[14]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), pp. 377-390.

[15]The calculation of Appendix A shows that $C_{\lambda\tau\eta} = (\alpha_0/4\pi)(\gamma_\eta\gamma_\tau\gamma_\lambda - \gamma_\lambda\gamma_\tau\gamma_\eta)$, but we will not need the explicit form for the argument which follows.

[16]This cancellation was first noted by S. Sen, Ref. 13.

[17]To the order in the fine-structure constant to which we are working, it makes no difference whether we use m_0 or m in Eq. (52).

[18]In rewriting the expression for I_2 in the final line of Eq. (56), we have used, in addition to Eq. (54) and the usual $r \rightarrow -r$ and reverse-ordering operations, the fact that I_2^m is a pseudotensor function of one four-vector q. This means that I_2^m has the form $\epsilon_{\mu\nu\sigma\tau}q^\sigma f(q^2/m^2)$, and therefore is odd under interchange of the indices μ and ν.

[19]Only the z integral enters into this distinction because the order in which the other Feynman-parameter integrals are done turns out to be immaterial.

[20]J. D. Bjorken and S. D. Drell, Ref. 14, pp. 220-223; R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966), pp. 31-33.

[21]The reason we expect the left-hand side of Eq. (61) to still be a polynomial is that taking absorptive parts renders the integrations in Eqs. (52) and (61) more convergent, and therefore should give the same result for both orders of integration. This may explain why integration-order problems do not seem to be present in the usual radiative correction calculations of quantum electrodynamics (QED): In QED the polynomial parts of the amplitudes are always readjusted at the end of a calculation to satisfy the renormalization conditions. Clearly, the question as to whether integration-order problems appear elsewhere deserves further investigation. If they do, they could be handled by the techniques which we have developed. We note that the integrals appearing in the earlier anomaly calculations of Adler and Bardeen (Ref. 5) and of Young *et al.* (Ref. 10) do *not* exhibit integration-order dependence.

[22]R. Arnowitt and S. Deser, Phys. Rev. 138, B712 (1965), Appendix B.

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Radius of Convergence for Perturbation Expansions in the SU_3 σ Model*

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The dependence of the tree-approximation solution of the SU_3 σ model for mesons on the $SU_3 \times SU_3$ symmetry-breaking parameters has been investigated in detail. From the explicit relations of the model we establish the existence of a radius of convergence of power series expansions about the symmetry limit. This radius of convergence is an order of magnitude smaller than the value determined by fitting the model to the pseudoscalar nonet masses. The mixing angles and scalar masses implied by the model are in surprisingly good agreement with (fragmentary) experimental data. Using the experimentally determined parameters, it is possible to compare the behavior of the several possible solutions in the limit of $SU_3 \times SU_3$ symmetry. Of these, only that having SU_3 symmetry of states and a pseudoscalar octet of massless Goldstone bosons is stable under symmetry-breaking perturbations. This result provides a dynamical reason for this frequently used *assumption*. The symmetry-breaking parameters turn out to be such that the Lagrangian has approximate $SU_2 \times SU_2$ symmetry. The physical point can be reached by first turning on either the SU_3-invariant or the $SU_2 \times SU_2$-invariant $SU_3 \times SU_3$ symmetry-breaking operators to their physical values and then using perturbation theory. In each case the perturbation expansion converges.

I. INTRODUCTION

The success of current algebras[1,2] has established the importance of the group $SU_3 \times SU_3$ for particle physics. Recently attempts have been made to understand the low-lying states in terms of effective Lagrangians with simple transformation properties under this group.[3] The question of how the underlying chiral symmetry is realized in light of the large mass splittings has been the subject of many investigations.[3-6] Difficulties arise when corrections to the symmetric limit are computed in perturbation theory.[6,7] These difficulties may be traced to the way in which chiral symmetry manifests itself by the spontaneous-breakdown mechanism and the associated occurrence of an octet of massless pseudoscalar mesons. Li and Pagels have recently shown[7] that perturbative closed loops involving massless bosons give rise to nonanalytic (logarithmic) behavior at the origin in symmetry-breaking parameters. In a recent letter[8] we suggested that the failure of perturbation expansions about the $SU_3 \times SU_3$ limit can be understood in the SU_3 σ model[9] without considering closed loops. Singularities were found in the symmetry-breaking parameters that rule out the use of perturbation expansions when the parameters are taken to their physical values. In this paper we give the details of the calculation reported in Ref. 8. We also go on to show that the stable solution in the $SU_3 \times SU_3$-symmetric limit is uniquely that spontaneously broken solution exhibiting SU_3 symmetry.

It may be questionable whether the SU_3 σ model should be taken seriously in describing the real world. Although it can be solved only in a very crude approximation, that crude solution gives surprisingly good predictions. Further, the allowed domains of symmetry-breaking parameters found by Okubo and Mathur[10] from very general considerations are contained in this model.[11] Our conclusions about the failure of perturbation expansions could very well be general since the troublesome singularity has a simple interpretation in terms of neighboring symmetric solutions.

We choose to examine a particularly simple form of the SU_3 σ model given by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\,\mathrm{Tr}\,\partial_\mu \mathfrak{M}^\dagger \partial^\mu \mathfrak{M} + f_1(\mathrm{Tr}\,\mathfrak{M}^\dagger \mathfrak{M})^2 + f_2\,\mathrm{Tr}\,\mathfrak{M}^\dagger \mathfrak{M}\mathfrak{M}^\dagger \mathfrak{M} + g(\det \mathfrak{M} + \mathrm{H.c.}) - \epsilon_0\sigma_0 - \epsilon_8\sigma_8 \,.$$

Here $\mathfrak{M}$ is a 3×3 matrix[12,13] transforming as $(3, \overline{3})$. In the limit $\epsilon_0 \to 0$ and $\epsilon_8 \to 0$ $\mathcal{L}$ has $SU_3 \times SU_3$ symmetry.[14] Note that there are no bare-mass terms. (The effect of the mass term was considered in Ref. 8.)

The basic approximation used to solve this model is the semiclassical or "Hartree" method.[15] The ground state of the Lagrangian is determined self-consistently where the fields are allowed to have nonzero vacuum expectation values. Small oscillations about these equilibrium values determine the masses.

The cubic term in $\mathcal{L}$, $g(\det \mathfrak{M} + \mathrm{H.c.})$, is essential for describing the pseudoscalar nonet. In the absence of this term the η' and π are degenerate.[3,11] Its contribution to the masses is interesting in that it pushes the corresponding scalar and pseudoscalar masses in opposite directions. It also has an interesting effect on the normal vacuum when ϵ_0 and ϵ_8 are small. (The "normal" vacuum is the one in which the vacuum expectations of the fields vanish in the limit of $\epsilon_8, \epsilon_0 \to 0$.) For small ϵ_0 and ϵ_8, the g term dominates the f_i terms and hence at least one $(\mathrm{mass})^2$ is negative. This means that although the normal vacuum is acceptable in the absence of the breaking terms, it is unstable under an infinitesimal breaking of the symmetry.

In Sec. II we calculate the masses in this model. Four masses are needed to fix all determinable parameters. (We choose them to be π, K, η, and η'.) The parameters of the $SU_3 \times SU_3$-symmetric Lagrangian are then held fixed at these values. In Sec. III we find the solutions to the Lagrangian in the $SU_3 \times SU_3$ limit. There are five possibilities, characterized by different vacuum expectation values of the field. Three of the possible solutions are ruled out from general considerations. In Sec. IV the problem of matching the physical solution to the symmetric limit is discussed. Only one solution remains that can be matched. Hence there is a unique symmetry limit in this model. It corresponds to a spontaneous breakdown of $SU_3 \times SU_3$ while preserving SU_3 and having an octet of massless pseudoscalar mesons. It has been conventional to assume that the limit of $SU_3 \times SU_3$ invariance has this property. To our knowledge this is the first dynamical explanation of why this solution is to be preferred over other possibilities.

II. MASSES AND NUMERICAL FIT

In order to find masses in this model we use the semiclassical approximation, as in our earlier papers.[8,11] We start from the Lagrangian (1.1) with no mass terms, allow for the fields to have nonzero equilibrium values (vacuum expectation values), redefine new fields that have zero equilibrium values and consider small oscillations of these fields. The only fields that get displaced are σ_0 and σ_8. Define the new fields σ_0' and σ_8' to be

$$\sigma_0' = \sigma_0 - \xi_0 \,, \qquad \sigma_8' = \sigma_8 - \xi_8 \,, \tag{2.1}$$

where $\xi_0 \equiv \langle \sigma_0 \rangle$ and $\xi_8 \equiv \langle \sigma_8 \rangle$. In terms of the new fields the Lagrangian contains powers of the fields from 1 to 4. The linear terms are eliminated by choosing ξ_0 and ξ_8 appropriately and then the masses are identified as coefficients of the bilinear terms.

Eliminating the linear terms σ_0' and σ_8' gives rise to equations for ϵ_0 and ϵ_8 in terms of ξ_0 and ξ_8. We define the variable $b = \xi_8/\sqrt{2}\,\xi_0$ and find these extremal conditions to be[16]

$$\epsilon_0 = \xi_0^{\,2}[\tfrac{4}{3}\xi_0 G(b) + \gamma(1-b^2)] \,, \qquad \epsilon_8/\sqrt{2} = \xi_0^{\,2} b[4\xi_0 H(b) - \gamma(1+b)] \,, \tag{2.2}$$

where

$$G(b) = 3f_1(1+2b^2) + f_2(1+6b^2-2b^3) \,, \qquad H(b) = f_1(1+2b^2) + f_2(1-b+b^2) \,, \qquad \gamma = 2g/\sqrt{3} \,. \tag{2.3}$$

We further define a variable a as in Ref. 10:

$$a = \epsilon_8/\sqrt{2}\,\epsilon_0 \,.$$

TABLE I. Mass terms in the Lagrangian. For unmixed fields the coefficient in the Lagrangian is $-\frac{1}{2}m^2\phi^2$, for mixed fields $-\frac{1}{2}(m_{00}^2\phi_0^2+m_{88}^2\phi_8^2+2m_{08}^2\phi_0\phi_8)$. The m^2 values listed in the first column are related to the A_i by the formula $m^2=-2\xi_0^2f_1A_1-2\xi_0^2f_2A_2-\gamma\xi_0A_3$.

m^2	A_1	A_2	A_3
π	$2(1+2b^2)$	$\frac{2}{3}(b+1)^2$	$(1-2b)$
K	$2(1+2b^2)$	$\frac{2}{3}(7b^2-b+1)$	$(1+b)$
η_{00}	$2(1+2b^2)$	$\frac{2}{3}(1+2b^2)$	-2
η_{88}	$2(1+2b^2)$	$\frac{2}{3}(3b^2-2b+1)$	$(1+2b)$
η_{08}	0	$\frac{2}{3}\sqrt{2}\,b(b-2)$	$\sqrt{2}\,b$
π_N	$2(1+2b^2)$	$2(b+1)^2$	$-(1-2b)$
κ	$2(1+2b^2)$	$2(b^2-b+1)$	$-(1+b)$
σ_{00}	$2(3+2b^2)$	$2(1+2b^2)$	2
σ_{88}	$2(1+6b^2)$	$2(3b^2-2b+1)$	$-(1+2b)$
σ_{08}	$4b\sqrt{2}$	$2\sqrt{2}\,b(2-b)$	$-\sqrt{2}\,b$

The masses are given by somewhat lengthy expressions and are recorded in Table I. For the unmixed fields the mass terms are of the form $-\frac{1}{2}m^2\phi^2$. For the mixed fields[17] we list the matrix elements of the quadratic form

$$-\tfrac{1}{2}(m_{00}^2\phi_0^2+m_{88}^2\phi_8^2+2m_{08}^2\phi_0\phi_8)\,.$$

The masses depend on four independent quantities. They can be taken to be $\xi_0^2f_1$, $\xi_0^2f_2$, $\xi_0\gamma$, and b. We fit these parameters to the pseudoscalar masses π, K, η, and η'. Two solutions were found. The predictions for both solutions are listed in Table II.

According to current wisdom, solution I is to be preferred since for solution II the κ and ϵ states occur at too low a mass. [A few years ago, when $\kappa(725)$ and $\sigma(400)$ were popular, we would clearly have made the opposite choice.] The π_N is somewhat lower than the value 980 MeV preferred by the experiment. κ and ϵ have been very elusive, though claimed to exist by pole extrapolators.[18] Again the predictions are slightly low compared to the mass values preferred by experimentalists. (However, some choose to identify κ with activity in the $K\pi$ mass range of 1100–1200 MeV.) The meson ϵ' has mass 1094 MeV in our model; its near-canonical mixing would give dominant $K\bar{K}$ decays. There is a good candidate for such a state at 1070 MeV: the η_0^+ or S^*, which does indeed have a predominant $K\bar{K}$ decay mode.[19] Considering the small number of parameters occurring in the model, and the probably crude nature of the approximate solution, the numerical results are surprisingly good. Further facets of the possible scalar nonet have been considered in Ref. 20.

The values of ϵ_8 and ϵ_0 give for the parameter $a=\epsilon_8/\sqrt{2}\,\epsilon_0$ the values -0.919 and -0.938 for solutions I and II. This is quite close to the value -0.89 given in Ref. 4. (The quantity c of that work is related to a by $c=\sqrt{2}\,a$.)

Solution I has some additional interesting results already noted in Ref. 8. The $\eta\eta'$ mixing angle is 2.4° and is smaller than that computed using the usual lowest-order mixing scheme. The SU_3 breaking in this model is not linear in ϵ_8 and so the higher-order corrections appear to make the η more purely octet than usually expected. The scalar mesons exhibit an unexpected correspondence with the quark model. The mixing angle θ_S corresponds to the linear combinations:

$$\begin{aligned}-\epsilon&=0.79\,\sigma_0'+0.61\,\sigma_8'\approx(\tfrac{2}{3})^{1/2}\sigma_0'+(\tfrac{1}{3})^{1/2}\sigma_8'\,,\\ -\epsilon'&=0.61\,\sigma_0'-0.79\,\sigma_8'\approx(\tfrac{1}{3})^{1/2}\sigma_0'-(\tfrac{2}{3})^{1/2}\sigma_8'\,.\end{aligned}\qquad(2.4)$$

If the fields σ_0' and σ_8' are replaced by quark scalar densities u_0 and u_8, the ϵ corresponds to the non-strange-quark combination while ϵ' corresponds to strange quarks only. Solution I also has a smaller absolute value of b, -0.2102 compared to -0.3610 for solution II, indicating that the states are more nearly SU_3 symmetric. For these reasons we feel that solution I is the preferred solution, and we will use it in the discussion below.

For the remainder of this paper we hold f_1, f_2,

TABLE II. Numerical fit to masses, and predicted mixing angles and scalar masses. θ_P and θ_S are the mixing angles for the $\eta\eta'$ system and $\epsilon\epsilon'$ system, respectively, defined in Ref. 17. ξ_{0P} is the physical value of the variable ξ_0. Only Solution I was used for the analysis in this paper. The input masses were (units of GeV^2) $m^2(\pi)=0.019\,06$, $m^2(K)=0.2450$, $m^2(\eta)=0.3003$, $m^2(\eta')=0.9178$. The unit is taken to be 1 GeV.

	Solution I	Solution II
$\tan\theta_P$	0.040 66	0.2371
$m(\pi_N)$	0.9110	0.9049
$m(\kappa)$	0.9025	0.7174
$m(\epsilon')$	1.094	0.9989
$m(\epsilon)$	0.6035	0.4106
$\tan\theta_S$	−1.283	−0.9903
$\xi_{0P}^2f_1$	−0.075 75	−0.085 69
$\xi_{0P}^2f_2$	−0.056 92	0.011 44
$\xi_{0P}\gamma$	0.2522	0.2362
b	−0.2102	−0.3610
ϵ_0/ξ_{0P}	−0.1861	−0.1974
$\epsilon_8/\sqrt{2}\,\xi_{0P}$	0.1710	0.1852
a	−0.9189	−0.9382

and γ fixed at the values determined by this solution. In a related paper[11] we needed the freedom to vary f_2/f_1 as we turned off the symmetry breaking, in order to get positive masses in the joint limit of scale and chiral invariance. We do not need that freedom here. Varying the parameters of the $SU_3\times SU_3$-invariant Lagrangian we regard as an unmotivated complication for the analysis of the present paper.

It is possible to express certain masses as simple functions of a and b. Using Eq. (2.2) we find

$$m_\pi^2=-\frac{\epsilon_0}{\xi_0}\,\frac{(a+1)}{(b+1)}\,,$$
$$m_K^2=-\frac{\epsilon_0}{\xi_0}\,\frac{(a-2)}{(b-2)}\,, \qquad (2.5)$$
$$m_\kappa^2=-\frac{\epsilon_0}{\xi_0}\left(\frac{a}{b}\right).$$

In terms of the variables a and b the domains of positive real mass are easily found and are shown in Fig. 1. There are further domain boundaries arising from the conditions that the other masses be positive, but they are complicated curves and do not particularly concern us. The boundaries in the absence of the term $g\det\mathfrak{M}+\text{H.c.}$ can be found in Ref. 11. The coordinates corresponding to the two fits are also marked on Fig. 1.

The $g\det\mathfrak{M}$ term is essential in fitting the masses. In fact, without this term the η' and π are degenerate. This term further has the property that it gives equal contributions with opposite signs to the corresponding scalars and pseudoscalars. The large value of g needed to split the (η', π) degeneracy then results in a large splitting of the scalars and pseudoscalars, which is in good agreement with experiment. This situation gives numerical detail to the qualitative arguments previously presented[21] in favor of such an interaction. The presence of this term has the interesting consequence of making the normal $SU_3\times SU_3$ solution unstable under small perturbations, and hence only spontaneously broken $SU_3\times SU_3$ is allowed, as we show in the following sections.

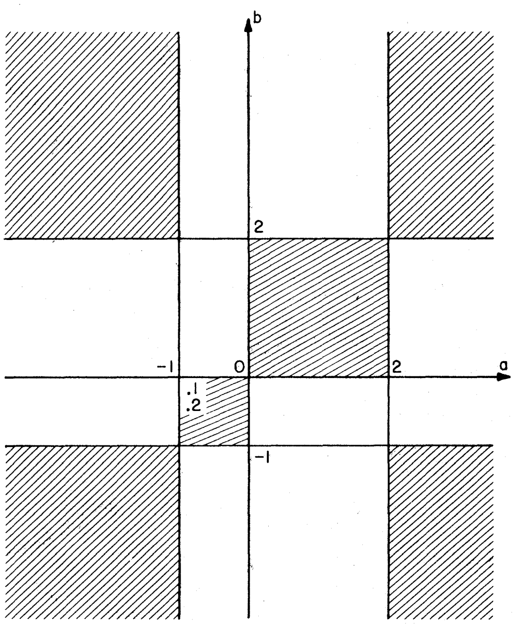

FIG. 1. Allowed domain plot in terms of the variables $a=\epsilon_8/\sqrt{2}\,\epsilon_0$ and $b=\xi_8/\sqrt{2}\,\xi_0$. The shaded areas correspond to positive π, K, and κ masses. The locations of the numerical solutions I and II are indicated. (The domains can be slightly sharpened by further stability conditions.)

III. SOLUTIONS IN THE ABSENCE OF EXPLICIT SYMMETRY BREAKING

The numerical fit in Sec. II contained no assumptions about the behavior of the model as a function of ϵ_0 and ϵ_8. In this section we find all solutions (five in number) in the symmetry limit $(\epsilon_8, \epsilon_0=0)$ in preparation for Sec. IV, where we learn how to make a smooth connection between the physical solution and a symmetric solution. In fact, it turns out that only one symmetric solution can be connected smoothly to the physical fit. This is a solution that has spontaneous breakdown of $SU_3\times SU_3$, but preserves SU_3 symmetry.

Let us examine Eq. (2.2) for $\epsilon_0, \epsilon_8=0$.

$$0=\xi_0^2[\tfrac{4}{3}\xi_0 G(b)+\gamma(1-b^2)]\,, \qquad (3.1a)$$
$$0=\xi_0^2 b[4\xi_0 H(b)-\gamma(1+b)]\,, \qquad (3.1b)$$

where G and H are given in Eq. (2.3). These equations have several solutions (see Table III for a summary):

(i) $\xi_0=0$, $|b|<\infty$. This is the normal solution in which the symmetry is realized by degeneracy. In

TABLE III. Allowed values of b and ξ_0 in the absence of explicit symmetry breaking $(\epsilon_0, \epsilon_8=0)$. Comments on solutions (i) and (iv) are discussed in Sec. IV; (iii) and (v) in Sec. III.

(i) $\lvert b\rvert<\infty$, $\xi_0=0$	Normal solution, unstable
(ii) $b=0$, $\xi_0=\dfrac{-3\gamma}{4(3f_1+f_2)}\equiv\hat{\xi}_0$	Good
(iii) $b=-1$, $\xi_0=0$	Normal solution, unstable
$f_1+f_2=0$	Disagrees with fit
(iv) $b=2$, $\xi_0=-\frac{1}{3}\hat{\xi}_0$	Acceptable, but too far from physical fit
(v) $b=\frac{1}{2}-\dfrac{3\gamma}{8\xi_0 f_2}$, ξ_0: Eq. (3.1b)	ξ_0 and b complex, unacceptable

fact, all masses vanish since there are no explicit mass terms in the Lagrangian.

(ii) $b=0,\ \xi_0\neq 0$. We can find ξ_0 from Eq. (3.1a):

$$\xi_0=\frac{-3\gamma}{4(3f_1+f_2)}\equiv\hat{\xi}_0. \tag{3.2}$$

This solution corresponds to the spontaneous breaking of $SU_3\times SU_3$, but preserves SU_3 symmetry ($b=0$). This defines $\hat{\xi}_0$. There are several cases for which $b\neq 0$ and $\xi_0\neq 0$. There are three solutions of this type. To find them take the difference of Eqs. (3.1a) and (3.1b):

$$(b-2)(b+1)[\tfrac{8}{3}(b-\tfrac{1}{2})\xi_0 f_2+\gamma]=0. \tag{3.3}$$

This equation together with either of Eqs. (3.1) gives the following results:

$$\text{(iii)}\quad b=-1,\ \ \xi_0(f_1+f_2)=0. \tag{3.4}$$

$\xi_0=0$ is the normal solution (already discussed) and $f_1+f_2=0$ conflicts with our experimental fit of Sec. II.

$$\text{(iv)}\quad b=2,\ \ 4\xi_0(3f_1+f_2)-\gamma=0. \tag{3.5}$$

This gives an expression for ξ_0.

$$\begin{aligned}\text{(v)}\quad & b=\tfrac{1}{2}-3/8\xi_0 f_2,\\ & 4(1+2b^2)\xi_0 f_1+4(b^2-b+1)\xi_0 f_2-(1+b)\gamma=0.\end{aligned} \tag{3.6}$$

If we take $f_2/f_1=0.74$ from the numerical fit, Eqs. (3.6) give complex b and ξ_0.

Table III summarizes these results. The only acceptable solution is the second [see Eq. (3.2)]. The reasons for rejecting (i) and (iv) will be discussed in Sec. IV. It is gratifying that there is only one acceptable solution and that it corresponds to that spontaneous breakdown of $SU_3\times SU_3$ invariance which preserves SU_3. In this limit the pseudoscalar octet appears as a set of massless Goldstone bosons.

IV. EXPLICIT BREAKING OF $SU_3\times SU_3$ SYMMETRY; ANALYTIC PROPERTIES IN THE PARAMETERS ϵ_0 AND ϵ_8

A. Introduction

Having found the physical values of the parameters in our model and the possible solutions in the $SU_3\times SU_3$ limit, the remaining task is to match the two. In order for the $SU_3\times SU_3$-symmetric Lagrangian to be a good starting point for describing physics, (1) there must be a path that corresponds to possible physical worlds in between. The criterion we use to satisfy this is that all (mass)^2 remain positive along the path. (2) We further demand that the masses be holomorphic functions of ϵ_0 and ϵ_8. If there are singularities in the path then perturbation theory is useless. We also are concerned with singularities near the path since they control the convergence of perturbation expansions.

As a preliminary step we must choose appropriate variables. The variables ξ_0 and ξ_8 (or equivalently ξ_0 and b) can be taken to describe the path of interest. However, the scale of ξ_0 is not determined in this model. We normalize ξ_0 to $\hat{\xi}_0$, its value at a symmetry point, $b=0,\ \xi_0\neq 0$ [see Eq. (3.2)]. We define the variable $x=\xi_0/\hat{\xi}_0$. The value ξ_{0P} (see Sec. II) refers to ξ_0 at the physical point. Using Eq. (3.2) and the numerical fit we find

$$\frac{1}{x_P}\equiv\frac{\hat{\xi}_0}{\xi_{0P}}=-\frac{3}{4}\frac{\gamma\xi_{0P}}{(3f_1+f_2)\xi_{0P}^{\,2}}=0.6656. \tag{4.1}$$

The variable b takes on the value -0.2102 at the physical point and 0 at the symmetry point.

B. Singularities in the Symmetry-Breaking Parameters ϵ_0 and ϵ_8

We will first show how to find the singularity structure of masses as a function of ϵ_0 and ϵ_8. The essential point of this discussion is explained in detail for the special case $\epsilon_8=0$ in Ref. 8. We give here a full discussion allowing $\epsilon_8\neq 0$.

The extremal conditions written in terms of x become

$$\begin{aligned}\bar{\epsilon}_0&\equiv\epsilon_0/\hat{\xi}_0\\ &=x^2[4(1+2b^2)xF_1+\tfrac{4}{3}(1+6b^2-2b^3)xF_2+(1-b^2)G],\\ \bar{\epsilon}_8/\sqrt{2}&\equiv\epsilon_8/\sqrt{2}\,\hat{\xi}_0\\ &=bx^2[4(1+2b^2)xF_1+4(b^2-b+1)xF_2-(1+b)G],\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}F_i&=\xi_{0P}^{\,2}f_i/x_P^{\,2},\\ G&=\xi_{0P}\gamma/x_P.\end{aligned} \tag{4.3}$$

The numbers F_i and G can be calculated from Table II and Eq. (4.1). Using Table I, the masses can be expressed in a similar fashion in terms of these variables. We give the π mass for purposes of illustration:

$$m_\pi^{\,2}=-x[4(1+2b^2)xF_1+\tfrac{4}{3}(b+1)^2xF_2+(1-2b)G]. \tag{4.4}$$

The masses are polynomials in x and b and are hence analytic in these variables. However, we are interested in the masses as functions of $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$, the parameters of the symmetry breaking. Equations (4.2) are thus a transformation of variables from x and b to $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$. The transformation is singular when the Jacobian vanishes:

$$J=\det\frac{\partial(\bar{\epsilon}_0,\bar{\epsilon}_8)}{\partial(x,b)}=0. \tag{4.5}$$

This expression is lengthy and we do not give it here. It is of the form

$$J = x^3[x - x_1(b)][x - x_2(b)]. \qquad (4.6)$$

The Jacobian vanishes at $x=0$ and at the real roots $x_1(b)$ and $x_2(b)$ shown in Fig. 2. Along these curves, x and b have branch points as a function of $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$. The functions x and b are clearly multivalued functions of $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$ since both spontaneously broken solutions marked on Fig. 2 have $\bar{\epsilon}_0, \bar{\epsilon}_8 = 0$.

The spontaneously broken solutions at $x=1$ and $b=0$ can be connected to the physical point by a path that satisfies our criterion of holomorphy. In Fig. 3 we show the mapping of this region onto the $(\bar{\epsilon}_0, \bar{\epsilon}_8)$ plane. The solid curve in Fig. 3 is the mapping of $x_1(b)$ in the neighborhood of the path. The dashed curve is also the mapping of $x_1(b)$ but it corresponds to large b. This singularity is not on the branch of the function that is exposed. This can be seen, for example, by holding $\bar{\epsilon}_0$ fixed and real, and looking in the $\bar{\epsilon}_8$ complex plane. There will be two square-root branch points given by the intersection of the $\bar{\epsilon}_0$ = constant line and the two curves. An examination of the complex plane shows that the branch point at the dashed curve lies on the second sheet of the branch point at the solid curve.

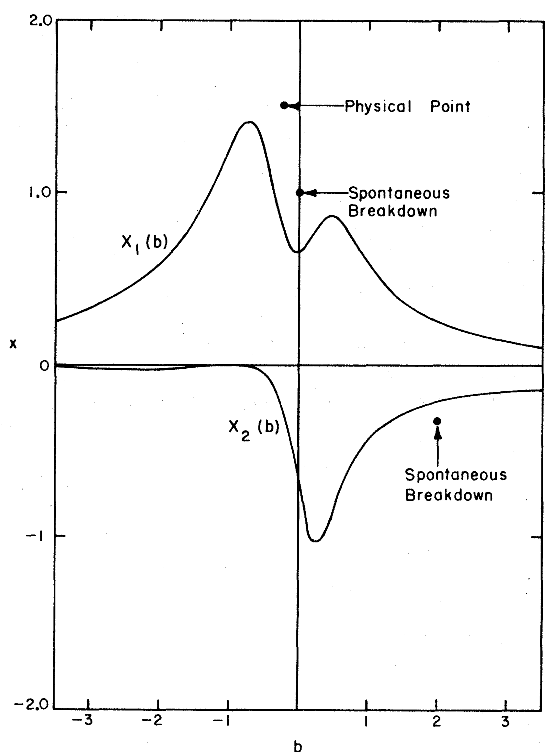

FIG. 2. $x_1(b)$ and $x_2(b)$ are the roots of the equation $\det[\partial(\bar{\epsilon}_0, \bar{\epsilon}_8)/\partial(x, b)] = 0$. In addition there is a third-order zero at $x=0$ independent of b.

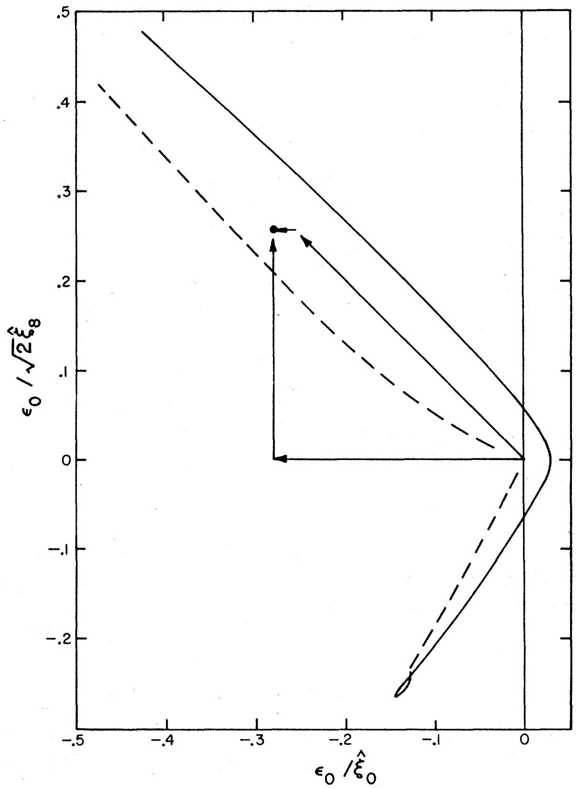

FIG. 3. This figure shows the curve of singularities in the (real) $(\bar{\epsilon}_0, \bar{\epsilon}_8)$ plane. The dashed part of the curve corresponds to second-sheet singularities. The two paths composed of straight-line segments show two important ways to reach the physical point (solid circle) from the origin (i.e., the $SU_3 \times SU_3$-symmetry point).

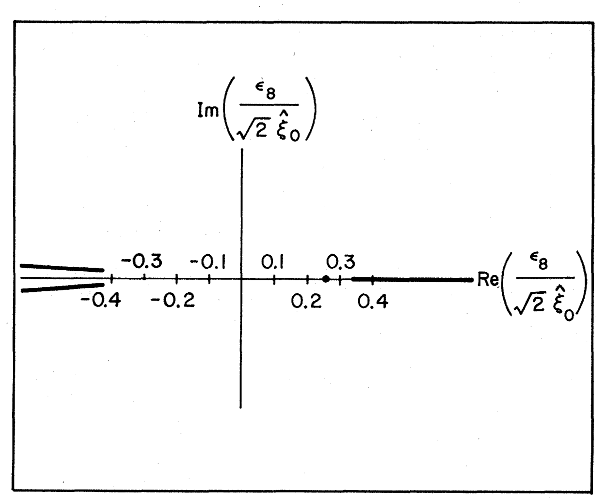

FIG. 4. Nearby branch points and associated cuts in the $\bar{\epsilon}_8$ complex plane for fixed (physical) value of $\bar{\epsilon}_0$. The origin corresponds to SU_3 symmetry. The physical point is denoted by a solid circle.

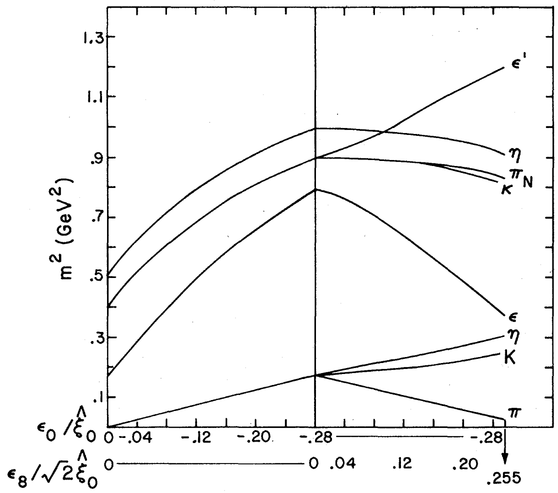

FIG. 5. Mass-level diagram as a function of symmetry-breaking parameters ϵ_0 and ϵ_8. The $SU_3 \times SU_3$-breaking parameter ϵ_0 is taken to its physical value first, keeping $b=0$ (conserving SU_3). Then the SU_3-breaking parameter ϵ_8 is taken to its physical value.

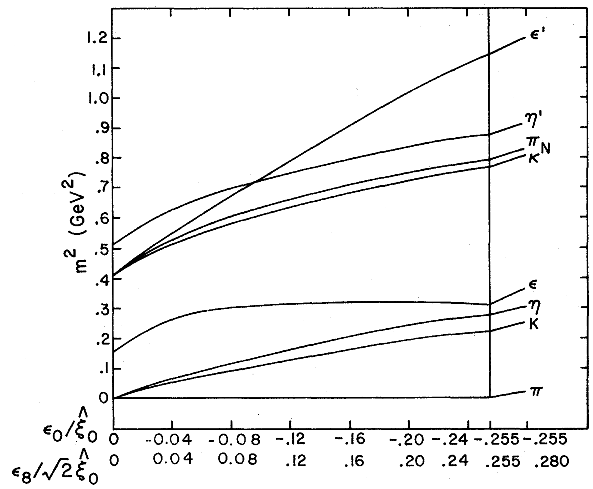

FIG. 6. Mass-level diagram as in Fig. 5 but with ϵ_0 and ϵ_8 constrained to their $SU_2 \times SU_2$ value for the first step; then $SU_2 \times SU_2$ is broken to SU_2.

The turnaround point of the singular curve in the third quadrant indicates that there is a complex-conjugate pair of branch points in the complex plane. These are found below for a particular choice of $\bar{\epsilon}_0$.

There are two paths joining the origin and point P which have special physical interest. (1) We first turn on ϵ_0 to its physical value, holding $\epsilon_8 = 0$, and then turn on ϵ_8. The first step has been discussed above. A power series will be limited by the singularity at (0.03, 0). The second step is a long distance on this scale, but the physical point lies within the radius of convergence. There are complex branch points nearby, as suggested by the lower part of the curve. All the nearby branch points and associated cuts are shown in Fig. 4. (2) In the second case we first turn on ϵ_0 and ϵ_8 together along the line $\epsilon_8 = -\sqrt{2}\,\epsilon_0$ (corresponding to $SU_2 \times SU_2$ symmetry) to the physical value of ϵ_8 and then take ϵ_0 to its physical value. The path corresponding to the first step has a singularity at (0.02, −0.02) and hence perturbation theory is useless. Having reached the value indicated by Fig. 3, one can safely reach the physical point by perturbation theory.

Our reasons for discarding the solution at $b = 2$, $x = -\frac{1}{3}$ are now clear from Fig. 2. In order to get from this point to the physical point we must pass through three points where the transformation is singular, i.e., $x = x_1$, x_2, and 0. In terms of the $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$ variables this means that we must encircle three branch points.

The normal solution also encounters difficulties. It corresponds to $x = 0$, $|b| < \infty$. The path to the physical point must still cross the curve $x = x_1(b)$.

C. Positivity of Masses, Stability of Solutions

The criterion that the symmetry solution be acceptable [i.e., that the path to the physical point be characterized by positive $(\text{mass})^2$] is now examined. Again the only symmetry-limit solution that satisfies this criterion is that having $b = 0$ and $x = 1$.

We first consider this solution. The results are shown in Figs. 5 and 6. We plot $(\text{mass})^2$ for the two paths indicated in Fig. 3. Figure 5 is the case in which we break $SU_3 \times SU_3$ explicitly while preserving SU_3 and then break SU_3 to SU_2. Figure 6 is the case in which $SU_2 \times SU_2$ is preserved in the initial breaking and then $SU_2 \times SU_2$ is broken to SU_2. In both cases the $(\text{mass})^2$ are all real and positive with the exception of the π which remains zero along the $SU_2 \times SU_2$ path.

In Sec. III we stated that the normal solution $\xi_0 = 0$, $|b| < \infty$ was unstable. For this solution the masses are all zero; however when ϵ_0 is turned on, at least one $(\text{mass})^2$ must go negative, which we now show.

When $\xi_0 \to 0$, i.e., $x \ll 1$, then the mass formulas are dominated by the term coming from $g \det \mathfrak{M}$. Four masses will illustrate our point:

$$m_\pi^{\,2} \approx -m_{\pi_N}^{\,2} \approx -x(1-2b)G\,,$$
$$m_K^{\,2} \approx -m_\kappa^{\,2} \approx -x(1+b)G\,. \tag{4.7}$$

There is no choice of b for which all four $(\text{mass})^2$ are positive.

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[13]$\mathfrak{M}$ is related to σ_i and ϕ_i by $\mathfrak{M}=\sum_{j=0}^{8}(\sigma_j+i\phi_j)\lambda_j/\sqrt{2}$, where λ_i are the usual SU_3 matrices.
[14]A discussion of the transformation properties of the various terms in $\mathcal{L}$ under $U_3\times U_3$ and dilations can be found in Ref. 11.
[15]This is discussed in many places – in particular Refs. 3, 6, and 11.
[16]This result is easily read from Table I of Ref. 11.
[17]The mixing angles are defined by

$$\eta=\phi_8\cos\theta_P-\phi_0\sin\theta_P\,,$$
$$\eta'=\phi_8\sin\theta_P+\phi_0\cos\theta_P\,.$$

We use the same convention for the scalars ϵ and ϵ' for the sake of uniformity.
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Matching Explicit and Spontaneous Scale-Invariance Breaking in Lagrangian Models*

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Effective Lagrangian models, constructed from fields transforming as $(3,\overline{3})+(\overline{3},3)$ and $(1,1)$ in $SU_3\times SU_3$, are studied in the tree approximation to learn how solutions having broken scale and chiral symmetries are related to underlying limit solutions exhibiting spontaneous breakdown of these symmetries. The requirement of a smooth transition to the limit of scale invariance leads to restrictive conditions on the structure of the model and its solutions. For a special case of the general model, it is possible to compute explicitly the squared masses of the single-particle excitations in terms of symmetry-breaking parameters. The condition of stability ($m^2\geq 0$) leads to the allowed domains of Okubo and Mathur, and hence, provides a physical interpretation to their result. It is noted that there are several ways to normalize the ninth axial-vector current by placing its divergence and the trace of the energy-momentum tensor in a representation of $U_1\times U_1$.

I. INTRODUCTION

The study of Lagrangian models has provided considerable insight into the nature of approximate chiral symmetry.[1] Much of this work has been concerned with "nonlinear realizations" since current-algebra soft-meson theorems are automatically satisfied, and the coordinates of heavy or dubious particles can be eliminated. A similar situation obtains in much recent work on models of scale invariance and its breaking.[2-4] For many purposes[5,6] it is useful to study ordinary linear realizations, not only because the particles in question might actually exist, but because the dynamics of the underlying spontaneous-breakdown mechanism is much clearer at this level. Further, the nonlinear realizations can be regarded[7] as certain limits of the usual linear theories.

The present paper is concerned with the phenomenon of spontaneous breakdown of scale invariance and the additional stringent requirement that such solutions change smoothly when scale-invariance-breaking operators are turned on. The calculations are carried out in the semiclassical (tree-graph) approximation, with canonical commutation relations assumed. (Modifications due to renormalization will occur in the next approximation to the self-consistent ground state used here.) The