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Factorization of Multi-Regge Amplitudes*

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In order to gain insight into the problem of factorization of multi-Regge amplitudes, the dual-resonance model is studied in detail. The full amplitude, as well as its total-energy discontinuity, are shown to factorize in the multi-Regge limit, although this is by no means a trivial consequence of the well-known factorization of a single generalized beta-function term. Factorization is found to depend crucially upon taking into proper account the singularities in energy variables which are dependent due to nonlinear Gram-determinant constraints. This result suggests how the apparent nonfactorization of the full multi-Regge amplitude recently pointed out by Dash may be generally circumvented. An application of the factorization of the total-energy discontinuity to the phenomenology of inclusive cross sections is discussed.

I. INTRODUCTION

In order to gain some insight into the important question of whether or not multi-Regge amplitudes factorize, we study here the dual-resonance model (DRM) in detail. It is well known that a single generalized beta-function term factorizes in the multi-Regge limit.[1] However, for reasons we will review below, factorization of the full amplitude, which is a sum of such terms, by no means follows trivially from the factorization of a single term. Nevertheless, we find that the full amplitude, as well as its discontinuity in the total energy, do factorize in the multi-Regge limit.

In order to bring out the possible relevance of this result to the general problem of factorization of multi-Regge amplitudes, let us review the issues involved. Consider the possibility of a generalization of the "proof" of factorization for single-Reggeon exchange[2] which proceeds along the same lines. That is, one first defines signatured amplitudes which allow a continuation of the partial-wave unitarity equation to complex angular momentum and then uses this equation to prove factorization of the Regge residues and thus of the signatured amplitude. The factorization of the full amplitude then follows directly, since it is asymptotically related to the signatured amplitude by the simple multiplicative signature factor $\xi = e^{-i\pi\alpha}+\tau$, where the signature τ is ± 1. The generalization of signature and the treatment of the unitarity equation for many-particle amplitudes certainly presents very formidable difficulties[3] which may never be overcome. In addition, Dash[4] has recently pointed out that even if we assume that this has been accomplished and factorization of the signatured amplitudes has been established, factorization of a single multi-Regge contribution to the full amplitude may not follow. Let us discuss this last difficulty in more detail.

Suppose we have an expression for the full amplitude, $A_{2\to n}$, in terms of generalized multi-Froissart-Gribov amplitudes,[5] $A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}$:

$$A_{2\to n}(s_{01}, s_{12}, \ldots, s_{n-2,n-1}; t_1, t_2, \ldots, t_{n-1}; \kappa_1, \kappa_2, \ldots, \kappa_{n-2})$$
$$= \sum_{\tau_i}\sum_{\nu_i} (\mu_1\mu_2\cdots\mu_{n-1}) A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}(\nu_1 s_{01}, \nu_2 s_{12}, \ldots, \nu_{n-1}s_{n-2,n-1}; t_1, t_2, \ldots, t_{n-1}; \kappa_1, \kappa_2, \ldots, \kappa_{n-2}) , \tag{1.1}$$

where $\tau_i = \pm 1$, $\nu_i = \pm 1$, and

$$\mu_i = \begin{cases} 1, & \nu_i = 1 \\ \tau_i, & \nu_i = -1. \end{cases}$$

In writing (1.1) we have used the usual complete set of $3n-4$ BCP variables[6] (see Fig. 1), where

$$\kappa_i = \frac{s_{i-1,i}s_{i,i+1}}{s_{i-1,i,i+1}} \sim \frac{\lambda(t_i, t_{i+1}; m_i^2)}{\sqrt{-t_i}\,\sqrt{-t_{i+1}}\,\cos\omega_i + m_i^2 - t_i - t_{i+1}} . \tag{1.2}$$

The remaining variables are dependent upon these either through linear relations (e.g., s_{02}, s_{a1}, etc.) or

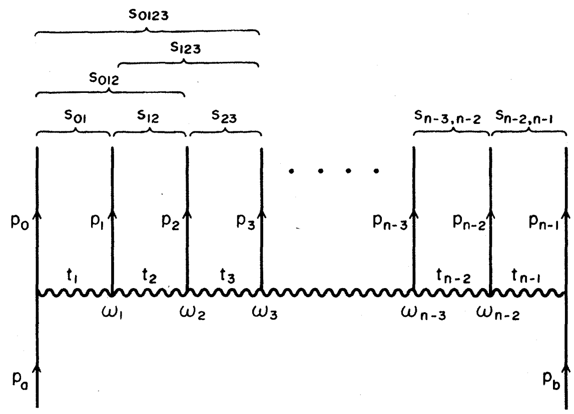

Fig. 1. Definition of the BCP variables.

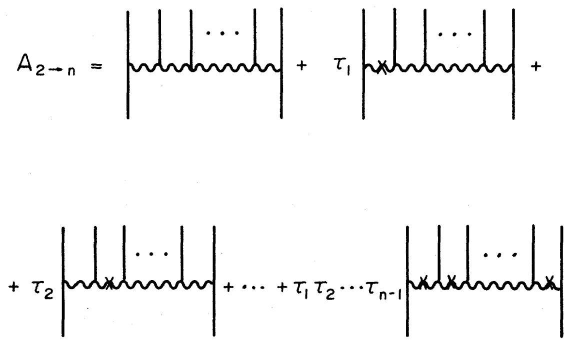

Fig. 2. Diagrammatical representation of Eq. (1.1).

through nonlinear relations (e.g., s_{ab}, s_{0123}, s_{a123}, etc.). A useful mnemonic for the sum over ν_i in (1.1) is shown in Fig. 2. The presence of a cross indicates a twist of the diagram about the crossed line. Each twist changes all the incoming (outgoing) lines to its right to outgoing (incoming). Asymptotically, this has the effect of changing the signs of the energy variables which span the twist (e.g., s_{ab}, s_{01}, s_{012}, s_{0123}, etc., in the term multiplied by τ_1).

We have motivated the expression (1.1) as a plausible generalization of the expression for $n=2$. However, it may be that such an expression is valid, at least in the multi-Regge asymptotic region, even in the absence of a generalization of signature to production amplitudes. Some further comments about this possibility are made in Sec. III. Therefore, the problem of the relationship between the factorization of the individual terms in (1.1) and that of the sum discussed here may be of relevance even if it is not possible to generalize all the other steps of the proof of factorization for $n=2$.

The signatured amplitudes $A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}$ are assumed to have only right-hand singularities and to be real for $s_{i,i+1}<0$ and $\kappa_i<0$ (or, equivalently, $s_{i-1,i,i+1}<0$). It thus appears plausible that the phases of the contributions to (1.1) for various ν_i can be determined by continuation of the appropriate $s_{i,i+1}$ and $s_{i-1,i,i+1}$ to positive values using the $+i\epsilon$ prescription.[4,7] The contribution to the full amplitude of a multi-Regge-pole contribution to the signatured amplitude can then be readily obtained. Thus, suppose the signatured amplitude has the factorized asymptotic behavior[8]

$$A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}(s_{01}, s_{12}, \ldots, s_{n-2,n-1}; t_1, \ldots, t_{n-1}; \kappa_1, \ldots, \kappa_{n-2})$$
$$\sim \beta(t_1)\Gamma(-\alpha_1)(-s_{01})^{\alpha_1}\beta(t_1, t_2; \kappa_1)\Gamma(-\alpha_2)(-s_{12})^{\alpha_2}\beta(t_2, t_3; \kappa_2)\cdots\Gamma(-\alpha_{n-1})(-s_{n-2,n-1})^{\alpha_{n-1}}\beta(t_{n-1}), \tag{1.3}$$

where $\alpha_i=\alpha_i(t_i)$. If the residues $\beta(t_i, t_{i+1}; \kappa_i)$ have no cuts in κ_i, Eq. (1.3) leads to a factorized contribution to (1.1) since the sum over ν_i merely generates the product of the signature factors $\xi_i=e^{-i\pi\alpha_i}+\tau_i$; thus,

$$A_{2\to n}\sim R(t_1)[\Gamma(-\alpha_1)\xi_1 s_{01}{}^{\alpha_1}]R(t_1 t_2; \kappa_1)[\Gamma(-\alpha_2)\xi_2 s_{12}{}^{\alpha_2}]R(t_2, t_3; \kappa_2)\cdots[\Gamma(-\alpha_{n-1})\xi_{n-1}s_{n-2,n-1}{}^{\alpha_{n-1}}]R(t_{n-1}), \tag{1.4}$$

where in this case $R=\beta$. However, if there are cuts in κ_i, Dash showed that factorization of the full amplitude will *not* generally obtain, since a given κ_i will in some terms be continued to above its cut and in other terms to below its cut.

We believe the study of the DRM in Sec. II suggests the solution to this difficulty. A single generalized beta-function term can be regarded as a model for the signatured amplitude since it has only right-hand singularities in the energy variables (s_{01}, s_{12}, s_{012}, s_{0123}, etc.). For negative values of the energy variables it indeed has the behavior (1.3). However, the continuation to positive values is not properly obtained by the simple prescription given above. When the singularities in variables dependent by virtue of the nonlinear constraints (related to the four-dimensionality of space-time) are properly taken into account, we find that the multi-Regge-pole contribution to the signatured amplitude does not factorize for every individual term in the sum over ν_i in (1.1), but the full amplitude does factorize.[9] Thus, as will be discussed further in Sec. III, we believe the key to the consistency between the factorization of the multi-Regge-pole residues and

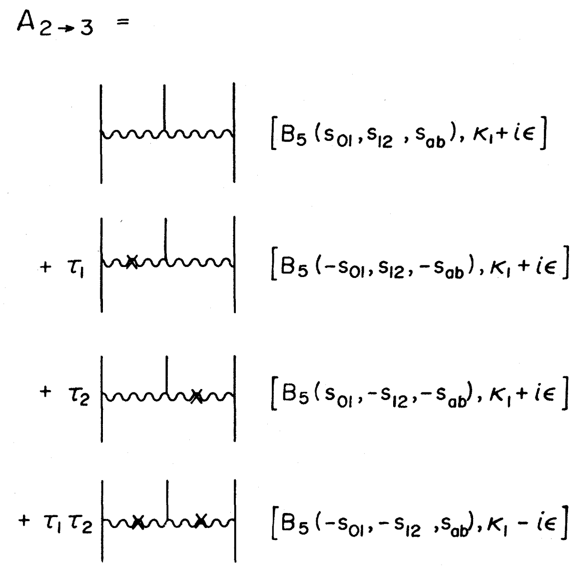

Fig. 3. Terms contributing to $A_{2\to 3}$.

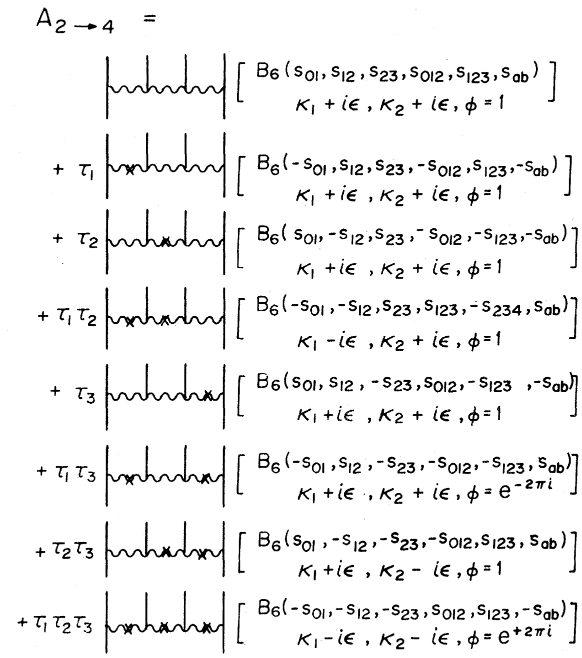

Fig. 4. Terms contributing to $A_{2\to 4}$.

the factorization of the full amplitude lies in the proper treatment of the cuts in nonlinearly dependent variables.[10]

We also show in Sec. II that in the DRM the total-energy discontinuity of the amplitude also factorizes in the multi-Regge limit[8,11]:

$$\Delta_{s_{ab}}A_{2\to n}\sim R(t_1)\,\frac{s_{01}{}^{\alpha_1}}{\Gamma(\alpha_1+1)}\,D(t_1,t_2;\kappa_1)\,\frac{s_{12}{}^{\alpha_2}}{\Gamma(\alpha_2+1)}$$
$$\times D(t_2,t_3;\kappa_3)\cdots\frac{s_{n-2,n-1}{}^{\alpha_{n-1}}}{\Gamma(\alpha_{n-1}+1)}\,R(t_{n-1})\,. \qquad (1.5)$$

This result is particularly interesting since it shows that the DRM is consistent with the result of multi-Regge or multiperipheral models for $\Delta_{s_{ab}}A_{2\to n}$. Factorization of $\Delta_{s_{ab}}A_{2\to n}$ in such models is expected to follow from the Fredholm nature of the integral equations involved. This factorization may well be a general result which again does not follow trivially from the factorization of (1.3).

The factorization of discontinuities of Regge-behaved amplitudes is also of interest in the phenomenology of inclusive cross sections. As one example, by a trivial generalization of Mueller's recent analysis for single-particle production,[12] one sees that the inclusive cross section for the production of $n-2$ particles in the "uncorrelated pionization" region is given by the total-energy discontinuity of a multi-Regge amplitude like that of Fig. 1 but with two particles at each internal vertex. Thus, the factorization of such a multi-Regge amplitude has a direct experimental consequence.

For the DRM this factorization follows directly from the results of Sec. II.[13] Asymptotically, the amplitude with two particles at each vertex can be shown to be an integral over the amplitude with one particle at each vertex[14]:

$$A_{n\to n}\sim\int_0^1 dz_1 z_1{}^{-\beta_1-1}(1-z_1)^{-\gamma_1-1}\int_0^1 dz_2 z_2{}^{-\beta_2-1}(1-z_2)^{-\gamma_2-1}\cdots\int_0^1 dz_{n-2}z_{n-2}{}^{-\beta_{n-2}-1}(1-z_{n-2})^{-\gamma_{n-2}-1}$$
$$\times A_{2\to n}(s_{01},\ldots,s_{n-2,n-1};t_1,\ldots,t_{n-1};\kappa_1/z_1(1-z_1),\kappa_2/z_2(1-z_2),\ldots,\kappa_{n-2}/z_{n-2}(1-z_{n-2}))\,,$$

where β_i and γ_i are certain Regge trajectories. Hence, factorization in the DRM follows from (1.5) with[15]

$$D(t_i,t_{i+1};\kappa_i)\to\int_0^1 dz_i z_i{}^{-\beta_i-1}(1-z_i)^{-\gamma_i-1}D(t_i,t_{i+1};\kappa_i/z_i(1-z_i))\,.$$

II. DUAL-RESONANCE MODEL

After reviewing the calculation of the two-Reggeon vertex function from the $2\to 3$ amplitude,[1,7] we prove factorization of the $2\to 4$ amplitude. This result is then generalized in a straightforward manner to prove factorization of the $2\to n$ amplitude.

As noted in Sec. I, the generalized beta function can be regarded as a model for the signatured amplitude. Indeed, the full amplitude is given in terms of the individual beta-function terms which contribute in the multi-Regge limit considered[16] by an expression of the form (1.1) (see also Fig. 2). This expression is given explicitly for the $2\to 3$ amplitude in Fig. 3. We show in Fig. 3 the relative signs of the asymptotic energies in the various terms. We also show the phase of κ_1 as determined[7] from Eq. (1.2) and the $+i\epsilon$ prescription for the energies, $s_{ij\cdots k}=(-s_{ij\cdots k})e^{-i\pi}$ when $s_{ij\cdots k}>0$. Thus, for example, for the first term we find

$$\kappa_1=\frac{s_{01}s_{12}}{s_{012}}=\frac{(-s_{01})e^{-i\pi}(-s_{12})e^{-i\pi}}{(-s_{012})e^{-i\pi}}=(-\kappa_1)e^{-i\pi}=\kappa_1+i\epsilon\,,$$

whereas for the last term

$$\kappa_1=\frac{s_{01}s_{12}}{(-s_{012})e^{-i\pi}}=(-\kappa_1)e^{i\pi}=\kappa_1-i\epsilon\,.$$

The generalized beta function also has multi-Regge behavior corresponding to (1.3) for the signatured amplitude. We find, using the usual exponentiation substitution in the Bardakci-Ruegg form,[1,17]

$$\begin{aligned}B_5(s_{01},s_{12},s_{ab})&\sim(-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}\int_0^\infty\int_0^\infty dy_1dy_2\,y_1^{-\alpha_1-1}y_2^{-\alpha_2-1}\exp\left(-y_1-y_2+\frac{y_1y_2}{\kappa_1}\right)\\&\equiv(-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}V(\alpha_1,\alpha_2;\kappa_1)\,.\end{aligned}\tag{2.1}$$

Using $\beta(t_j)=1$ (determined from the $2\to 2$ amplitude), we obtain the two-Reggeon residue,

$$\beta(t_j,t_{j+1};\kappa_j)=[\Gamma(-\alpha_j)\Gamma(-\alpha_{j+1})]^{-1}V(\alpha_j,\alpha_{j+1};\kappa_j).\tag{2.2}$$

The full amplitude is written in the factorized form (1.4) in order to *define* the two-Reggeon vertex function[7]

$$\begin{aligned}R(t_j,t_{j+1};\kappa_j)=\xi_j^{-1}\xi_{j+1}^{-1}[\Gamma(-\alpha_j)\Gamma(-\alpha_{j+1})]^{-1}[&e^{-i\pi\alpha_j}V(\alpha_j,\alpha_{j+1};\kappa_j+i\epsilon)e^{-i\pi\alpha_{j+1}}+\tau_jV(\alpha_j,\alpha_{j+1};\kappa_j+i\epsilon)e^{-i\pi\alpha_{j+1}}\\&+e^{-i\pi\alpha_j}V(\alpha_j;\alpha_{j+1};\kappa_j+i\epsilon)\tau_{j+1}+\tau_j\,V(\alpha_j,\alpha_{j+1};\kappa_j-i\epsilon)\tau_{j+1}].\end{aligned}\tag{2.3}$$

We will find it convenient to rewrite (2.3) using the following expression for V:

$$\begin{aligned}V(\alpha_j,\alpha_{j+1};\kappa_j\pm i\epsilon)&=\Gamma(-\alpha_j)\int_0^\infty dy\,y^{-\alpha_{j+1}-1}\left(1-\frac{y}{\kappa_j}\right)^{\alpha_j}e^{-y}\\&=\Gamma(-\alpha_j)\int_0^{\kappa_j}dy\,y^{-\alpha_{j+1}-1}\left(1-\frac{y}{\kappa_j}\right)^{\alpha_j}e^{-y}+e^{\pm i\pi\alpha_j}\Gamma(-\alpha_j)\int_{\kappa_j}^\infty dy\,y^{-\alpha_{j+1}-1}\left(\frac{y}{\kappa_j}-1\right)^{\alpha_j}e^{-y}\end{aligned}\tag{2.4}$$

$$\equiv\Gamma(-\alpha_j)[I_1(\alpha_j,\alpha_{j+1};\kappa_j)+e^{\pm i\pi\alpha_j}I_2(\alpha_j,\alpha_{j+1};\kappa_j)]\,,\tag{2.5}$$

where I_1 and I_2 are real for $\kappa_j\geqslant 0$. Therefore, we have

$$\begin{aligned}R(t_j,t_{j+1};\kappa_j)&=[\xi_{j+1}\Gamma(-\alpha_{j+1})]^{-1}[e^{-i\pi\alpha_{j+1}}(I_1+e^{i\pi\alpha_j}I_2)+\tau_{j+1}(I_1+\tau_jI_2)]\\&=[\xi_{j+1}\Gamma(-\alpha_{j+1})]^{-1}[\xi_{j+1}I_1+(e^{i\pi\alpha_j}e^{-i\pi\alpha_{j+1}}+\tau_j\tau_{j+1})I_2]\,,\end{aligned}\tag{2.6}$$

where we have used $\tau_j^2=1$.

The total-energy discontinuity of $A_{2\to3}$ comes only from the first term in Fig. 3 [or Eq. (2.3)], since only this term has singularities in $s_{ab}=s_{012}$. From (1.2) we see that asymptotically the cut in κ_1 represents these singularities. Hence[11]

$$\Delta_{s_{ab}}A_{2\to3}\sim R(t_1)\frac{s_{01}^{\alpha_1}}{\Gamma(\alpha_1+1)}D(t_1,t_2;\kappa_1)\frac{s_{12}^{\alpha_2}}{\Gamma(\alpha_2+1)}R(t_2),\tag{2.7}$$

where

$$\begin{aligned}D(t_j,t_{j+1};\kappa_j)&=e^{-i\pi\alpha_j}e^{-i\pi\alpha_{j+1}}\Gamma(\alpha_j+1)\Gamma(\alpha_{j+1}+1)\Delta_{\kappa_j}V(\alpha_j,\alpha_{j+1};\kappa_j)\\&=e^{+i\pi\alpha_j}e^{-i\pi\alpha_{j+1}}\Gamma(\alpha_{j+1}+1)I_2\,.\end{aligned}\tag{2.8}$$

From (2.4) and (2.8) we see that (2.7) has no poles for integral values of α_1 and α_2 as expected.

We now turn to the proof of factorization of the $2\to 4$ amplitude. The eight terms contributing in the triple-

Regge limit are shown in Fig. 4.[16] The usual exponentiation substitution gives the asymptotic behavior

$$B_6(s_{01}, s_{12}, s_{23}, s_{012}, s_{123}, s_{ab}) \sim (-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}(-s_{23})^{\alpha_3} \int_0^\infty \int_0^\infty \int_0^\infty dy_1 dy_2 dy_3\, y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} y_3^{-\alpha_3-1}$$
$$\times \exp\left(-y_1-y_2-y_3+\frac{\alpha_{012}}{\alpha_{01}\alpha_{12}}\, y_1y_2+\frac{\alpha_{123}}{\alpha_{12}\alpha_{23}}\, y_2y_3-\frac{\alpha_{0123}}{\alpha_{01}\alpha_{12}\alpha_{23}}\, y_1y_2y_3\right), \tag{2.9}$$

where $\alpha_{ij\cdots k}=s_{ij\cdots k}+a_{ij\cdots k}$. It is useful to derive a formula analogous to (2.4) by doing the integral over y_1 and making the change of variables

$$y_2 \to y_2(1-y_3/\kappa_2)^{-1},$$

$$B_6 \sim (-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}(-s_{23})^{\alpha_3}\Gamma(-\alpha_1)\int_0^\infty\int_0^\infty dy_2 dy_3 y_2^{-\alpha_2-1} y_3^{-\alpha_3-1} e^{-y_2-y_3}\left(1-\frac{y_2}{\kappa_1}\,\frac{1-(y_3/\kappa_2)\phi}{1-y_3/\kappa_2}\right)^{\alpha_1}\left(1-\frac{y_3}{\kappa_2}\right)^{\alpha_2}$$
$$\equiv (-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}(-s_{23})^{\alpha_3}\Gamma(-\alpha_1) I^{(2)}(\alpha_1, \alpha_2, \alpha_3; \kappa_1, \kappa_2; \phi), \tag{2.10}$$

where

$$\phi \equiv \frac{\alpha_{0123}\alpha_{23}}{\alpha_{012}\alpha_{123}}. \tag{2.11}$$

In the triple-Regge limit, $\phi=1$, as a consequence of the nonlinear Gram-determinant conditions on the invariants.[18] For $\phi=1$, clearly (2.10) factorizes[1] into the product of two double-Regge residues [see Eq. (2.4)]. However, since (2.10) has a cut in ϕ we must be careful to specify the phase of ϕ. Indeed, two of the terms in Fig. 4 have $\phi=e^{\pm 2\pi i}$. We discuss the interpretation of this phase below.

To obtain the asymptotic behavior of B_6 in the general case, we must study the continuation of the factors

$$\left(1-\frac{y_2}{\kappa_1}\,\frac{[1-(y_3/\kappa_2)\phi]}{[1-y_3/\kappa_2]}\right)^{\alpha_1}\left(1-\frac{y_3}{\kappa_2}\right)^{\alpha_2} \tag{2.12}$$

as the appropriate energy variables are continued into their right-half planes from their left-half planes where (2.9) and (2.10) are initially defined. We see that the sub-brackets in the first factor of (2.12), and thus ϕ, only have an effect when $y_3 \geq \kappa_2$ and also $y_2 \geq \kappa_1$. We show the phase of (2.12) in Fig. 5 for $\phi=e^{n 2\pi i}$ where it factorizes into separate functions of y_2 and y_3. Comparing with Eqs. (2.4) and (2.5) we see that $I^{(2)}$ can be expressed as

$$I^{(2)}(\alpha_1, \alpha_2, \alpha_3; (-\kappa_1)e^{i n_1\pi}, (-\kappa_2)e^{i n_2\pi}; \phi)$$
$$= I_1(1)I_1(2)+e^{-in_2\pi\alpha_2}I_1(1)I_2(2)+e^{-in_1\pi\alpha_1}I_2(1)I_1(2)+e^{-in_1\pi\alpha_1}e^{-in_2\pi\alpha_2}\phi^{\alpha_1}I_2(1)I_2(2), \tag{2.13}$$

where $I_i(1)$ is shorthand for $I_i(\alpha_1, \alpha_2; \kappa_1)$ and $I_i(2)$ is shorthand for $I_i(\alpha_2, \alpha_3; \kappa_2)$.

We now group the eight contributions to $A_{2\to4}$ into pairs of successive terms in Fig. 4 and express them using (2.13) and $\tau_1^2=1$:

$$A_{2\to4} \sim [\Gamma(-\alpha_1)\xi_1 s_{01}^{\alpha_1}]\, s_{12}^{\alpha_2} s_{23}^{\alpha_3}\{e^{-i\pi\alpha_2}e^{-i\pi\alpha_3}[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+e^{i\pi\alpha_1}e^{i\pi\alpha_2}I_2(1)I_2(2)]$$
$$+\tau_2 e^{-i\pi\alpha_3}[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+\tau_1 I_2(1)I_1(2)+\tau_1 e^{i\pi\alpha_2}I_2(1)I_2(2)]$$
$$+e^{-i\pi\alpha_2}\tau_3[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+\tau_1 e^{i\pi\alpha_2}I_2(1)I_2(2)]$$
$$+\tau_2\tau_3[I_1(1)I_1(2)+e^{-i\pi\alpha_2}I_1(1)I_2(2)+\tau_1 I_2(1)I_1(2)+e^{i\pi\alpha_1}e^{-i\pi\alpha_2}I_2(1)I_2(2)]\}. \tag{2.14}$$

Grouping the terms in (2.14) again by pairs, we find using $\tau_2^2=1$,

$$A_{2\to4} \sim [\Gamma(-\alpha_1)\xi_1 s_{01}^{\alpha_1}]\, s_{12}^{\alpha_2} s_{23}^{\alpha_3}\{[\xi_2 I_1(1)+(e^{i\pi\alpha_1}e^{-i\pi\alpha_2}+\tau_1\tau_2)I_2(1)]\, e^{-i\pi\alpha_3}[I_1(2)+e^{i\pi\alpha_2}I_2(2)]$$
$$+[\xi_2 I_1(1)+(e^{i\pi\alpha_1}e^{-i\pi\alpha_2}+\tau_1\tau_2)I_2(1)]\tau_3[I_1(2)+\tau_2 I_2(2)]\}. \tag{2.15}$$

Comparing with (2.6), we see that (2.15) has the factorized form (1.4) as asserted. Factorization would not have been obtained if we had not taken into account the phase of ϕ.

The cuts in κ_1, κ_2, and ϕ in (2.10) are asymptotic representations of the poles in s_{012}, s_{123}, and $s_{0123}=s_{ab}$. In particular, the poles in s_{ab} are represented exclusively by the cut in ϕ, since, when only the phase of s_{ab} changes, only the phase of ϕ changes and conversely. Thus, the correct phase of ϕ means we are taking the asymptotic limit with the proper $+i\epsilon$ relationship to the cut in s_{ab}. For example, taking $\phi=1$ in the third-to-last and last terms in Fig. 4 would correspond to taking $s_{ab}\to+\infty-i\epsilon$ in $A_{2\to4}$ as far as these two terms are concerned. Therefore, the essential ingredient in the proof of factorization is the correct relationship of the asymptotic limit to the cut representing the singularities in s_{ab}, a variable which indeed does not even occur in the BCP set due to the nonlinear constraint (2.11).

From the above discussion we see that the total-energy discontinuity of $A_{2\to4}$ is given by the discontinuity in ϕ of the first term in Fig. 4. From (2.10) and (2.13) we easily obtain

$$\Delta_{s_{ab}}A_{2\to4}\sim s_{01}{}^{\alpha_1}s_{12}{}^{\alpha_2}s_{23}{}^{\alpha_3}e^{i\pi\alpha_1}e^{-i\pi\alpha_3}\frac{1}{\Gamma(\alpha_1+1)}I_2(1)I_2(2)$$

$$=R(t_1)\frac{s_{01}{}^{\alpha_1}}{\Gamma(\alpha_1+1)}D(t_1,t_2;\kappa_1)\frac{s_{12}{}^{\alpha_2}}{\Gamma(\alpha_2+1)}D(t_2,t_3;\kappa_2)\frac{s_{23}{}^{\alpha_3}}{\Gamma(\alpha_3+1)}R(t_3)\,, \tag{2.16}$$

which factorizes as asserted.

We now prove factorization for arbitrary multi-Regge amplitudes by induction, using the proof of factorization of $A_{2\to4}$ as a guide. We follow steps analogous to those leading to (2.14) and (2.15) in order to show that the amplitude with n Reggeon exchanges factorizes into the two-Reggeon vertex and the amplitude for $(n-1)$ Reggeon exchanges.

The usual exponentiation substitution gives the asymptotic behavior

$$B_{n+3}\sim(-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}\cdots(-s_{n-1,n})^{\alpha_n}\int_0^\infty\int_0^\infty\cdots\int_0^\infty dy_1dy_2\cdots dy_n\,y_1{}^{-\alpha_1-1}y_2{}^{-\alpha_2-1}\cdots y_n{}^{-\alpha_n-1}$$

$$\times\exp\left(-y_1-y_2-\cdots-y_n+\frac{\alpha_{012}}{\alpha_{01}\alpha_{12}}y_1y_2+\cdots+\frac{\alpha_{n-2,n-1,n}}{\alpha_{n-2,n-1}\alpha_{n-1,n}}y_{n-1}y_n+\cdots+(-1)^n\frac{\alpha_{0\cdots n}}{\alpha_{01}\alpha_{12}\cdots\alpha_{n-1,n}}y_1y_2\cdots y_n\right). \tag{2.17}$$

The generalization of (2.10) is obtained by integrating over y_1 and making successive changes of variables on y_2, y_3, up to y_{n-1} in order to remove the α dependence from the exponential:

$$B_{n+3}\sim(-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}\cdots(-s_{n-1,n})^{\alpha_n}\Gamma(-\alpha_1)$$

$$\times\int_0^\infty\int_0^\infty\cdots\int_0^\infty dy_2dy_3\cdots dy_n\,y_2{}^{-\alpha_2-1}y_3{}^{-\alpha_3-1}\cdots y_n{}^{-\alpha_n-1}e^{-y_2-y_3-\cdots-y_n}X_1{}^{\alpha_1}X_2{}^{\alpha_2}\cdots X_{n-1}{}^{\alpha_{n-1}}\,, \tag{2.18}$$

where X_l is defined recursively by

$$X_l=1-\frac{\alpha_{l-1,\ldots,l+1}}{\alpha_{l-1,l}\alpha_{l,l+1}}y_{l+1}\frac{1}{X_{l+1}}\left(1-\frac{\alpha_{l-1,\ldots,l+2}}{\alpha_{l-1,\ldots,l+1}\alpha_{l+1,l+2}}y_{l+2}\frac{1}{X_{l+2}}\left(\cdots\frac{1}{X_{n-1}}\left(1-\frac{\alpha_{l-1,\ldots,n}}{\alpha_{l-1,\ldots,n-1}\alpha_{n-1,n}}y_n\right)\cdots\right)\right), \tag{2.19}$$

with

$$X_{n-1}=1-\frac{\alpha_{n-2,\ldots,n}}{\alpha_{n-2,n-1}\alpha_{n-1,n}}y_n\,.$$

The nonlinear Gram-determinant conditions are numerous in this case.[18] We have

$$\frac{\alpha_{j,\ldots,l+1}}{\alpha_{j,\ldots,l}\alpha_{l,l+1}}\sim\frac{\alpha_{l-1,\ldots,l+1}}{\alpha_{l-1,l}\alpha_{l,l+1}}=\frac{1}{\kappa_l}\quad(0\le j\le l-1)\,, \tag{2.20}$$

and thus the coefficient of each y_{l+1} in (2.19) is just $\kappa_l{}^{-1}$ and $X_l=1-y_{l+1}/\kappa_l$. Thus, with this naive application of the conditions (2.20), B_6 clearly factorizes into the product of $n-1$ double-Reggeon vertices. However, just as in the case of the $2\to4$ amplitude, we must be careful to take into account phases $e^{n2\pi i}$ in (2.20).

In order to study the phase of (2.18) we break each integration into two ranges, $0\le y_{l+1}\le\kappa_l$ and $\kappa_l\le y_{l+1}<\infty$. With (2.19) and (2.20) the integral in (2.18) can be written in a form generalizing (2.13):

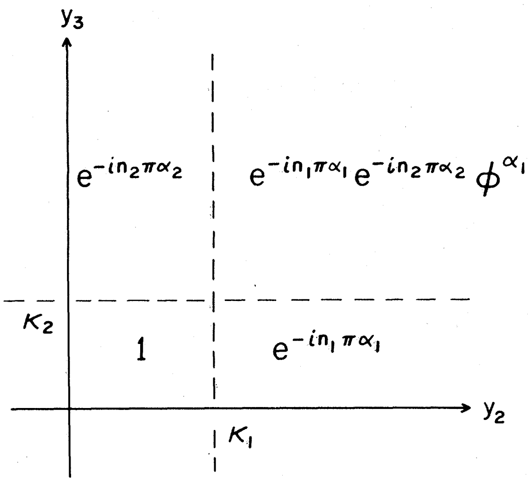

Fig. 5. Phase of the bracket (2.12) where $\kappa_j=(-\kappa_j)e^{in_j\pi}(n_j \text{ odd})$.

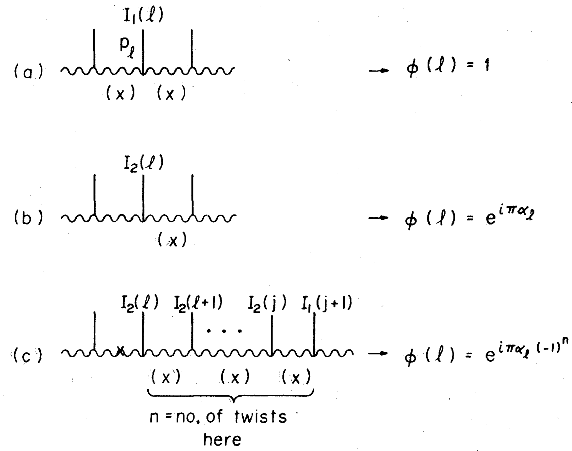

Fig. 6. Phases in Eq. (2.21). Only the ranges of integration that effect $\phi(l)$ are shown. An (x) indicates that a twist may or may not be present.

$$I^{(n-1)}=\sum_{i_l=1,2}[\phi(1)I_{i_1}(1)][\phi(2)I_{i_2}(2)]\cdots[\phi(n-1)I_{i_{n-1}}(n-1)]\,. \tag{2.21}$$

We must now determine each phase $\phi(l)$ which arises from the bracket X_l. Since for $i_l=1$, $0\leq y_{l+1}\leq\kappa_l$, we see clearly from (2.19) that $\phi(l)=1$. For $i_l=2$, we see from (2.19) that the phase will depend upon the ranges of integration of the variables y_j for $j>l+1$. If $y_{j+1}\leq\kappa_j$, then the nested parentheses in (2.19) beyond y_j have no effect. With this in mind, it is not difficult to see that the phase of $\phi(l)$ in the term

$$\cdots I_2(l)I_2(l+1)\cdots I_2(j)I_1(j+1)\cdots$$

of (2.21) is determined by the phase of $\alpha_{l-1,\ldots,j+1}/\alpha_{l-1,l}\alpha_{l,\ldots,j+1}$:

$$\phi(l)=\left(-\frac{\alpha_{l-1,\ldots,j+1}}{\alpha_{l-1,l}\alpha_{l,\ldots,j+1}}\Big/\left|-\frac{\alpha_{l-1,\ldots,j+1}}{\alpha_{l-1,l}\alpha_{l,\ldots,j+1}}\right|\right)^{\alpha_l}. \tag{2.22}$$

It is easy to see using the $+i\epsilon$ prescription that $\phi(l)=e^{+i\pi\alpha_l}$ unless the lth Reggeon is twisted. If the lth Reggeon is twisted, $\phi(l)=e^{i\pi\alpha_l(-1)^n}$, where n is the number of twisted Reggeons from $l+1$ up to and including $j+1$. We summarize these results pictorially in Fig. 6. It is important to note that the phase $\phi(l)$ does not depend on the ranges of integration of variables to the left or on twists in nonadjacent Reggeons to the left.

We are now in a position to carry out the calculation analogous to that leading to (2.14) and (2.15):

$$\begin{aligned}A_{2\to n+1}\sim{}&\Gamma(-\alpha_1)s_{01}{}^{\alpha_1}s_{12}{}^{\alpha_2}\cdots s_{n-1,n}{}^{\alpha_n}\\&\times\Big\{e^{-i\pi\alpha_1}e^{-i\pi\alpha_2}[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+e^{i\pi\alpha_1}e^{i\pi\alpha_2}I_2(1)I_2(2)][\cdots]\\&+\tau_1e^{-i\pi\alpha_2}\left[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+\begin{pmatrix}e^{i\pi\alpha_1}\\e^{-i\pi\alpha_1}\end{pmatrix}e^{i\pi\alpha_2}I_2(1)I_2(2)\right][\cdots]\\&+e^{-i\pi\alpha_1}\tau_2\left[I_1(1)I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\e^{-i\pi\alpha_2}\end{pmatrix}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+e^{i\pi\alpha_1}\begin{pmatrix}e^{i\pi\alpha_2}\\e^{-i\pi\alpha_2}\end{pmatrix}I_2(1)I_2(2)\right][\cdots]\\&+\tau_1\tau_2\left[I_1(1)I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\e^{-i\pi\alpha_2}\end{pmatrix}I_1(1)I_2(2)+e^{-i\pi\alpha_1}I_2(1)I_1(2)+\begin{pmatrix}e^{-i\pi\alpha_1}e^{i\pi\alpha_2}\\e^{i\pi\alpha_1}e^{-i\pi\alpha_2}\end{pmatrix}I_2(1)I_2(2)\right][\cdots]\Big\}.\end{aligned} \tag{2.23}$$

We have suppressed the $I_i(l)$ for $l>2$, since these affect the phases above only through the number of twists contained among the I_2 in the sense of Fig. 6(c). Where this affects (2.23) we have shown the phase for an even (odd) number by the upper (lower) term in a bracket. In (2.23) we show only one choice for the remaining $I_i(l)$ and twists; the sum over all choices is to be imagined.

Grouping the terms in (2.23) pairwise, we arrive at expressions analogous to (2.14) and then (2.15):

$$A_{2\to n+1}\sim[\Gamma(-\alpha_1)\xi_1 s_{01}{}^{\alpha_1}]s_{12}{}^{\alpha_2}s_{23}{}^{\alpha_3}\cdots s_{n-1,n}{}^{\alpha_n}$$
$$\times\Bigg\{e^{-i\pi\alpha_2}\Bigg[I_1(1)I_1(2)+e^{i\pi\alpha_2}I_1(1)I_2(2)+e^{i\pi\alpha_1}I_2(1)I_1(2)+\begin{pmatrix}e^{i\pi\alpha_1}\\ \tau_1\end{pmatrix}e^{i\pi\alpha_2}I_2(1)I_2(2)\Bigg][\cdots]$$
$$+\tau_2\Bigg[I_1(1)I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\ e^{-i\pi\alpha_2}\end{pmatrix}I_1(1)I_2(2)+\tau_1 I_2(1)I_1(2)+\begin{pmatrix}\tau_1 e^{i\pi\alpha_2}\\ e^{i\pi\alpha_1}e^{-i\pi\alpha_2}\end{pmatrix}I_2(1)I_2(2)\Bigg][\cdots]\Bigg\} \tag{2.24}$$
$$=[\Gamma(-\alpha_1)\xi_1 s_{01}{}^{\alpha_1}]\{[\xi_2\Gamma(-\alpha_2)]^{-1}[\xi_2 I_1(1)+(e^{i\pi\alpha_1}e^{-i\pi\alpha_2}+\tau_1\tau_2)I_2(1)]\}$$
$$\times\Bigg\{\Gamma(-\alpha_2)s_{12}{}^{\alpha_2}s_{23}{}^{\alpha_3}\cdots s_{n-1,n}{}^{\alpha_n}\xi_2\Bigg[I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\ \tau_2\end{pmatrix}I_2(2)\Bigg][\cdots]\Bigg\}. \tag{2.25}$$

In (2.25) we have been able to factor out a signatured propagator and a two-Reggeon vertex function (2.6). The remaining bracket is the $2\to n$ amplitude:

$$A_{2\to n}\sim\Gamma(-\alpha_2)s_{12}{}^{\alpha_2}s_{23}{}^{\alpha_3}\cdots s_{n-1,n}{}^{\alpha_n}\Bigg\{e^{-i\pi\alpha_2}[I_1(2)+e^{i\pi\alpha_2}I_2(2)][\cdots]+\tau_2\Bigg[I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\ e^{-i\pi\alpha_2}\end{pmatrix}I_2(2)\Bigg][\cdots]\Bigg\}$$
$$=\Gamma(-\alpha_2)s_{12}{}^{\alpha_2}s_{23}{}^{\alpha_3}\cdots s_{n-1,n}{}^{\alpha_n}\xi_2\Bigg[I_1(2)+\begin{pmatrix}e^{i\pi\alpha_2}\\ \tau_2\end{pmatrix}I_2(2)\Bigg][\cdots].$$

This completes the general proof of factorization by induction.

Factorization of the total-energy discontinuity follows easily from (2.21). From (2.19) and (2.22) we see that $s_{ab}=s_{0,\ldots,n}$ only enters in X_1 and only has an effect for all $y_{l+1}\geq\kappa_l$. We obtain

$$\Delta_{s_{ab}}A_{2\to n+1}\sim s_{01}{}^{\alpha_1}s_{12}{}^{\alpha_2}\cdots s_{n-1,n}{}^{\alpha_n}e^{i\pi\alpha_1}e^{-i\pi\alpha_n}\frac{1}{\Gamma(\alpha_1+1)}I_2(1)I_2(2)\cdots I_2(n-1), \tag{2.26}$$

which has the form (1.5) as asserted.

III. DISCUSSION

The calculations in the DRM of Sec. II have shown that factorization of the full amplitude and its total-energy discontinuity depends crucially on the proper treatment of the cuts in variables dependent by virtue of the nonlinear constraints. It may appear to the reader that these constraints were treated rather cavalierly in Sec. II in that we effectively treated all the variables in B_n as independent, i.e., treated the dimension of space-time as greater than four. However, this was only a convenience for the sake of discussion and not a necessity. Our loose description of specifying the phase of the dependent variables as if they were independent is just a shorthand way of describing where the limit is to be taken in relation to the cuts in the independent variables which arise from the normal thresholds in the dependent variables through (2.20). This is illustrated by an example in Fig. 7. Thus, what is important is where the asymptotic limit is taken in relation to the cuts due to the normal thresholds in these dependent variables. We therefore expect that in a general proof of factorization, it will be crucial to distinguish between various contributions to the singularities in the independent set of variables according to their origin from singularities in the dependent variables, and in the separate terms of (1.1) take different limits

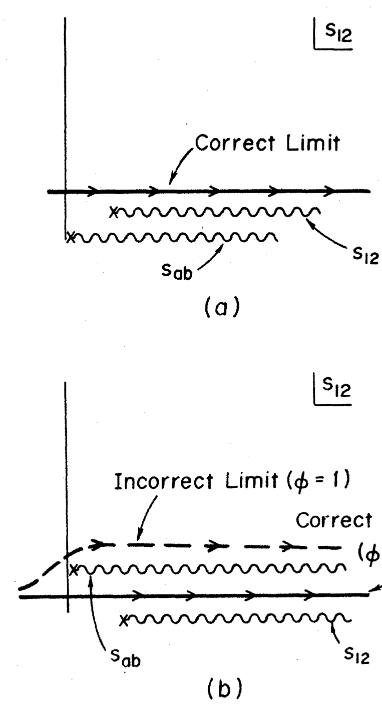

Fig. 7. Cuts in s_{12} due to thresholds in s_{12} and s_{ab} for (a) $B_6(s_{01},s_{12},s_{23},s_{012},s_{123},s_{ab})$ and (b) $B_6(-s_{01},s_{12},-s_{23},-s_{012},-s_{123},s_{ab})$. For visual clarity the cuts have been displaced slightly from the real axis.

in relation to these singularities as illustrated in Fig. 7.

Let us speculate briefly on one possible mechanism by which factorization could be generally true which is suggested by the above discussion. We first suppose that a decomposition of the form (1.1) can be made in which $A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}$ contains asymptotically only right-hand cuts corresponding to cuts in the variables $s_{i,i+1,\ldots,i+j}$. In other words the right-hand singularities in variables corresponding to nonadjacent lines in Fig. 1 (which are *linearly* dependent on the above set) are "flopped over" in forming the "signatured" amplitude when necessary. As already noted, the DRM indeed incorporates the expansion (1.1) and this property of the signatured amplitude. It appears that other known models of Regge behavior – the ladder model in perturbation theory, Gribov's Reggeon calculus,[19] and the van Hove–Durand model[20] – also naturally give an expansion of the form (1.1) with the above property for the signatured amplitudes. Thus, for reasons not yet understood, (1.1) might be true in the multi-Regge asymptotic region even if signature cannot be generalized to production processes.

Factorization may then be derivable if the various contributions to the cuts in the independent variables are properly taken into account as described above. To distinguish between the origin of these various contributions, it may be useful to write a Laplace- (or Fourier-) transform expression which exhibits the left-half-plane analyticity assumed for the signatured amplitude[21]:

$$A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}=\int_0^\infty\int_0^\infty\cdots\int_0^\infty dx_{01}\cdots dx_{n-2,n-1}dx_{012}\cdots dx_{0,\ldots,n-1}f(x_{01},\ldots,x_{0,\ldots,n-1};t_1,\ldots,t_{n-1})$$
$$\times\exp(x_{01}s_{01}+\cdots+x_{n-2,n-1}s_{n-2,n-1}+x_{012}s_{012}+\cdots+x_{0,\ldots,n-1}s_{0,\ldots,n-1}). \quad (3.1)$$

With the changes of variables

$$y_1=x_{01}(-s_{01}),\ldots,y_{n-1}=x_{n-2,n-1}(-s_{n-2,n-1}),$$
$$z_1=\frac{x_{012}}{x_{01}x_{12}},\ldots,z_{12}=\frac{x_{0123}}{x_{01}x_{12}x_{23}},\ldots,z_{1,\ldots,n-2}=\frac{x_{0,\ldots,n-1}}{x_{01}x_{12}\cdots x_{n-2,n-1}},$$

we obtain

$$A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}=(-s_{01})^{-1}(-s_{12})^{-1}\cdots(-s_{n-2,n-1})^{-1}\int_0^\infty\int_0^\infty\cdots\int_0^\infty dy_1\cdots dy_{n-1}dz_1\cdots dz_{1,\ldots,n-2}$$
$$\times g[(-s_{01})^{-1}y_1,\ldots,(-s_{n-2,n-1})^{-1}y_{n-1};z_1,\ldots,z_{1,\ldots,n-2};t_1,\ldots,t_{n-1}]$$
$$\times\exp\left(-y_1-y_2-\cdots-y_n+\frac{s_{012}}{s_{01}s_{12}}z_1y_1y_2+\cdots+(-1)^{n-1}\frac{s_{0,\ldots,n-1}}{s_{01}s_{12}\cdots s_{n-2,n-1}}z_{1,\ldots,n-2}y_1\cdots y_{n-1}\right), \quad (3.2)$$

where g is related to f in an obvious way. A sufficient condition for Regge behavior of the signatured amplitude would be

$$g\sim[(-s_{01})^{-1}y_1]^{-\alpha_1-1}\cdots[(-s_{n-2,n-1})^{-1}y_{n-1}]^{-\alpha_{n-1}-1}h(z_1,\ldots,z_{1,\ldots,n-2};t_1,\ldots,t_{n-1}) \quad (3.3)$$

for $s_{01},\ldots,s_{n-2,n-1}\to\infty$ with the assumption that the limit could be taken inside the integral. Factorization would then follow, if, for example,

$$h=f(z_1;t_1,t_2)\cdots f(z_{n-2};t_{n-2},t_{n-1})\delta(z_{12}-z_1z_2)\delta(z_{23}-z_2z_3)\cdots\delta(z_{1,\ldots,n-2}-z_1\cdots z_{n-2}). \quad (3.4)$$

Combining the conjectural expressions (3.2) to (3.4), we have

$$A_{2\to n}^{\tau_1\tau_2\cdots\tau_{n-1}}\sim(-s_{01})^{\alpha_1}(-s_{12})^{\alpha_2}\cdots(-s_{n-2,n-1})^{\alpha_{n-1}}$$
$$\times\int_0^\infty\int_0^\infty\cdots\int_0^\infty dz_1dz_2\cdots dz_{n-2}f(z_1;t_1,t_2)f(z_2;t_2,t_3)\cdots f(z_{n-2};t_{n-2},t_{n-1})$$
$$\times\int_0^\infty\int_0^\infty\cdots\int_0^\infty dy_1dy_2\cdots dy_{n-1}y_1^{-\alpha_1-1}y_2^{-\alpha_2-1}\cdots y_{n-1}^{-\alpha_{n-1}-1}$$
$$\times\exp\left[-y_1-y_2-\cdots-y_{n-1}+\left(\frac{s_{012}}{s_{01}s_{12}}z_1\right)y_1y_2+\cdots+(-1)^n\left(\frac{s_{0,\ldots,n-1}}{s_{01}s_{12}\cdots s_{n-2,n-1}}z_1\cdots z_{n-2}\right)y_1y_2\cdots y_{n-1}\right]. \quad (3.5)$$

Comparing with (2.17) we see that (3.5) is an integral over the DRM result and

$$\beta(t_j,t_{j+1};\kappa_j)=\int_0^\infty dz_j f(z_j;t_j,t_{j+1})[\Gamma(-\alpha_j)\Gamma(-\alpha_{j+1})]^{-1}V(\alpha_j,\alpha_{j+1};\kappa_j/z_j)\,.$$

If the integrations over z_j in (3.5) introduce no further singularities in κ_j, then the arguments of Sec. II are sufficient to prove factorization as before.

We have not attempted to prove the validity of a representation like (3.5) or (3.1) in models other than the DRM. However, it is at least plausible, since it has been shown (with some reasonable additional assumptions) for $n=3$ in the ladder, Gribov, and van Hove–Durand models by Drummond *et al.*[7] Factorization of the full amplitude in the Gribov model has in fact been shown by Campbell,[22] but the relationship of his derivation to the preceding arguments is not obvious. Finally, we note that in writing (3.1) and applying the results of the preceding discussion, we have essentially taken into account only normal threshold singularities. It may (or may not) be that only such singularities are important in the multi-Regge asymptotic limit in the physical region.

In conclusion, we believe the results of the DRM calculation provide some source of optimism for the validity of factorization of multi-Regge amplitudes and also suggest some directions for further work on the problem.

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[1]K. Bardakci and H. Ruegg, Phys. Rev. 181, 1884 (1969). Factorization is proven for energy variables in their left-half complex planes. As we shall see in Sec. II, we do not always have factorization when some energy variables are in the right-half planes. Here we consider factorization on only the leading Regge trajectory.

[2]For a nice formulation of this proof and references to the original papers, see F. Arbab and J. D. Jackson, Phys. Rev. 176, 1796 (1968). Proof has been written in quotation marks since it involves a "dynamical" assumption which some people regard as tantamount to assuming the result.

[3]For early work on this problem, see I. T. Drummond, Phys. Rev. 140, 1368 (1965); 153, 1565 (1967).

[4]J. W. Dash, Phys. Rev. D 3, 1016 (1971).

[5]We must, of course, be vague about precisely what such amplitudes are since a rigorous formulation has never been given. For more discussion see Dash, Ref. 4.

[6]N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967); referred to hereafter as BCP.

[7]A clear discussion of this prescription for $n=3$ is given by I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Phys. Letters 28B, 676 (1969); Nucl. Phys. B11, 383 (1969). See also Sec. II below.

[8]The use of a gamma function as a "Reggeon propagator" here contains no physics; it only specifies a convenient normalization of the residues.

[9]The importance of the cuts in dependent variables is particularly obvious in the study of the total-energy (s_{ab}) discontinuity of (1.1). The naive phase prescription discussed above (1.3) would completely neglect the phase associated with cuts in s_{ab} leading to an apparent vanishing of this discontinuity.

[10]I. Halliday (see note added in proof to Ref. 4) has suggested the possible crucial role of such cuts.

[11]We define the discontinuity symbol $\Delta_x f(x)=(2\pi i)^{-1}\times[f(x+i\epsilon)-f(x-i\epsilon)]$.

[12]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[13]This has also been shown for two-particle production by C. E. DeTar and J. H. Weis (unpublished); D. Gordon and G. Veneziano (unpublished); C. S. Hsue, B. Hasslacher, and D. K. Sinclair (unpublished); C. L. Jen, K. Kang, P. Shen, and C.-I Tan (unpublished); R. C. Arnold and S. Fenster (unpublished).

[14]For single-particle production this expression is derived in C. E. DeTar, K. Kang, C.-I Tan, and J. H. Weis, Phys. Rev. D 4, 33 (1971).

[15]Actually, since the term of interest in this application is the one with twists in the Reggeons on each end of the chain, D differs from the explicit form (2.8) by the absence of the phase factors. This gives a real (and positive) cross section.

[16]The vanishing of the terms without poles in the momentum transfers which are held fixed has not been shown in general but is expected to follow from Plahte's relations [E. Plahte, Nuovo Cimento 66A, 713 (1970)] when the energies approach infinity in their upper complex planes. For the 2 →2 amplitude, see G. Veneziano, Nuovo Cimento 57A, 196 (1968).

[17]These expressions are initially defined for the $s_{i,i+1}$ in their left-half planes. That we can continue this asymptotic behavior into the right-half planes (i.e., the absence of "Stoke's phenomena") seems very plausible, but to our knowledge an explicit proof has not appeared in the literature.

[18]These can be straightforwardly derived in the multi-Regge region using the BCP variables.

[19]V. N. Gribov, Zh. Eksperim i Teor. Fiz. 56, 818 (1968)

[Soviet Phys. JETP 26, 414 (1968)].
[20]L. van Hove, Phys. Letters 24B, 193 (1967); L. Durand, Phys. Rev. 154, 1537 (1967); R. Blankenbecler and R. L. Sugar, *ibid.* 168, 1597 (1968).
[21]This approach is suggested by the work of Ref. 7.
[22]D. K. Campbell, Phys. Rev. 188, 2471 (1969).

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Vanishing of the Second-Order Correction to the Triangle Anomaly in Landau-Gauge, Zero-Fermion-Mass Quantum Electrodynamics

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We investigate possible second-order radiative corrections to the axial-vector divergence anomaly in spinor electrodynamics. Our procedure is to directly evaluate the shift terms which lead to the anomaly in zero-fermion-mass, Landau-gauge electrodynamics. This version of electrodynamics is chosen because it makes the electron propagator and vertex function finite, so that no infinite renormalizations need be dealt with in the anomaly calculation. We find that the second-order corrections to the anomaly do vanish, in agreement with the work of Adler and Bardeen and with calculations (by different methods) of other authors.

I. INTRODUCTION AND SUMMARY

Ward-identity anomalies have been a subject of considerable theoretical interest during the past few years.[1] The simplest example of an anomaly arises from the axial-vector–vector–vector (A-V-V) triangle graph in spinor electrodynamics, illustrated in Fig. 1. When this diagram is calculated in a manner which preserves vector-current conservation, its axial-vector divergence does not have the expected, naive canonical value. The deviation from the expected value (the so-called anomaly) is compactly summarized by the statement that the axial-vector current in spinor electrodynamics does not satisfy the usual divergence equation

$$\partial^\lambda j_\lambda^5(x) = 2im_0 j^5(x) , \tag{1}$$

where

$$j_\lambda^5(x) = \bar{\psi}(x)\gamma_\lambda\gamma_5\psi(x), \quad j^5(x) = \bar{\psi}(x)\gamma_5\psi(x) ,$$

but rather obeys the modified equation

$$\partial^\lambda j_\lambda^5(x) = 2im_0 j^5(x) + \frac{\alpha_0}{4\pi} F^{\mu\sigma}(x)F^{\nu\tau}(x)\epsilon_{\mu\sigma\nu\tau} , \tag{2}$$

with $F^{\mu\sigma}$ the electromagnetic field-strength tensor. Equation (2) has a direct analog in the σ model, where one finds that the usual PCAC (partially conserved axial-vector current) equations for the neutral axial-vector octet currents also have anomalous electromagnetic modifications proportional to $F^{\mu\sigma}F^{\nu\tau}\epsilon_{\mu\sigma\nu\tau}$, a result which implies remarkable low-energy theorems for the decays $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$.[2] More complicated examples of anomalies arise from box and pentagon diagrams,[3] and they in turn lead to low-energy theorems for such processes as the colliding-beam reaction $\gamma + \gamma \to 3\pi$.[4] Clearly, the significance of the low-energy theorems derived from anomalies depends greatly on whether the coefficient of the anomalous term is given just by the contribution of the relevant lowest-order ring diagram [e.g.,