28) are proportional to one another [see Eq. (3.29)]. Thus for all practical purposes we can drop $b\mathfrak{L}_8$ from the Lagrangian. In this case A is given by

$$A = m_\pi^2[\tfrac{103}{140} + \tfrac{9}{70}\mu(\mu+3)] + 2m_K^2[\tfrac{2}{35} - \tfrac{1}{70}\mu(\mu+3)], \quad \mu_2 = 0. \tag{4.17}$$

Except for the simple case of (p) a triplet with $\mu = 1$, the predictions of this set of representations are unsatisfactory: For $\mu = 2$, a_2 is much larger than a_0 in magnitude, and for higher values of μ, the objections are much the same as in the self-adjoint case. Thus we have grounds for ruling out symmetry breaking by triangular representations larger than the triplet, i.e., $p = 6, 10, 15, \ldots$, .

If (p) is neither self-adjoint, nor triangular, then we have no reason to drop $b\mathfrak{L}_8$ from the Lagrangian. We therefore have one free parameter, and we can choose it in many ways. For example, we could fix b so that

$$A = \tfrac{5}{4} m_\pi^2 \tag{4.18}$$

no matter what the representation (p) may be. In this case the π-π scattering lengths would be exactly the same as for $\{(3,\overline{3}) + (\overline{3},3)\}$ breaking [see Eq. (4.13a)], and the model would have to be tested in K-K and π-K scattering. The simplest representation of this kind is a 15-dimensional one with parameters $\mu_1 = 2$, $\mu_2 = 1$.

We see from this analysis that our present knowledge of π-π scattering lengths allows us to put fairly strong restrictions on the manner of chiral SU(3) breaking. Knowledge of π-K and K-K scattering would help us even more.

V. WEAK CURRENTS

The currents associated with weak decays of pseudoscalar mesons are usually assumed to be members of the (8, 1) representation of SU(3) $\times$SU(3). We can construct them directly from Z_α, $\overline{Z}_\beta$, and their derivatives, or we can derive them from the Lagrangian of Sec. IV [see Eqs. (4.1)–(4.6)] by means of Noether's theorem. Whichever method we use, we obtain the same answer:

$$J_\mu^{(k)} = g[if_{mlk} + d_{mlk} + (\tfrac{2}{3})^{1/2}(\delta_{m0}\delta_{lk} + \delta_{l0}\delta_{mk} + \delta_{k0}\delta_{lm})] Z_l \overleftrightarrow{\partial}_\mu \overline{Z}_m, \tag{5.1}$$

where g is a constant chosen to yield the correct normalization for the vector part of $J_\mu^{(k)}$, and

$$A\overleftrightarrow{\partial}_\mu B \equiv A(\partial_\mu B) - (\partial_\mu A)B. \tag{5.2}$$

Using the analysis of Sec. II [see Eqs. (2.69)–(2.74)], we can expand the current in powers of the meson field. To third order we obtain

$$J_\mu^{(k)} = \tfrac{1}{2}[if_{mlk}\pi_l \overleftrightarrow{\partial}_\mu \pi_m + 2iF_0\partial_\mu \pi_k - iX\partial_\mu \pi_k(2F_0F_x + \tfrac{4}{3}) - i\pi_k\partial_\mu X(2F_0F_x) - id_{mlk}\Pi_l \overleftrightarrow{\partial}_\mu \pi_m], \tag{5.3}$$

where the first term belongs to the vector current, and the remaining ones to the axial-vector current. From the linear term we see that K-meson decay and pion decay are described by a single constant, namely, F_0:

$$f_K = f_\pi = F_0 \approx 95 \text{ MeV}, \tag{5.4}$$

and from the cubic term we see that F_x is determined by the form factors of K_{l4} decay.

Because there is only one vector term in $J_\mu^{(k)}$ [Eq. (5.3)], the K_{l3} decay modes are described by a single form factor which corresponds to f_+ in the standard notation. In fact we obtain from Eq. (5.3) the result that

$$f_+ = 1, \quad f_- = 0. \tag{5.5}$$

Now it is known from experiment[13] that f_- is not a small quantity, and so, if we are to accommodate this result within the framework of chiral SU(3), we must find an additional $\Delta S = 1$ current somewhere.

If we limit ourselves to the (8, 1) representation, the only possibility for another current is given by Eq. (5.1) with $\partial_\mu(Z_l\bar{Z}_m)$ replacing $Z_l\overleftrightarrow{\partial}_\mu\bar{Z}_m$. The power series for this modified current begins with a term of degree three in the meson field, and so it cannot contribute to K_{l3} decay in the tree approximation. Therefore we must introduce $\Delta S=1$ currents belonging to representations other than the (8, 1). This can be done in many ways, and we shall not attempt to pick a particular one here.

The need to go outside the (8,1) may seem rather surprising when compared with the SU(2) analysis. In chiral SU(2) we had no trouble constructing two independent $(\frac{1}{2},0)$ currents corresponding to the two form factors of K_{l3} decay.[1] However, we must realize that, from the viewpoint of SU(3), K_{l3} decay is a transition between two members of the basic meson multiplet. The analogous transition in SU(2) is not K_{l3} decay itself, but rather pion β-decay $\pi^+\to\pi^0+e^++\nu$.

In chiral SU(2), we can construct only one current of the type (1, 0), and so we have only one form factor for pion β decay instead of the two that are allowed by kinematics. The reason for this is, of course, conservation of the vector current—a feature which is automatically built into chiral-symmetry theories, and which makes it difficult to construct extra currents. Thus our difficulties with K_{l3} in SU(3)×SU(3) come from having an octet of conserved vector currents.

The breaking of this conservation law by the additional $\Delta S=1$ current has some interesting consequences for the Lagrangian. We remarked before that the current $J_\mu^{(k)}$ of Eq. (5.1) can be derived from the Lagrangian of the previous section; in fact it comes solely from the kinetic term $\mathcal{L}_{\text{K.E.}}$. Now, if we are to derive the additional current from the Lagrangian, then we must add extra terms to $\mathcal{L}_{\text{K.E.}}$. These terms cannot be chiral invariant, and so our hypothesis of a symmetric $\mathcal{L}_{\text{K.E.}}$ in Sec. IV may be subject to some doubt.[14]

To conclude this section, we wish to draw attention to a curious feature of the current in Eq. (5.1). In addition to the octet of (8, 1) currents, it also gives rise to a ninth, chirally invariant axial-vector current:

$$A_\mu^{(0)}=\sum_{l=0}^{8}g(\tfrac{2}{3})^{1/2}(z_l\overleftrightarrow{\partial}_\mu z_l). \tag{5.6}$$

When we come to expand $A_\mu^{(0)}$ in powers of the meson field, we expect, on grounds of parity and SU(3) invariance, that the leading term will be $\partial_\mu Y$. However, when we actually substitute the expansions of Eqs. (2.67) – (2.74) in Eq. (5.6), we find that the cubic term cancels out, and that the lowest term is of fifth degree in the meson field:

$$A_\mu^{(0)}=\frac{i\sqrt{2}}{36\sqrt{3}\,F_0^{\,3}}\,(2X\partial_\mu Y-Y\partial_\mu X)+\cdots. \tag{5.7}$$

This suggests that some selection rule is at work, but we are not sure what it is.

VI. SUMMARY

We have constructed the analog for SU(3) of the nonlinear σ model of Gell-Mann and Lévy.[2] That is, given the most general action of chiral operators upon the octet of pseudoscalar meson fields[6] [see Eqs. (1.5) and (2.1)], we have been able to determine the elements, Z_α and $\bar{Z}_\beta$, of the $(3,\bar{3})$ and $(\bar{3},3)$ representations. Since we have not specified the commutation tensor $F_{ab}(\pi)$ any more than is necessary, our results are valid for all definitions of the meson field. Consequently they must be covariant with respect to redefinitions of the meson field. The formal proof of this is given in the Appendix.

From the Z_α and $\bar{Z}_\beta$ we have constructed higher representations of SU(3)×SU(3) and we have used them to construct a model for meson-meson scattering. In this model, all symmetry breaking occurs in the mass term and is made to fit the observed masses of pion and K meson. Our results show that if the S-wave, π-π scattering lengths are such that a_0 is always positive and much larger than a_2, then the symmetry breaking cannot occur in representations like (8, 8), (27, 27),..., or $\{(6,\bar{6})+(\bar{6},6)\}$, $\{(10,\overline{10})+(\overline{10},10)\}$, ..., . There are, however, many other representations consistent with our requirements, and the choice between them will depend upon the properties of K-K and π-K scattering.

We have also studied the form of weak currents, and have found that it is not possible to allow for the form factor f_- of K_{l3} decay without breaking the chiral symmetry in the kinetic term of the Lagrangian. The consequences of this have yet to be explored.

The problems we have considered so far involve only the transformation properties of the meson field itself. In a subsequent paper we shall consider the transformation of other fields, with application to such problems as π-nucleon scattering.

APPENDIX: REDEFINITION OF THE MESON FIELD

Suppose that we redefine the meson field by means of the expression

$$\tilde{\pi}_i=\lambda\pi_i+\mu\Pi_i\,, \tag{A1}$$

where λ and μ are, in general, functions of X and Y. We must then determine how the other quantities in the theory, especially $F_{ab}(\pi)$, behave.

The dual vector presents no problem:

$$\tilde{\Pi}_i = d_{ijk}\,\tilde{\pi}_j\,\tilde{\pi}_k = \tfrac{2}{3}\,\mu(\lambda X + \mu Y)\pi_i + (\lambda^2 - \tfrac{1}{3}X\mu^2)\Pi_i\,. \tag{A2}$$

For convenience we combine this with Eq. (A1) into a matrix expression

$$\begin{bmatrix}\tilde{\pi}_i\\ \tilde{\Pi}_i\end{bmatrix} = R\begin{bmatrix}\pi_i\\ \Pi_i\end{bmatrix},\quad R = \begin{bmatrix}\lambda & \mu\\ \tfrac{2}{3}\mu(\lambda X + \mu Y) & \lambda^2 - \tfrac{1}{3}X\mu^2\end{bmatrix}. \tag{A3}$$

To invert this relation, we define the determinant of R

$$\det R \equiv \Delta \equiv \lambda^3 - \lambda\mu^2 X - \tfrac{2}{3}Y\mu^3 \tag{A4}$$

and an associated matrix

$$\overline{R} = -\mathcal{C}R\mathcal{C} = \begin{bmatrix}\lambda^2 - \tfrac{1}{3}X\mu^2 & \tfrac{2}{3}\mu(\lambda X + \mu Y)\\ -\mu & \lambda\end{bmatrix}, \tag{A5}$$

where $\mathcal{C}$ is given in Eq. (2.14). We then find that

$$(\overline{R})^{\mathrm{Tr}}R = R^{\mathrm{Tr}}\overline{R} = I\Delta, \tag{A6}$$

where I is the unit, 2×2 matrix, and hence that

$$\begin{bmatrix}\pi_i\\ \Pi_i\end{bmatrix} = \frac{1}{\Delta}(\overline{R})^{\mathrm{Tr}}\begin{bmatrix}\tilde{\pi}_i\\ \tilde{\Pi}_i\end{bmatrix}. \tag{A7}$$

To determine how $F_{ab}(\pi)$ transforms, we evaluate the commutator $[K_a, \tilde{\pi}_b] = \tilde{F}_{ab}$ in terms of the original fields using Eqs. (A1), (1.5), and (2.1). We then use Eq. (A7) to recast the expression in terms of the new fields, and we identify $\tilde{F}$, $\tilde{B}$, and $\tilde{C}$ by their roles as coefficients of δ_{ab}, $d_{abc}\tilde{\pi}_c$, and $d_{abc}\tilde{\Pi}_c$, respectively. In this way we find that

$$\begin{bmatrix}\tfrac{1}{3}(\tilde{B}\tilde{X} + \tilde{C}\tilde{Y})\\ -\tilde{F}\end{bmatrix} = \overline{R}\begin{bmatrix}\tfrac{1}{3}(BX + CY)\\ -F\end{bmatrix}, \tag{A8}$$

$$\begin{bmatrix}\tilde{B} & 2\tilde{F} - \tfrac{1}{3}\tilde{X}\tilde{C}\\ \tilde{C} & -\tilde{B}\end{bmatrix} = \frac{1}{\Delta}\overline{R}\begin{bmatrix}B & 2F - \tfrac{1}{3}XC\\ C & -B\end{bmatrix}R^{\mathrm{Tr}}, \tag{A9}$$

$$\begin{bmatrix}\tfrac{2}{3}\tilde{b}(\tilde{a}\tilde{X} + \tilde{b}\tilde{Y})\\ \tilde{a}^2 - \tfrac{1}{3}\tilde{X}\tilde{b}^2\end{bmatrix} = \frac{1}{\Delta}\overline{R}\begin{bmatrix}\tfrac{2}{3}b(aX + bY)\\ a^2 - \tfrac{1}{3}Xb^2\end{bmatrix}, \tag{A10}$$

$$\begin{pmatrix}a\\ b\end{pmatrix} \equiv \begin{bmatrix}\tfrac{1}{3}(BX + CY)\\ -F\end{bmatrix}.$$

These results are sufficient to show that our expressions for Z_α and $\overline{Z}_\beta$ in Sec. II are covariant with respect to redefinitions of the meson field.

It is helpful in the analysis of Eqs. (2.27)–(2.29) to note that

$$\begin{aligned}\tilde{F} &= \lambda F + \tfrac{1}{3}\mu(BX + CY),\\ \tilde{C} &= [C\lambda^2 - 2\lambda\mu B - (2F - \tfrac{1}{3}XC)\mu^2]/\Delta\,.\end{aligned} \tag{A11}$$

Also, if we take the determinant of both sides of Eq. (A9), we find that

$$\tilde{B}^2 + 2\tilde{F}\tilde{C} - \tfrac{1}{3}\tilde{X}\tilde{C}^2 = B^2 + 2FC - \tfrac{1}{3}XC^2 = -1. \tag{A12}$$

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