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[6]Strictly speaking, the amplitude involved in Eq. (3) is that for the process $\pi+$"$\pi$"$\to\pi+\omega$, where "$\pi$" is a heavy pion of mass equal to $2m_p$. Clearly, we are faced with the problem of continuation in the pion mass from its physical value $m_\pi$ to the value $2m_p$. However, the process $\pi+$"$\pi$"$\to\pi+\omega$ has the same Regge trajectories as the physical process $\pi+\pi\to\pi+\omega$. We therefore assume that mass extrapolation will affect only the parameter $\bar\beta$ of Eq. (22) (see Ref. 3). This will be taken care of by the coefficient $b_1$ of $\tau_1$ in Eq. (20). Similarly, the amplitude involved in Eq. (6) is really that of the process $\pi+$"$\omega$"$\to\pi+\omega$, where "$\omega$" is a heavy $\omega^0$ meson of mass $2m_p$. Again, the Regge trajectories are the same as those in (24)–(27), though the coefficients in the Veneziano form are affected by mass extrapolation from the physical mass $m_\omega$ to the mass $2m_p$. Since there does not exist, to our knowledge, any satisfactory theory of estimating the effect of this mass extrapolation, we shall simply assume that the amplitude for $\pi+$"$\omega$"$\to\pi+\omega$ is proportional to that of $\pi+\omega\to\pi+\omega$, the constant of proportionality containing the off-mass-shell effects. This means that $m_\omega$ in Eq. (30) is the physical $\omega^0$ mass. The constant of proportionality is absorbed in the coefficient $b_3$ of $\tau_3$ in Eq. (20). Notice that with the above assumption, the factorization constraints, Eq. (28), remain intact.

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PHYSICAL REVIEW D VOLUME 4, NUMBER 1 1 JULY 1971

# $\pi^0$-$\eta^0$-$X^0$ Mixing in Broken Chiral Symmetry

Ranabir Dutt* and Shalom Eliezer
*Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel*
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The mixing of $\pi^0$, $\eta$, and $X(958)$ mesons has been investigated within the framework of the broken-chiral-symmetry model of Gell-Mann, Oakes, and Renner. Considering $SU(2)$ symmetry breaking of the Hamiltonian due to the nonelectromagnetic isospin-violation term (so-called $u_3$) and imposing the smoothness assumption on certain three-point vertex functions, we calculate the mixing angles and the symmetry-breaking parameters in terms of the known masses of the pseudoscalar mesons and one input variable, $F=f_K/f_\pi$. We obtain two sets of solutions (e.g., $\eta$-$X$ mixing angle $\omega\approx 10.5°$ or 19° for $F=1$) and predict negligible mixing between $\pi^0$-$X$ compared to $\pi^0$-$\eta$. The effect of the vacuum symmetry breaking on these mixing parameters has been studied as well. In the case of an $SU(3)$-invariant vacuum, we reproduce separately the known results about $\eta$-$X$ and $\pi^0$-$\eta$ mixings and obtain the corrections to the Gell-Mann–Okubo mass formulas.

## I. INTRODUCTION

In most previous works, the problem of the pseudoscalar-meson mixing has been treated separately for $\eta$-$X$ (Refs. 1–12) and $\eta$-$\pi$ (Refs. 13–16). In a recent paper[17] Oneda, Umezawa, and Matsuda have determined $\pi^0$-$\eta$-$X$ mixing parameters using the concept of asymptotic symmetry in broken $SU(3)$ and $SU(2)$ symmetries. Here we shall present a unified treatment of these mixings based on a broken $SU(3)\otimes SU(3)$ model, given by Gell-Mann, Oakes, and Renner (GOR).[18]

The importance of $SU(2)$ violation of the Hamiltonian due to a nonelectromagnetic isospin-breaking term ($u_3$) has been pointed out by several authors.[19–23] One of the reasons for such a consideration is the large failure of the Dashen sum rule (DSR)[24] $m_{K^+}{}^2-m_{K^0}{}^2=m_{\pi^+}{}^2-m_{\pi^0}{}^2$, which is derived from electromagnetic effects alone. Moreover, if one considers the $\pi^0$-$\eta$ mixing angle from the diagonalization of the mass matrix and uses this sum rule (DSR), one gets a vanishing contribution of the electromagnetic effect to this particular mixing. As a consequence, one may look for the contributions of the nonelectromagnetic effects to the whole phenomenon of pseudoscalar-meson mixing.

In this work, we start with a broken $SU(3)\otimes SU(3)$ Hamiltonian which contains $SU(3)$ and $SU(2)$ violations explicitly. In Sec. II, the model as well as the necessary notations and the conventions are explained. In Sec. III, the smoothness assumption is introduced to get sum rules for the renormalization constants. Other sets of sum rules, derived from our Hamiltonian, are given in Sec. IV. In Sec. V, we solve the equations for the most general case, i.e., a broken Hamiltonian and noninvariant vacuum. For all the mixing angles, two sets of solutions are obtained for different $F=f_K/f_\pi$ (see Table I). The large solutions for the $\eta$-$X$ mixing angle $\omega$ are effectively insensitive whereas the smaller ones vary appreciably with the variation of $F$. However, the $SU(2)$ mixing parameters ($\theta$ and $\phi$) are less dependent on $F$. Besides, a negligible mixing effect between $\pi^0$ and $X$ is predicted. Section VI contains a few illustrations of special cases with an $SU(3)$-symmetric vacuum and we discuss separately the mixing of $\eta$-$X$, $\pi^0$-$\eta$, and $\pi^0$-$X$. In these cases, the expressions for the mixing angles and the modified Gell-Mann–Okubo (GMO) mass sum rules, as given by other authors,[1,14,17] are reproduced. In the last section, we conclude with some remarks about our results.

## II. THE FORMALISM, NOTATIONS, AND DEFINITIONS

1. We define a set of scalar and pseudoscalar nonets $u_i(x)$ and $v_i(x)$ $(i=0,1,2,\ldots,8)$ which transform according to the $(3,\bar{3})\oplus(\bar{3},3)$ representation of the group $SU(3)\otimes SU(3)$, and choose the symmetry-breaking (SB) Hamiltonian as

$$H_{\rm SB}=\epsilon_0[u_0+\alpha u_8+\beta u_3]. \qquad (1)$$

The scalar and the pseudoscalar densities satisfy the equal-time commutation relations[25]

$$[F_i, u_j(x)]=if_{ijk}u_k(x), \qquad (2a)$$

$$[F_i, v_j(x)]=if_{ijk}v_k(x), \qquad (2b)$$

$$[F_i^5, u_j(x)]=id_{ijk}v_k(x), \qquad (2c)$$

$$[F_i^5, v_j(x)]=-id_{ijk}u_k(x), \qquad (2d)$$

where $Q_i=F_i\pm F_i^5$ are the generators of the $SU(3)\otimes SU(3)$ group and $i=1,2,\ldots,8$, $j,k=0,1,2,\ldots,8$.

Using the local generalization of the equation of motion for the axial-vector currents,

$$\partial_\mu A_i^\mu=-i[F_i^5, H_{\rm SB}], \qquad (3)$$

we obtain the following explicit form of the current divergences:

$$\partial_\mu A^\mu_{1\pm i2}=(\tfrac{1}{3})^{1/2}\epsilon_0(\sqrt{2}+\alpha)v_{1\pm i2}, \qquad (4a)$$

$$\partial_\mu A^\mu_{4\pm i5}=(\tfrac{1}{3})^{1/2}\epsilon_0[(\sqrt{2}-\tfrac{1}{2}\alpha)+\tfrac{1}{2}\sqrt{3}\,\beta]v_{4\pm i5}, \qquad (4b)$$

$$\partial_\mu A^\mu_{6\pm i7}=(\tfrac{1}{3})^{1/2}\epsilon_0[(\sqrt{2}-\tfrac{1}{2}\alpha)-\tfrac{1}{2}\sqrt{3}\,\beta]v_{6\pm i7}, \qquad (4c)$$

$$\partial_\mu A^\mu_3=(\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\beta v_0+\beta v_8+(\sqrt{2}+\alpha)v_3], \qquad (4d)$$

$$\partial_\mu A^\mu_8=(\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\alpha v_0+(\sqrt{2}-\alpha)v_8+\beta v_3]. \qquad (4e)$$

From Eqs. (4d) and (4e) it is evident that $\alpha$ gives the mixing of the singlet to the eighth component of the octet, whereas the mixing of the third to the eighth and the singlet components of the pseudoscalar densities is due to nonvanishing $\beta$.

2. We define here the matrix elements of the divergences of the axial currents, given in Eqs. (4), between vacuum and the appropriate physical pseudoscalar-meson states

$$\langle 0|\partial_\mu A^\mu_{1-i2}|\pi^+\rangle=f_\pi m_{\pi^+}{}^2, \qquad (5a)$$

$$\langle 0|\partial_\mu A^\mu_{4-i5}|K^+\rangle=f_{K^+}m_{K^+}{}^2, \qquad (5b)$$

$$\langle 0|\partial_\mu A^\mu_{6-i7}|K^0\rangle=f_{K^0}m_{K^0}{}^2, \qquad (5c)$$

$$\langle 0|\partial_\mu A^\mu_a|\pi^0\rangle=f_{\pi^0_a}m_{\pi^0}{}^2, \qquad (6a)$$

$$\langle 0|\partial_\mu A^\mu_a|\eta\rangle=f_{\eta_a}m_\eta{}^2, \qquad (6b)$$

$$\langle 0|\partial_\mu A^\mu_a|X\rangle=f_{X_a}m_X{}^2 \quad (a=3 \text{ or } 8). \qquad (6c)$$

Since the pseudoscalar densities $v_i$ are probably not physical fields, we also need to define the matrix elements

$$\langle 0|v_{1-i2}|\pi^+\rangle=Z_{\pi^+}{}^{1/2}, \qquad (7a)$$

$$\langle 0|v_{4-i5}|K^+\rangle=Z_{K^+}{}^{1/2}, \qquad (7b)$$

$$\langle 0|v_{6-i7}|K^0\rangle=Z_{K^0}{}^{1/2}, \qquad (7c)$$

$$\langle 0|v_b|\pi^0\rangle=Z_{\pi^0_b}{}^{1/2}, \qquad (8a)$$

$$\langle 0|v_b|\eta\rangle=Z_{\eta_b}{}^{1/2}, \qquad (8b)$$

$$\langle 0|v_b|X\rangle=Z_{X_b}{}^{1/2} \quad (b=0,8, \text{ or } 3), \qquad (8c)$$

where the $Z^{1/2}$'s are the wave-function renormalization factors.

3. As we are considering a Hamiltonian which breaks the $SU(3)$ and $SU(2)$ symmetries, mixing must appear among the physical fields of $\pi^0$, $\eta$, and $X$. Denoting an octet of Hermitian pseudoscalar fields by $P_i$ $(i=1,2,\ldots,8)$ and a singlet by $P_0$, the physical pseudoscalar meson states may be expressed as linear combinations[26]:

$$\pi^+=(\tfrac{1}{2})^{1/2}(P_1+iP_2), \qquad (9a)$$

$$K^+=(\tfrac{1}{2})^{1/2}(P_4+iP_5), \qquad (9b)$$

$$K^0=(\tfrac{1}{2})^{1/2}(P_6+iP_7), \qquad (9c)$$

$$\pi^0=\cos\theta\cos\phi\,P_3+(\sin\theta\cos\omega-\cos\theta\sin\phi\sin\omega)P_8+(\sin\theta\sin\omega+\cos\theta\sin\phi\cos\omega)P_0, \qquad (10a)$$

$$\eta=-\sin\theta\cos\phi\,P_3+(\cos\theta\cos\omega+\sin\theta\sin\phi\sin\omega)P_8+(\cos\theta\sin\omega-\sin\theta\sin\phi\cos\omega)P_0, \qquad (10b)$$

$$X=-\sin\phi\,P_3-\cos\theta\sin\omega\,P_8+\cos\phi\cos\omega\,P_0, \qquad (10c)$$

where $\omega$, $\theta$, and $\phi$ are the mixing angles between the pairs $(\eta,X)$, $(\pi^0,\eta)$, and $(\pi^0,X)$, respectively.

4. Because the symmetry can be broken by the vacuum as well as by the Hamiltonian, we shall assume that the vacuum not only transforms as a pure singlet under $SU(3)$ transformation, but also includes a mixture of the eighth and third components of an octet. Symbolically it can be represented as

$$|0\rangle \equiv |\text{vac}\rangle = C_1|\text{singlet}\rangle + C_2|8,\text{octet}\rangle + C_3|3,\text{octet}\rangle\,. \tag{11}$$

The matrix elements of Eqs. (7) and (8) may be expressed in terms of the $C$'s by using the Wigner-Eckart theorem:

$$\langle 0|v_i|P_j\rangle = (\tfrac{2}{3})^{1/2}\delta_{ij}C_1g_1 + C_2g_2d_{8ij} + C_3g_2d_{3ij} \quad (i,j=1,2,\ldots,8), \tag{12a}$$

$$\langle 0|v_i|P_0\rangle = (\tfrac{2}{3})^{1/2}g_3(C_2\delta_{i8}+C_3\delta_{i3}) \quad (i=1,2,\ldots,8), \tag{12b}$$

$$\langle 0|v_0|P_i\rangle = (\tfrac{2}{3})^{1/2}g_4(C_2\delta_{i8}+C_3\delta_{i3}) \quad (i=1,2,\ldots,8), \tag{12c}$$

$$\langle 0|v_0|P_0\rangle = (\tfrac{2}{3})^{1/2}C_1g_5, \tag{12d}$$

where the $g$'s are the reduced matrix elements

$$\begin{aligned}
&\langle 0,\text{sing}\|v_i\|P_j\rangle \equiv g_1,\\
&\langle 0,\text{oct}\|v_i\|P_j\rangle \equiv g_2,\\
&\langle 0,\text{oct}\|v_i\|P_0\rangle \equiv g_3,\\
&\langle 0,\text{oct}\|v_0\|P_i\rangle \equiv g_4,\\
&\langle 0,\text{sing}\|v_0\|P_0\rangle \equiv g_5 \quad (i,j=1,2,\ldots,8).
\end{aligned} \tag{13}$$

Combining Eqs. (7)–(10) and (12), we finally get

$$\begin{aligned}
Z_{\pi^+}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[2C_1g_1+\sqrt{2}\,C_2g_2],\\
Z_{K^+}{}^{1/2} &= (\tfrac{1}{6})^{1/2}[2\sqrt{2}\,C_1g_1 - C_2g_2 + \sqrt{3}\,C_3g_2],\\
Z_{K^0}{}^{1/2} &= (\tfrac{1}{6})^{1/2}[2\sqrt{2}\,C_1g_1 - C_2g_2 - \sqrt{3}\,C_3g_2]\,;
\end{aligned} \tag{14}$$

$$\begin{aligned}
Z_{\pi_3^0}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[\sqrt{2}\,C_1g_1 + C_2g_2 + C_3g_2(\theta\cos\omega - \phi\sin\omega) + \sqrt{2}\,C_3g_3(\theta\sin\omega+\phi\cos\omega)],\\
Z_{\pi_8^0}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[C_3g_2 + (\sqrt{2}\,C_1g_1 - C_2g_2)(\theta\cos\omega - \phi\sin\omega) + \sqrt{2}\,C_2g_3(\theta\sin\omega+\phi\cos\omega)],\\
Z_{\pi_0^0}{}^{1/2} &= (\tfrac{2}{3})^{1/2}[C_3g_4 + C_2g_4(\theta\cos\omega - \phi\sin\omega) + C_1g_5(\theta\sin\omega+\phi\cos\omega)];
\end{aligned} \tag{15}$$

$$\begin{aligned}
Z_{\eta_3}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[-(\sqrt{2}\,C_1g_1 + C_2g_2)\theta + C_3g_2\cos\omega + \sqrt{2}\,C_3g_3\sin\omega],\\
Z_{\eta_8}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[-C_3g_2\theta + (\sqrt{2}\,C_1g_1 - C_2g_2)\cos\omega + \sqrt{2}\,C_2g_3\sin\omega],\\
Z_{\eta_0}{}^{1/2} &= (\tfrac{2}{3})^{1/2}[-C_3g_4\theta + C_2g_4\cos\omega + C_1g_5\sin\omega];\ .
\end{aligned} \tag{16}$$

$$\begin{aligned}
Z_{X_3}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[-(\sqrt{2}\,C_1g_1 + C_2g_2)\phi - C_3g_2\sin\omega + \sqrt{2}\,C_3g_3\cos\omega],\\
Z_{X_8}{}^{1/2} &= (\tfrac{1}{3})^{1/2}[-C_3g_2\phi - (\sqrt{2}\,C_1g_1 - C_2g_2)\sin\omega + \sqrt{2}\,C_2g_3\cos\omega],\\
Z_{X_0}{}^{1/2} &= (\tfrac{2}{3})^{1/2}[-C_3g_4\phi - C_2g_4\sin\omega + C_1g_5\cos\omega].
\end{aligned} \tag{17}$$

Here we have made the approximations $\sin\phi \approx \phi$, $\cos\phi\approx 1$ and $\sin\theta\approx\theta$, $\cos\theta\approx 1$ assuming that $\theta$ and $\phi$ are small compared to $\omega$.

Finally, when the vacuum is $SU(3)$-noninvariant, we may have nonvanishing vacuum expectation values of the scalar densities $u_0$, $u_8$, and $u_3$. These expectation values may be denoted by

$$\begin{aligned}
\xi_0 &= \langle 0|u_0|0\rangle\,,\\
\xi_8 &= \langle 0|u_8|0\rangle\,,\\
\xi_3 &= \langle 0|u_3|0\rangle\,.
\end{aligned} \tag{18}$$

## III. SMOOTHNESS ASSUMPTION AND $Z$ RELATIONS

Here we shall derive relations among the $Z$'s, as defined in Sec. II, by assuming smoothness condition for certain vertex functions, considered by several authors.[27,28] In other words, this means that the vertex functions are as smooth functions of momenta as possible after the removal of the pole singularities. We proceed in two steps: (1) without mixing, (2) including mixing.

(1) As defined in Ref. 27, we also consider the three-point functions

$$\begin{aligned}
G_{ijk}(p_i^2,p_j^2,q^2) \equiv{}& (p_i^2-M_i^2)(p_j^2-M_j^2)(q^2-M_k^2)\\
&\times(Z_iZ_jZ_k)^{-1/2}\int d^4x\,d^4y\,e^{ip_i\cdot x - ip_j\cdot y}\\
&\times\langle 0|T\{v_i(x)v_j(y)u_k(0)\}|0\rangle\,,
\end{aligned} \tag{19}$$

$$\begin{aligned}
i(p_i+p_j)_\mu F^+_{ijl}(p_i^2,p_j^2,q^2) + iq_\mu F^-_{ijl}(p_i^2,p_j^2,q^2)
&= -(p_i^2-M_i^2)(p_j^2-M_j^2)(Z_iZ_j)^{-1/2}\\
&\quad\times\int d^4x\,d^4y\,e^{ip_i\cdot x-ip_j\cdot y}\langle 0|T\{v_i(x)v_j(y)V^l_\mu(0)\}|0\rangle\,,
\end{aligned} \tag{20}$$

where $q=p_i-p_j$ and $i,j$ are the pseudoscalar indices and $k$ is the scalar index.

When the particles are on the mass shells, $G_{ijk}$ is the physical coupling constant of two pseudoscalar fields and one scalar field, and the $F^\pm_{ijk}$'s are the physical form factors of the vector currents between two pseudoscalar states.

We use the general form of the divergence of the vector current,

$$\partial_\mu V_i^\mu = -\epsilon_0(\alpha f_{8ik}+\beta f_{3ik})u_k\,, \tag{21}$$

in Eq. (20), and follow the same procedure given by Auvil and Deshpande,[27] to get the relation

$$Z_i = Z_j \quad (i,j=1,2,4,5,6,7). \tag{22}$$

This equation implies

$$Z_{\pi^+} = Z_{K^+} = Z_{K^0}. \tag{23}$$

(2) In the case of mixing, we define appropriate interpolating fields for $\pi^0$, $\eta$, and $X$ as

$$\Phi_j = \sum_{a=0,8,3} A_j^a v_a \quad (j=\pi^0,\eta,\text{ or } X), \tag{24}$$

where the components of $A_j^a$ are given by

$$\begin{aligned}
A^0_{\pi^0} &= (Z_{\eta_8}{}^{1/2} Z_{X_3}{}^{1/2} - Z_{X_8}{}^{1/2} Z_{\eta_3}{}^{1/2})/D,\\
A^8_{\pi^0} &= (Z_{\eta_3}{}^{1/2} Z_{X_0}{}^{1/2} - Z_{X_3}{}^{1/2} Z_{\eta_0}{}^{1/2})/D,\\
A^3_{\pi^0} &= (Z_{\eta_0}{}^{1/2} Z_{X_8}{}^{1/2} - Z_{X_0}{}^{1/2} Z_{\eta_8}{}^{1/2})/D;
\end{aligned} \tag{25a}$$

$$\begin{aligned}
A^0_{\eta} &= (Z_{\pi^0_3}{}^{1/2} Z_{X_8}{}^{1/2} - Z_{X_3}{}^{1/2} Z_{\pi^0_8}{}^{1/2})/D,\\
A^8_{\eta} &= (Z_{\pi^0_0}{}^{1/2} Z_{X_3}{}^{1/2} - Z_{X_0}{}^{1/2} Z_{\pi^0_3}{}^{1/2})/D,\\
A^3_{\eta} &= (Z_{\pi^0_8}{}^{1/2} Z_{X_0}{}^{1/2} - Z_{X_8}{}^{1/2} Z_{\pi^0_0}{}^{1/2})/D;
\end{aligned} \tag{25b}$$

$$\begin{aligned}
A^0_{X} &= (Z_{\pi^0_8}{}^{1/2} Z_{\eta_3}{}^{1/2} - Z_{\eta_8}{}^{1/2} Z_{\pi^0_3}{}^{1/2})/D,\\
A^8_{X} &= (Z_{\pi^0_3}{}^{1/2} Z_{\eta_0}{}^{1/2} - Z_{\eta_3}{}^{1/2} Z_{\pi^0_0}{}^{1/2})/D,\\
A^3_{X} &= (Z_{\pi^0_0}{}^{1/2} Z_{\eta_8}{}^{1/2} - Z_{\eta_0}{}^{1/2} Z_{\pi^0_8}{}^{1/2})/D;
\end{aligned} \tag{25c}$$

$$D = \begin{vmatrix} Z_{\pi^0_0}{}^{1/2} & Z_{\pi^0_8}{}^{1/2} & Z_{\pi^0_3}{}^{1/2} \\ Z_{\eta_0}{}^{1/2} & Z_{\eta_8}{}^{1/2} & Z_{\eta_3}{}^{1/2} \\ Z_{X_0}{}^{1/2} & Z_{X_8}{}^{1/2} & Z_{X_3}{}^{1/2} \end{vmatrix} \tag{25d}$$

The expressions in Eqs. (25) have been obtained by using the orthogonality condition

$$\langle 0|\Phi_i|P_j\rangle = \delta_{ij} \quad (i,j=\pi^0,\ \eta,\text{ or } X). \tag{26}$$

In Eqs. (19) and (20), we replace $v_j/Z_j{}^{1/2}$ by $\Phi_j$ given in Eq. (24) and follow the analogous procedure as mentioned in the nonmixing case to obtain

$$\begin{aligned}
(p_i^2-p_j^2)F^+_{ijl} = {}&\epsilon_0 \frac{Z_k{}^{1/2}}{M_K{}^2}(\alpha f_{8lk}+\beta f_{3lk})G_{ijk} - f_{il8}(p_i{}^2-M_i{}^2)Z_i{}^{-1/2}(Z_{\pi^0_8}{}^{1/2}\delta_{j\pi^0}+Z_{\eta_8}{}^{1/2}\delta_{j\eta}+Z_{X_8}{}^{1/2}\delta_{jX})\\
&-f_{il3}(p_i{}^2-M_i{}^2)Z_i{}^{-1/2}(Z_{\pi^0_3}{}^{1/2}\delta_{j\pi^0}+Z_{\eta_3}{}^{1/2}\delta_{j\eta}+Z_{X_3}{}^{1/2}\delta_{jX}) + (p_j{}^2-M_j{}^2)Z_i{}^{1/2}\sum_{a=0,8,3} f_{lai}A^a_j\,,
\end{aligned} \tag{27}$$

where $i=\pi^+,K^+$, or $K^0$ and $j=\pi^0,\eta$, or $X$. To derive Eq. (27), we have already taken the limit $q^2\to 0$ and assumed no momentum dependence of the $F$'s and $G$ in this limit, as required by the smoothness condition.

Now equating the coefficients of the independent variables $p_i{}^2$ and $p_j{}^2$ in Eq. (27), we get

$$\begin{aligned}
Z_i A^8_j &= Z_{\pi^0_8}{}^{1/2}\delta_{j\pi^0} + Z_{\eta_8}{}^{1/2}\delta_{j\eta} + Z_{X_8}{}^{1/2}\delta_{jX},\\
Z_i A^3_j &= Z_{\pi^0_3}{}^{1/2}\delta_{j\pi^0} + Z_{\eta_3}{}^{1/2}\delta_{j\eta} + Z_{X_3}{}^{1/2}\delta_{jX}\\
&(i=\pi^+,K^+,\text{ or } K^0;\ j=\pi^0,\eta,\text{ or } X).
\end{aligned} \tag{28}$$

These equations may be simplified to give the following $Z$ relations:

$$Z_{\pi^0_8} + Z_{\eta_8} + Z_{X_8} = \tfrac{1}{2} Z_{\pi^+}, \tag{29a}$$

$$Z_{\pi^0_3} + Z_{\eta_3} + Z_{X_3} = \tfrac{1}{2} Z_{\pi^+}, \tag{29b}$$

$$Z_{\pi^0_8}{}^{1/2} Z_{\pi^0_3}{}^{1/2} + Z_{\eta_8}{}^{1/2} Z_{\eta_3}{}^{1/2} + Z_{X_8}{}^{1/2} Z_{X_3}{}^{1/2} = 0, \tag{29c}$$

$$Z_{\pi^0_0}{}^{1/2} Z_{\pi^0_3}{}^{1/2} + Z_{\eta_0}{}^{1/2} Z_{\eta_3}{}^{1/2} + Z_{X_0}{}^{1/2} Z_{X_3}{}^{1/2} = 0, \tag{29d}$$

$$Z_{\pi^0_0}{}^{1/2} Z_{\pi^0_8}{}^{1/2} + Z_{\eta_0}{}^{1/2} Z_{\eta_8}{}^{1/2} + Z_{X_0}{}^{1/2} Z_{X_8}{}^{1/2} = 0. \tag{29e}$$

## IV. BROKEN $SU(2)$ AND $SU(2)$ SUM RULES

1. For the first set of sum rules, we consider the matrix elements of the divergences of the axial-vector currents given in Eqs. (4), between the vacuum and the single-meson states. For this purpose, we use further Eqs. (5)–(8) and thus obtain

$$f_\pi m_{\pi^+}{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0(\sqrt{2}+\alpha)Z_{\pi^+}{}^{1/2}, \tag{30a}$$

$$f_{K^+} m_{K^+}{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[(\sqrt{2}-\tfrac{1}{2}\alpha)+\tfrac{1}{2}\sqrt{3}\,\beta]Z_{K^+}{}^{1/2}, \tag{30b}$$

$$f_{K^0} m_{K^0}{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[(\sqrt{2}-\tfrac{1}{2}\alpha)-\tfrac{1}{2}\sqrt{3}\,\beta]Z_{K^0}{}^{1/2}, \tag{30c}$$

$$f_{\pi^0_8} m_{\pi^0}{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\alpha Z_{\pi^0_0}{}^{1/2} + (\sqrt{2}-\alpha)Z_{\pi^0_8}{}^{1/2} + \beta Z_{\pi^0_3}{}^{1/2}], \tag{31a}$$

$$f_{\pi_3^0} m_{\pi^0}{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\beta Z_{\pi_0^0}{}^{1/2} + \beta Z_{\pi_8^0}{}^{1/2} + (\sqrt{2}+\alpha) Z_{\pi_3^0}{}^{1/2}], \tag{31b}$$

$$f_{\eta_8} m_\eta{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\alpha Z_{\eta^0}{}^{1/2} + (\sqrt{2}-\alpha) Z_{\eta_8}{}^{1/2} + \beta Z_{\eta_3}{}^{1/2}], \tag{32a}$$

$$f_{\eta_3} m_\eta{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\beta Z_{\eta_0}{}^{1/2} + \beta Z_{\eta_8}{}^{1/2} + (\sqrt{2}+\alpha) Z_{\eta_3}{}^{1/2}], \tag{32b}$$

$$f_{X_8} m_X{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\alpha Z_{X_0}{}^{1/2} + (\sqrt{2}-\alpha) Z_{X_8}{}^{1/2} + \beta Z_{X_3}{}^{1/2}], \tag{33a}$$

$$f_{X_3} m_X{}^2 = (\tfrac{1}{3})^{1/2}\epsilon_0[\sqrt{2}\,\beta Z_{X_0}{}^{1/2} + \beta Z_{X_8}{}^{1/2} + (\sqrt{2}+\alpha) Z_{X_3}{}^{1/2}]. \tag{33b}$$

2. The next set of sum rules results trivially when one considers the vacuum expectation values of both sides of the Eq. (2d) and introduces a complete set of states, commonly believed to be dominated by the single-meson states, between the $F$ and $v$'s on the left-hand side.[8,11,27] In this manner we obtain the relations

$$f_{\pi^+} Z_{\pi^+}{}^{1/2} = -2(\tfrac{1}{3})^{1/2}(\sqrt{2}\,\xi_0 + \xi_8), \tag{34a}$$

$$f_{K^+} Z_{K^+}{}^{1/2} = -(\tfrac{1}{3})^{1/2}(2\sqrt{2}\,\xi_0 - \xi_8 + \sqrt{3}\,\xi_3), \tag{34b}$$

$$f_{K^0} Z_{K^0}{}^{1/2} = -(\tfrac{1}{3})^{1/2}(2\sqrt{2}\,\xi_0 - \xi_8 - \sqrt{3}\,\xi_3), \tag{34c}$$

$$f_{\pi_3^0} Z_{\pi_3^0}{}^{1/2} + f_{\eta_3} Z_{\eta_3}{}^{1/2} + f_{X_3} Z_{X_3}{}^{1/2} = -(\tfrac{1}{3})^{1/2}(\xi_8 + \sqrt{2}\,\xi_0), \tag{35a}$$

$$f_{\pi_3^0} Z_{\pi_8^0}{}^{1/2} + f_{\eta_3} Z_{\eta_8}{}^{1/2} + f_{X_3} Z_{X_8}{}^{1/2} = -(\tfrac{1}{3})^{1/2}\xi_3, \tag{35b}$$

$$f_{\pi_3^0} Z_{\pi_0^0}{}^{1/2} + f_{\eta_3} Z_{\eta_0}{}^{1/2} + f_{X_3} Z_{X_0}{}^{1/2} = -(\tfrac{2}{3})^{1/2}\xi_3, \tag{35c}$$

$$f_{\pi_8^0} Z_{\pi_3^0}{}^{1/2} + f_{\eta_8} Z_{\eta_3}{}^{1/2} + f_{X_8} Z_{X_3}{}^{1/2} = -(\tfrac{1}{3})^{1/2}\xi_3, \tag{35d}$$

$$f_{\pi_8^0} Z_{\pi_8^0}{}^{1/2} + f_{\eta_8} Z_{\eta_8}{}^{1/2} + f_{X_8} Z_{X_8}{}^{1/2} = -(\tfrac{1}{3})^{1/2}(\sqrt{2}\,\xi_0 - \xi_8), \tag{35e}$$

$$f_{\pi_8^0} Z_{\pi_0^0}{}^{1/2} + f_{\eta_8} Z_{\eta_0}{}^{1/2} + f_{X_8} Z_{X_0}{}^{1/2} = -(\tfrac{2}{3})^{1/2}\xi_8. \tag{35f}$$

3. To obtain a relation between the parameters $\alpha$ and $\beta$, we now make use of the concept that in the limit of exact $SU(3)\otimes SU(3)$ symmetry, the octet of pseudoscalar mesons becomes massless[18] while they are raised to finite mass when the chiral symmetry is broken by the $u_0$ term [in Eq. (1)]. Consequently, one may assume that $u_8$ and $u_3$ terms in the Hamiltonian give the mass splitting within the $SU(3)$ and $SU(2)$ multiplets, respectively.[23] Thus we write, in the lowest order of $H_{\text{SB}}$,

$$\begin{aligned} m_{\pi^+}{}^2 = m_{\pi^0}{}^2 &= \langle \pi^{+,0} | H_{\text{SB}} | \pi^{+,0}\rangle \\ &= \epsilon_0[(\tfrac{2}{3})^{1/2}h_1 + (\tfrac{1}{3})^{1/2}\alpha h_2], \\ m_{K^+}{}^2 &= \langle K^+ | H_{\text{SB}} | K^+\rangle \\ &= \epsilon_0[(\tfrac{2}{3})^{1/2}h_1 - \tfrac{1}{2}(\tfrac{1}{3})^{1/2}\alpha h_2 + \tfrac{1}{2}\beta h_2], \\ m_{K^0}{}^2 &= \langle K^0 | H_{\text{SB}} | K^0\rangle \\ &= \epsilon_0[(\tfrac{2}{3})^{1/2}h_1 - \tfrac{1}{2}(\tfrac{1}{3})^{1/2}\alpha h_2 - \tfrac{1}{2}\beta h_2], \end{aligned} \tag{36}$$

where $h_1$ and $h_2$ are the reduced matrix elements of $u_0$ and $u_i$ $(i=1,2,\ldots,8)$ between pseudoscalar octet states, respectively.

Eliminating $h_1$ and $h_2$ from (36), we get

$$\frac{\beta}{\alpha} = \sqrt{3}\,\frac{m_{K^0}{}^2 - m_{K^+}{}^2}{m_{K^+}{}^2 + m_{K^0}{}^2 - 2m_\pi{}^2}\,. \tag{37}$$

## V. THE GENERAL SOLUTIONS

In this section, the relations given in Eqs. (14)–(17), (23), (29)–(35), and (37) will be used systematically to estimate the mixing angles $\omega$, $\theta$, and $\phi$ and the symmetry-breaking parameters $\alpha$, $\beta$, $\xi_8/\xi_0$, and $\xi_3/\xi_0$. As we are here dealing with *an asymmetric degenerate vacuum,* $\xi_8$ *and* $\xi_3$ *(and consequently* $C_2$ *and* $C_3$*) are, in general, nonzero.* For consistency of Eqs. (14) and (23) one then requires $g_2 = 0$. Similarly, Eqs. (29a) and (29d) together with Eqs. (15)–(17) give $g_3 = g_4 = 0$. These conditions simplify our equations greatly. Just for the convenience of the readers, we shall rewrite Eqs.(30)–(33) eliminating the $Z^{1/2}$'s from Eqs. (14)–(17):

$$f_\pi m_{\pi^+}{}^2 = \sqrt{2}\,A(\sqrt{2}+\alpha), \tag{38a}$$

$$f_{K^+} m_{\pi^+}{}^2 = \sqrt{2}\,A(\sqrt{2} - \tfrac{1}{2}\alpha + \tfrac{1}{2}\sqrt{3}\,\beta), \tag{38b}$$

$$f_{K^0} m_{K^0}{}^2 = \sqrt{2}\,A(\sqrt{2} - \tfrac{1}{2}\alpha - \tfrac{1}{2}\sqrt{3}\,\beta), \tag{38c}$$

$$f_{\pi_8^0} m_{\pi^0}{}^2 = A[\beta + (\sqrt{2}-\alpha)(\theta\cos\omega - \phi\sin\omega) + \sqrt{2}\,\alpha(\theta\sin\omega + \phi\cos\omega) g_5/g_1], \tag{39a}$$

$$f_{\pi_3^0} m_{\pi^0}{}^2 = A(\sqrt{2}+\alpha), \tag{39b}$$

$$f_{\eta_8} m_\eta{}^2 = A[(\sqrt{2}-\alpha)\cos\omega + \sqrt{2}\,\alpha(\sin\omega) g_5/g_1], \tag{40a}$$

$$f_{\eta_3} m_\eta^2 = A[-(\sqrt{2}+\alpha)\theta + \beta\cos\omega + \sqrt{2}\,\beta(\sin\omega)g_5/g_1], \tag{40b}$$

$$f_{X_8} m_X^2 = A[-(\sqrt{2}-\alpha)\sin\omega + \sqrt{2}\,\alpha(\cos\omega)g_5/g_1], \tag{41a}$$

$$f_{X_3} m_X^2 = A[-(\sqrt{2}+\alpha)\phi - \beta\sin\omega + \sqrt{2}\,\beta(\cos\omega)g_5/g_1], \tag{41b}$$

where

$$A = \tfrac{1}{3}\sqrt{2}\,\epsilon_0 C_1 g_1. \tag{42}$$

It should be mentioned that $\beta$, $\theta$, and $\phi$ are of the order of $e^2$ [$SU(2)$ breaking strength] and that in obtaining the above equations we have neglected terms of order $e^4$ in the presence of the lower-order terms. We shall follow this approximation throughout this calculation. From Eqs. (39a), (40b), and (41b), it is observed that the couplings $f_{\pi_8^0}$, $f_{\eta_3}$, and $f_{X_3}$ are also of the order of $SU(2)$ violation.

Elimination of the $Z$'s from Eqs. (34) and (35) gives further

$$\sqrt{2}\,Bf_{\pi^+} = -2(\tfrac{1}{3})^{1/2}(\sqrt{2}\,\xi_0 + \xi_8), \tag{43a}$$

$$\sqrt{2}\,Bf_{K^+} = -(\tfrac{1}{3})^{1/2}(2\sqrt{2}\,\xi_0 - \xi_8 + \sqrt{3}\,\xi_3), \tag{43b}$$

$$\sqrt{2}\,Bf_{K^0} = -(\tfrac{1}{3})^{1/2}(2\sqrt{2}\,\xi_0 - \xi_8 - \sqrt{3}\,\xi_3), \tag{43c}$$

$$Bf_{\pi_3^0} = -(\tfrac{1}{3})^{1/2}(\sqrt{2}\,\xi_0 + \xi_8), \tag{44a}$$

$$B[f_{\pi_3^0}(\theta\cos\omega - \phi\sin\omega) + f_{\eta_3}\cos\omega - f_{X_3}\sin\omega] = -(\tfrac{1}{3})^{1/2}\xi_3, \tag{44b}$$

$$(g_5/g_1)B[f_{\pi_3^0}(\theta\sin\omega + \phi\cos\omega) + f_{\eta_3}\sin\omega + f_{X_3}\cos\omega] = -(\tfrac{2}{3})^{1/2}\xi_3, \tag{44c}$$

$$B[f_{\pi_8^0} - \theta f_{\eta_8} - \phi f_{X_8}] = -(\tfrac{1}{3})^{1/2}\xi_3, \tag{44d}$$

$$B[f_{\eta_8}\cos\omega - f_{X_8}\sin\omega] = -(\tfrac{1}{3})^{1/2}(\sqrt{2}\,\xi_0 - \xi_8), \tag{44e}$$

$$(g_5/g_1)B[f_{\eta_8}\sin\omega + f_{X_8}\cos\omega] = -(\tfrac{2}{3})^{1/2}\xi_8, \tag{44f}$$

with

$$B = (\tfrac{2}{3})^{1/2} C_1 g_1. \tag{45}$$

From Eqs. (38a), (39b), (43a), and (44a) we see that, within our approximation, one gets

$$f_{\pi_3^0} = (\tfrac{1}{2})^{1/2} f_\pi, \quad m_{\pi^+}{}^2 = m_{\pi^0}{}^2.$$

Further, the expressions for the symmetry-breaking parameters are obtained in terms of the known quantities given in Eqs. (46)–(49).

Equations (37), (38a), and (38b) give

$$\alpha = \sqrt{2}\left(F\frac{m_{K^+}{}^2}{m_\pi{}^2} - 1\right)\bigg/\left(\frac{m_{K^0}{}^2 - 2m_{K^+}{}^2 + m_\pi{}^2}{m_{K^+}{}^2 + m_{K^0}{}^2 - 2m_\pi{}^2} - F\frac{m_{K^+}{}^2}{m_\pi{}^2}\right), \tag{46}$$

where $F \equiv f_{K^+}/f_\pi$.

Equations (38b) and (38c) give

$$X \equiv \frac{f_{K^0}}{f_{K^+}} = \frac{m_{K^+}{}^2}{m_{K^0}{}^2}\left(1 - \frac{\sqrt{3}\,\beta}{\sqrt{2} - \frac{1}{2}\alpha + \frac{1}{2}\sqrt{3}\,\beta}\right). \tag{47}$$

Equations (43) give

$$\frac{\xi_8}{\xi_0} = 2\sqrt{2}\left(1 - \frac{3}{2}\,\frac{F(1+X)}{1+F+FX}\right), \tag{48}$$

$$\frac{\xi_3}{\xi_0} = \sqrt{6}\,\frac{F(1-X)}{1+F+FX}. \tag{49}$$

For the calculation of $\omega$, Eqs. (40a), (41a), (44e), and (44f) are used to give

$$f_{\eta_8} = K_1/\cos\omega, \quad f_{X_8} = K_2/\sin\omega, \tag{50}$$

$$K_1 = \frac{f_\pi m_\pi{}^2}{\sqrt{2}\,(m_X{}^2 - m_\eta{}^2)}\left[\frac{1}{3\sqrt{2}}\left(\sqrt{2} - \frac{\xi_8}{\xi_0}\right)(1+F+FX)\frac{m_X{}^2}{m_\pi{}^2} - \frac{\sqrt{2}-\alpha}{\sqrt{2}+\alpha}\right],$$

$$K_2 = \frac{f_\pi m_\pi{}^2}{\sqrt{2}\,(m_X{}^2 - m_\eta{}^2)}\left[\frac{1}{3\sqrt{2}}\left(\sqrt{2} - \frac{\xi_8}{\xi_0}\right)(1+F+FX)\frac{m_\eta{}^2}{m_\pi{}^2} - \frac{\sqrt{2}-\alpha}{\sqrt{2}+\alpha}\right], \tag{51}$$

$$T^2 \equiv (\tan\omega)^2 = \tfrac{1}{2}\left\{\pm\left[\frac{K_2}{K_1}\left(1+\frac{m_X}{m_\eta}\right)^2 + L\right]^{1/2}\left[\frac{K_2}{K_1}\left(1-\frac{m_X}{m_\eta}\right)^2 + L\right]^{1/2} - \left[\frac{K_2}{K_1}\left(1+\frac{m_X{}^2}{m_\eta{}^2}\right) + L\right]\right\}, \tag{52}$$

$$L = -\frac{1}{3\sqrt{2}\,K_1{}^2}\,\frac{\alpha}{(\sqrt{2}+\alpha)}\,\frac{f_\pi{}^2 m_\pi{}^2}{m_\eta{}^2}\,(1+F+FX)\frac{\xi_8}{\xi_0}. \tag{53}$$

In deriving Eqs. (50)–(53) we have used

$$A=\frac{f_\pi m_\pi^2}{\sqrt{2}\,(\sqrt{2}+\alpha)}\,,\qquad \frac{B}{\xi_0}=-\frac{2\sqrt{3}}{f_\pi(1+F+FX)} \tag{54}$$

obtained from Eqs. (38a) and (43).

To calculate the $SU(2)$ mixing parameters $\theta$ and $\phi$, we must look for the linear equations in these variables because we have neglected throughout quadratic terms such as $\phi^2$, $\theta\phi$, etc.

Eliminating the $f$'s from Eqs. (40b), (41b), (44b), and (44c), one gets the required linear solutions:

$$\theta=-\frac{1}{(m_\eta^2-m_\pi^2)}\left[\left(\cos\omega+\sqrt{2}\,(\sin\omega)\frac{g_5}{g_1}\right)\frac{\beta m_\pi^2}{\sqrt{2}+\alpha}-\frac{1}{\sqrt{3}}\left(\cos\omega+\sqrt{2}\,(\sin\omega)\frac{g_1}{g_5}\right)F(1-X)m_\eta^2\right], \tag{55}$$

$$\phi=\frac{1}{(m_X^2-m_\pi^2)}\left[\left(\sin\omega-\sqrt{2}\,(\cos\omega)\frac{g_5}{g_1}\right)\frac{\beta m_\pi^2}{\sqrt{2}+\alpha}-\frac{1}{\sqrt{3}}\left(\sin\omega-\sqrt{2}\,(\cos\omega)\frac{g_1}{g_5}\right)F(1-X)m_X^2\right], \tag{56}$$

where

$$\frac{g_5}{g_1}=\frac{K_1 m_\eta^2}{\sqrt{2}\,\alpha T}\left(T^2+\frac{K_2}{K_1}\frac{m_X^2}{m_\eta^2}\right), \tag{57}$$

as given by Eqs. (40a), (41a), and (50).

For the numerical estimate, we use the experimental masses of the pseudoscalar mesons[29] and take $f_\pi=133$ MeV. With $F$ as a parameter the mixing angles have been computed for different values of it. The results are quoted in Table I. We observe that there are two sets of solutions for $\omega$ (sets I and II), corresponding to the positive and negative signs in Eq. (52). The former gives a consistent but large value $\approx 19°$ compared to the experimental observations $\approx 10.5°$.[4,29] In the second set, $\omega$ corresponding to $F=1$ is in agreement with the value predicted by the quadratic mass formula; however, it is quite sensitive to the variation of $F$ and falls off rapidly as $F$ increases.

The numerical values of $\theta$ and $\phi$ are found to be consistent and insensitive to the variation of $F$, as expected. It is found that, in general, $\phi$ is smaller than $\theta$; this is reasonable because of large mass splitting between the pair $\pi^0$ and $X$ mesons. Also, the order of magnitudes of both $\theta$ and $\phi$ are compatible with our assumption that the strength of $SU(2)$ breaking is much smaller than that of $SU(3)$ violation.

## VI. SPECIAL CASES WITH $SU(3)$-INVARIANT VACUUM

In the previous sections we have given the general form of the equations and the sum rules with the condition of degeneracy in an asymmetric vacuum. Here we shall present a few special cases with an $SU(3)$-invariant vacuum, which means $C_2=C_3=0$ (and correspondingly, $\xi_8=\xi_3=0$). With this constraint we obtain the same set of equations (38)–(45). Equations (43) give

$$f_{\pi^+}=f_{K^+}=f_{K^0},\qquad F=X=1. \tag{58}$$

TABLE I. Predictions of the mixing angles and symmetry-breaking parameters corresponding to different values of $f_{K^+}/f_\pi$.

| | | | | | | Set I | | | Set II | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F=\frac{f_{K^+}}{f_\pi}$ | $\alpha$ | $\beta$ | $X=\frac{f_{K^0}}{f_{K^+}}$ | $\frac{\xi_8}{\xi_0}$ | $\frac{\xi_3}{\xi_0}$ | $\tan\omega$ ($\omega$) | $\tan\theta$ | $\tan\phi$ | $\tan\omega$ ($\omega$) | $\tan\theta$ | $\tan\phi$ |
| 1.0 | −1.26 | $-1.9\times10^{-2}$ | 1.0000 | 0 | 0 | 0.32 (≈18°) | 0.008 | −0.0008 | 0.19 (≈10.5°) | 0.008 | −0.0005 |
| 1.1 | −1.27 | $-1.9\times10^{-2}$ | 1.0001 | −0.09 | $-0.8\times10^{-4}$ | 0.36 (≈20°) | 0.008 | −0.0010 | 0.10 (≈6°) | 0.009 | −0.0002 |
| 1.2 | −1.28 | $-2.0\times10^{-2}$ | 1.0003 | −0.17 | $-2.6\times10^{-4}$ | 0.35 (≈20°) | 0.010 | −0.0011 | 0.03 (≈2°) | 0.010 | 0 |
| 1.28 | −1.29 | $-2.0\times10^{-2}$ | 1.0003 | −0.22 | $-2.6\times10^{-4}$ | 0.35 (≈19°) | 0.010 | −0.0012 | 0.01 (≈1°) | 0.011 | 0 |

### A. No $SU(2)$ Violation

First we wish to consider the case *with no* $SU(2)$ *violation*, i.e.,

$$\beta=\theta=\phi=0 \quad (\alpha \neq 0),$$
$$m_{K^+}=m_{K^0}\equiv m_K, \tag{59}$$
$$m_{\pi^+}=m_{\pi^0}\equiv m_\pi .$$

Using Eqs. (58) and (59) in Eqs. (46), (51), and (52), one obtains

$$\alpha=-\sqrt{2}\left(1-\frac{3m_\pi^2}{2m_K^2+m_\pi^2}\right), \tag{60}$$

$$K_1=\frac{f_\pi}{3\sqrt{2}\,(m_X^2-m_\eta^2)}\,(3m_X^2+m_\pi^2-4m_K^2), \tag{61a}$$

$$K_2=\frac{f_\pi}{3\sqrt{2}\,(m_X^2-m_\eta^2)}\,(3m_\eta^2+m_\pi^2-4m_K^2), \tag{61b}$$

$$(\tan^2\omega)_+=-\frac{m_X^2}{m_\eta^2}\,\frac{K_2}{K_1}, \tag{62a}$$

$$(\tan^2\omega)_-=-K_2/K_1. \tag{62b}$$

Combining Eqs. (61) and (62), we get the modified forms of the GMO mass formula:

$$4m_K^2-3m_\eta^2-m_\pi^2-3\sin^2\omega_+\,(m_X^2-m_\eta^2)\frac{4m_K^2-m_\pi^2}{3m_X^2}=0, \tag{63a}$$

$$4m_K^2-3m_\eta^2-m_\pi^2-3\sin^2\omega_-\,(m_X^2-m_\eta^2)=0. \tag{63b}$$

Equation (63b) has been obtained by several authors, such as Dalitz and Sutherland,[1] and Oneda *et al.*,[17] working with different formalisms. Equation (63a) is new and is related to the large solution of $\omega$.

In the absence of all mixings (i.e., $\omega$ is also zero), both of the equations (63) give the well-known mass relation

$$4m_K^2=3m_\eta^2+m_\pi^2. \tag{64}$$

We must point out a major difference between our calculation and the approach of Auvil and Deshpande.[27] When the vacuum is $SU(3)$-symmetric and there is *no mixing*, these authors need $m_X\to\infty$ for the consistency of their Eq. (8). However, in our case, the corresponding Eq. (41a) [together with Eq. (50)], viz.,

$$K_2m_X^2=A\sin\omega[-(\sqrt{2}-\alpha)\sin\omega+\sqrt{2}\,\alpha(\cos\omega)g_5/g_1], \tag{65}$$

is self-consistent, because when $\omega=0$, Eq. (61b) gives $K_2=0$ identically, since in this case we have the GMO formula [Eq. (64)]. Thus our equations do not require an abnormally high mass for the $X$ meson.

### B. $\pi^0$-$\eta$ Mixing

In principle we may study a very special situation when mixing takes place only between $\pi^0$ and $\eta$.[13–16] Thus, we set $\omega=\phi=0$ and find from Eq. (55)

$$\theta=-\frac{\beta m_\pi^2}{(\sqrt{2}+\alpha)(m_\eta^2-m_\pi^2)}\,. \tag{66}$$

Now replacing the factor $\beta m_\pi^2/(\sqrt{2}+\alpha)$ from Eqs. (38) by

$$\frac{\beta m_\pi^2}{\sqrt{2}+\alpha}=-\frac{1}{\sqrt{3}}\,(m_{K^0}^2-m_{K^+}^2), \tag{67}$$

one obtains

$$\theta=\frac{1}{\sqrt{3}}\,\frac{m_{K^0}^2-m_{K^+}^2}{m_\eta^2-m_\pi^2}\,. \tag{68}$$

This expression may be compared with the form

$$\theta=\frac{1}{\sqrt{3}}\,\frac{(m_{K^0}^2-m_{K^+}^2)+(m_{\pi^+}^2-m_{\pi^0}^2)}{m_\eta^2-m_\pi^2} \tag{69}$$

given by Okubo and Sakita[13] since, as we already mentioned, in our approximation, $m_{\pi^+}^2-m_{\pi^0}^2\approx 0$.

We may consider that $\pi^0$-$\eta$ mixing receives contributions from the electromagnetic interaction, $u_3$ term and other effects, so that one can write symbolically

$$\theta_{\text{total}}=\theta_{\text{em}}+\theta_{u_3\,\text{term}}+\text{“other contributions.”} \tag{70}$$

If $\theta$ given in Eq. (69) corresponds to $\theta_{\text{total}}$, then using the Dashen sum rule,[24]

$$(m_{K^0}^2-m_{K^+}^2+m_{\pi^+}^2-m_{\pi^0}^2)_{\text{em}}=0, \tag{71}$$

we find $\theta_{\text{em}}\approx 0$. It may be then concluded that $\pi^0$-$\eta$ mixing gets a significant contribution from the $u_3$ term as given in Eq. (68).

Another interesting aspect of this special case is that the $SU(2)$-breaking correction to the GMO formula is obtained easily. Equations (43a) and (44c) give $f_{\eta_8}=f_\pi/\sqrt{2}$ and combining this result with Eqs. (38a) and (40a) we get

$$\alpha=-\sqrt{2}\,\frac{m_\eta^2-m_\pi^2}{m_\eta^2+m_\pi^2}\,. \tag{72}$$

It is to be noted that when $m_\pi\to 0$ in this equation, $\alpha$ becomes $-\sqrt{2}$, which is its value in the limit of exact $SU(2)\otimes SU(2)$ symmetry.[18] Comparing Eqs. (46) (for $F=1$) with (72), we get

$$2(m_{K^+}^2+m_{K^0}^2)-3m_\eta^2-m_\pi^2=0. \tag{73}$$

This formula was first given by Glashow, as pointed out in Ref. 1.

### C. $\pi^0$-$X$ Mixing

Similar to the case of Sec. VI B, we may consider $\pi^0$-$X$ mixing alone by setting $\omega=\theta=0$. Equation (56) then gives

$$\phi=-\sqrt{2}\,\frac{g_5}{g_1}\,\frac{\beta m_\pi^2}{(m_X^2-m_\pi^2)}. \tag{74}$$

From Eq. (62) it follows that when $T\to 0$, $K_2$ vanishes as $T^2$; consequently $g_5/g_1$ in Eq. (57) vanishes when there is no $\eta$-$X$ mixing. Thus from Eq. (74) we predict

$$\phi=0.$$

One may test that our equations are consistent with this solution.

## VII. SUMMARY OF THE RESULTS AND DISCUSSIONS

In this paper we have outlined a general method for determining the mixing effects of the pseudoscalar mesons on the basis of a kinematical approach. We have considered a symmetry-breaking Hamiltonian which transforms as the $(3,\bar{3})\oplus(\bar{3},3)$ representation of the group $SU(3)\otimes SU(3)$ and contains explicitly $SU(3)$ and $SU(2)$ violations, the latter being manifested by the $u_3$ term. With the further assumptions of maximal smoothness behavior of three-point vertex functions and meson dominance of certain commutators between the vacuum and physical states, we need to use only one input parameter $f_K/f_\pi$, the value of $f_\pi$, and the well-known masses of the pseudoscalar particles to calculate the mixing angles and the symmetry-breaking parameters.

In our model, we find two different solutions for the $\eta$-$X$ mixing angle $\omega$, both for an $SU(3)$-invariant and -noninvariant vacuum. In the case of an $SU(3)$-symmetric vacuum, we have only one choice, $F=1$ [see Eq. (58)], which gives (a) $\omega\approx 19°$ and (b) $\omega\approx 10.5°$. The second solution is exactly the same as obtained from the quadratic mass formula. It is interesting to point out here that Schülke[6] has also reported the existence of a small and a large solution from a different point of view.[30]

When the vacuum is asymmetric, we have two sets of solutions as functions of the parameter $F$. In this case when $F=1$, we have approximately the same solution as obtained in the case of an invariant vacuum (because $F=1$ gives $X\approx 1$ and $\xi_8\approx\xi_3\approx 0$). Our large solution is not sensitive to the variation of $F$ whereas the smaller solution decreases rapidly as $F$ increases (see Table I). A plausible explanation may be given in the following way: When $F=1$, the contribution to $\omega$ comes from $\alpha$ alone [$SU(3)$ breaking of the Hamiltonian]; when $F\neq 1$ and consequently $\xi_8\neq 0$ [$SU(3)$ breaking of the vacuum], the terms involving $\xi_8$ are found to contribute in the same order as given by $\alpha$ alone. The larger solutions are due to the additive effect of these two comparable contributions ($\alpha$ and $\xi_8$) while the smaller ones are due to their difference.

Regarding the predictions of the $\pi^0$-$\eta$ and $\pi^0$-$X$ mixing angles, $\theta$ and $\phi$, we notice a few important points:

(a) For the general case, the magnitude and the sign of $\theta$ are in agreement with the value for $\theta=0.0105\pm 0.0013$ given by Dalitz and von Hippel.[14] This value is reproduced in our calculation for $f_K/f_\pi\approx 1.2$.

(b) In the special case when the vacuum is $SU(3)$-*invariant* and *$\eta$-$X$ mixing is absent,* we give a form for $\theta$ which corresponds to that obtained by the diagonalization of the $\eta$-$\pi$ mass matrix.

(c) In our theory, $\phi$ turns out to be much smaller (e.g., by a factor of 8–10 in Set I) than $\theta$, as expected due to the larger mass splitting between the pair $\pi^0$-$X$ relative to $\pi^0$-$\eta$. However, we are in contradiction with Oneda *et al.*,[17] who predict $\theta\approx 0.03$, $\phi\approx 0.03$ or $\theta\approx -0.01$, $\phi\approx -0.04$.

(d) Considering *$\pi^0$-$X$ mixing alone in the case of an invariant vacuum,* we obtain the solution $\phi=0$. The reason why we get different results for $\theta$ and $\phi$ in similar special situations is perhaps that they are not treated on the same footing in an $SU(3)$ symmetry model.

Finally, we mention that when vacuum symmetry is not broken, our model gives modified GMO mass formulas including $\eta$-$X$ mixing and $SU(2)$ breaking separately [see Eqs. (63) and (73)]. One of these is the standard GMO relation [Eq. (63b)], which corresponds to the small solution for $\omega$ in our case. The other [Eq. (63a)] is a new one and corresponds to the large solution for $\omega$. In principle, one must have such modifications in the general case when vacuum symmetry is broken; however, the forms of the mass relations are not as simple as in the special case.

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*Postdoctoral research fellow from the Department of Physics and Astrophysics, University of Delhi, Delhi-7, India.

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PHYSICAL REVIEW D VOLUME 4, NUMBER 1 1 JULY 1971

# $K_{l3}$ Decay, Elastic Unitarity, and the Symmetry-Breaking Parameter of the Vacuum States

Francis C. P. Chan
*Saginaw Valley College, University Center, Michigan 48710*
(Received 22 March 1971)

$K_{l3}$ form factors are derived in the hard-pion approach without using vector- or axial-vector-meson dominance and with more general consideration of symmetry breaking. Elastic unitarity is employed for the vector form factor. The ratio $f_-(0)/f_+(0)$ is expressed in terms of the symmetry-breaking parameter, $b=\langle 0|u_8|0\rangle/\langle 0|u_0|0\rangle$. It is found that current algebra can provide a satisfactory explanation for the decay parameters.

## I. INTRODUCTION

The current-algebra approach towards hadron physics has proved to be most fruitful. Applied to the problem of $K_{l3}$ decay there have been numerous theoretical treatises since the work of Callan and Treiman.[1] However, a good deal remains to be said on the subject because of the ill-defined nature of symmetry breaking and the uncertainty in the experimental data involved. Recently, the hard-pion approach of Schnitzer, Weinberg, and Gerstein[2] has been used to derive an effective-range formula for the pion form factor with the application of the principles of unitarity.[3] Also, the symmetry-breaking argument of Gell-Mann, Oakes, and Renner[4] has been challenged and extended.[5] With this and the updated experimental numbers in mind, we intend to review and formulate the $K_{l3}$ problem without using the vector-meson or axial-vector-meson dominance approximation.