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Duality and the Lorentz Group*

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We investigate the suggestion that the internal SL(2, C) invariance of the generalized Virasoro model is in fact related to the *same* SL(2, C) group which generates Lorentz transformations on the external momenta. We use fields which transform according to the representations of the homogeneous Lorentz group and derive the n-point Virasoro model.

I. INTRODUCTION

Some time ago Domokos *et al.*[1] obtained the Veneziano model from a consideration of the scattering amplitude in terms of representations of the Lorentz group. Although some clarification and justification of the concepts and methods in that paper would be welcome, its main theme is very suggestive. The authors make the important point that the internal group of invariance in dual models may not be independent from Lorentz transformations which affect the momenta and spins of external particles.

If indeed the internal symmetry group(s) of dual-resonance models can be reinterpreted as the Lorentz group of space-time transformations, new avenues might be opened for the understanding of the duality concept, as well as for the construction of physical models and the classification of their spectrum.

In this paper we do not discuss the implications of this idea. Instead, we give an example for the construction of a dual model, through the use of irreducible representations of the homogeneous Lorentz group, in such a way that the internal SL(2, C) symmetry group is simultaneously identified with the Lorentz transformations of the external momenta.

We use a somewhat unconventional formalism for the Lorentz group which is briefly presented in the Appendix, the full details of which are given in a separate paper.[2] Using fields which transform in a definite way under the Lorentz transformations we construct a dual vertex, which has the same form as the conventional vertex, with the exception of an important property. That is, under a Lorentz transformation *both* the momentum and the internal integration variable get changed. Moreover, taking fields which transform according to the supplementary series of the Lorentz group and then going to the limit of integer points, we find that the vertex also transforms like a representation. Using these facts we construct a dual model in the conventional way[3] which gives the Virasoro-Shapiro[4] model.

II. THE DUAL-RESONANCE MODEL AND THE LORENTZ GROUP

We would like to realize the dual-resonance model by using representations of the Lorentz group in such a way that the SL(2, C) transformations of the Koba-Nielsen variables are identified with the simultaneous Lorentz transformations of the external momenta.

Our notation for the representations of the Lorentz group is explained in the Appendix, so we urge the reader to study it before proceeding further.

We work with the supplementary series of the Lorentz group for which

$$2j_1 = 2j_2 = \rho - 1 \equiv \epsilon\,, \quad -2 < \epsilon < 0\,, \tag{1}$$

where ρ is also given in the Appendix as $\rho = j_1 + j_2^* + 1$. We will be interested in the limit $\epsilon = 0$, which takes us to one of the integer-point representations.

We denote the states as $|\epsilon, z\rangle$, and define the operator-valued functions (see Appendix)

$$\phi_\epsilon^\mu(z) = \langle \phi^\mu | \epsilon, z\rangle, \quad \mu = 0,1,2,3. \tag{2}$$

We demand that $\phi_\epsilon^\mu(z)$ transform under Lorentz transformations like the direct product of two representations, namely,

$$U(\Lambda)\phi_\epsilon^\mu(z)U^\dagger(\Lambda) = (\Lambda^{-1})^\mu_\nu \phi_\epsilon^\nu\left(\frac{az+b}{cz+d}\right)|cz+d|^{2\epsilon}\,. \tag{3}$$

It is due to the extra Λ^{-1} that we will be able to unify the transformations of the momenta and the z variables. $\Lambda_{\mu\nu}$ is the 4×4 representation of the Lorentz transformation defined by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We also define the Hermitian conjugate of $\phi_\epsilon^\mu(z)$,

$$\phi_\epsilon^{\mu\,\dagger}(z) = (\langle \phi^\mu | \epsilon, z\rangle)^\dagger\,, \tag{4}$$

and demand that the following commutation relation be satisfied:

$$[\phi_\epsilon^\mu(z), \phi_\epsilon^{\nu\,\dagger}(z')] = G_\epsilon(z', z) g^{\mu\nu}\,. \tag{5}$$

That is, the right-hand side of Eq. (5) is the product of the metrics of the two representations. $G_\epsilon(z', z)$ is defined in the Appendix, Eq. (A17), and is given as

$$G_\epsilon(z', z) = \langle \epsilon, z' | G | \epsilon, z\rangle = \frac{\Gamma(-\epsilon)}{\pi\Gamma(1+\epsilon)}|z'-z|^{2\epsilon}. \tag{6}$$

We use the metric $g^{\mu\nu} = (-1, 1, 1, 1)$. Notice that as ϵ approaches zero, $G_\epsilon(z', z)$ diverges:

$$G_\epsilon(z', z) \underset{\epsilon\to 0}{\longrightarrow} \frac{-1}{\pi\epsilon}[1 + 2\epsilon \ln|z'-z| + O(\epsilon^2)]\,. \tag{7}$$

We can also define the field $Q_\epsilon^\mu(z) = \phi_\epsilon^\mu(z) + \phi_\epsilon^{\mu\,\dagger}(z)$ which obeys the commutation relation

$$[Q_\epsilon^\mu(z), Q_\epsilon^\nu(z')] = 0\,. \tag{8}$$

The infinitesimal form of Eq. (3), gives the commutators

$$[J^{\alpha\beta}, \phi_\epsilon^\mu(z)] = -\Sigma_{\mu\nu}^{\alpha\beta}\phi_\epsilon^\nu(z) + J^{\alpha\beta}\left(z, \frac{\partial}{\partial z}\right)\phi_\epsilon^\mu(z)\,, \tag{9a}$$

where the differential operators $J^{\alpha\beta}(z, \partial/\partial z)$ are given in Refs. 2 and 5. They will not be used explicitly in what follows. $\Sigma_{\mu\nu}^{\alpha\beta}$ is given as

$$\Sigma_{\mu\nu}^{\alpha\beta} = -i(g_\mu^\alpha g_\nu^\beta - g_\nu^\alpha g_\mu^\beta)\,. \tag{9b}$$

The generators of the Lorentz group which satisfy Eqs. (3) and (9) can be written in terms of $\phi_\epsilon^\mu(z)$ and $\phi_\epsilon^{\nu\,\dagger}(z)$ as follows:

$$J^{\alpha\beta} = \int d^2z\, d^2z'\, \phi_\epsilon^{\mu\,\dagger}(z)[J_\epsilon^{\alpha\beta}(z, z')]_{\mu\nu}\phi_\epsilon^\nu(z')\,, \tag{10}$$

where

$$[J_\epsilon^{\alpha\beta}(z, z')]_{\mu\nu} = \Sigma_{\mu\nu}^{\alpha\beta}\langle \epsilon, z' | G^{-1} | \epsilon, z\rangle - g_{\mu\nu}\langle \epsilon, z' | J^{\alpha\beta}G^{-1} | \epsilon, z\rangle\,. \tag{11}$$

The matrix elements of $J^{\alpha\beta}G^{-1}$ can be evaluated by introducing $1 = \int d^2z\, |z\rangle\langle z|$ and using the fact that $J^{\alpha\beta}$ on $|z\rangle$ are differential operators given in Refs. 2 and 5. We do not do it here, since it will not be needed explicitly in this paper.

Equations (2)–(11) could also be realized in terms of an infinite number of harmonic oscillators as in previous works.[3] Although it is not necessary to introduce an infinite number of degrees of freedom we wish to present it here in order to make contact with other formalisms in the literature. This is done by introducing a complete set of states which characterize a representation of the Lorentz group

$$\sum_n |n\rangle\langle n| = 1\,. \tag{12}$$

The states $|n\rangle$ could be chosen as the canonical representation of the Lorentz group $|\epsilon, jm\rangle$, where $j(j+1)$ is the eigenvalue of $\vec{J}^2$ = angular momentum squared, and m is the eigenvalue of J_3. A different set for $|n\rangle$ could be the set $|\epsilon, m, n\rangle$, where m, n are eigenvalues of X_3 and $X_3^\dagger$ (the operators X_i are introduced in the Appendix). Using Eq. (12) we rewrite Eq. (2) as

$$\phi_\epsilon^\mu(z) = \langle \phi^\mu | \epsilon, z\rangle = \sum_n \langle \phi^\mu | n\rangle\langle n | \epsilon, z\rangle\,, \tag{13}$$

and then identify the harmonic-oscillator operator

$$a_n^\mu = \langle \phi^\mu | n\rangle\,. \tag{14}$$

Similarly,

$$\phi_\epsilon^\mu(z)^\dagger = (\langle \phi^\mu | \epsilon, z\rangle)^\dagger = \sum_n a_n^{\mu\,\dagger}\langle n | \epsilon, z\rangle^*\,. \tag{15}$$

Thus, writing

$$[a_n^\mu, a_m^{\nu\,\dagger}] = g^{\mu\nu}\delta_{mn}\,, \tag{16}$$

we get

$$[\phi_\epsilon^\mu(z), \phi_\epsilon^{\nu\,\dagger}(z')] = \sum_n \langle n|\epsilon, z'\rangle^* \langle n|\epsilon, z\rangle g^{\mu\nu}$$
$$= (|\epsilon, z\rangle)^\dagger\, |\epsilon, z\rangle\, g^{\mu\nu}$$
$$= \langle \epsilon, z'\,|G|\epsilon, z\rangle\, g^{\mu\nu}$$
$$= G_\epsilon(z', z) g^{\mu\nu}, \qquad (17)$$

which is the same as Eq. (5).

The generators $J^{\alpha\beta}$ could also be written in terms of a_n^μ and $a_n^{\mu\,\dagger}$ as follows:

$$J^{\alpha\beta} = \Sigma a_n^{\mu\,\dagger}(J^{\alpha\beta})_{\mu n,\, \nu m} a_m^\nu, \qquad (18)$$

where

$$(J^{\alpha\beta})_{\mu n,\, \nu m} = \Sigma_{\mu\nu}^{\alpha\beta}\delta_{nm} - g_{\mu\nu}\langle n|J^{\alpha\beta}|m\rangle. \qquad (19)$$

$\langle n|J^{\alpha\beta}|m\rangle$ can be evaluated according to the complete set of states chosen.

We would like to mention that the set $|n\rangle$ contains a state (e.g., $j=0$ and $m=0$ in the case of canonical basis) for which $\langle n|\epsilon, z\rangle$ diverges like $\epsilon^{-1/2}$ as ϵ approaches zero; therefore, $\phi_\epsilon^\mu(z)$ also diverges like $\epsilon^{-1/2}$. This is also obvious from Eqs. (5) and (7) (compare also with Clavelli and Ramond[3]). This infinity will be canceled by a zero when we build our dual model, and take the limit properly. In what follows we will not use the formalism in terms of an infinite number of harmonic oscillators since it is not necessary. We will only need Eqs. (2)–(7).

To obtain a dual model we construct a vertex in the standard way. We do not have a particular reason for choosing this vertex except for the fact that it is similar to the one that has "worked" in the literature. Our vertex will have an important additional property, namely, the momentum will transform simultaneously with the Koba-Nielsen variable under an SL(2, C) transformation. We define the vertex as

$$V_\epsilon(k, z) = \exp[i\sqrt{\pi}k\cdot\phi_\epsilon^\dagger(z_+)]\exp[i\sqrt{\pi}k\cdot\phi_\epsilon(z_-)]e^{k^2/2\epsilon} \qquad (20a)$$
$$= :\exp[i\sqrt{\pi}\,Q_\epsilon(z)\cdot k]:e^{k^2/2\epsilon}, \qquad (20b)$$
$$\xrightarrow[\epsilon\to 0]{} \exp[i\sqrt{\pi}\,Q_\epsilon(z)\cdot k], \qquad (20c)$$

where $z_\pm = z \pm \epsilon$ and $Q_\epsilon^\mu(z)$ has been defined in Eq. (8). The reason for the point splitting ($z_\pm$) is to have well-defined unambiguous limits as we let $\epsilon\to 0$. Notice that as $\epsilon\to 0$, $z_+ \to z_- \to z$ simultaneously with the representation approaching the integer point. The factor $e^{k^2/2\epsilon}$ also is necessary to obtain a well-defined amplitude. By using Eq. (5) and $e^Ae^B = e^Be^Ae^{[A,B]}$ when $[A,B]$ is a c number, we can verify that when $z_+ = z_- = z$ (or as $\epsilon\to 0$),

$$[V_\epsilon(k_1, z_1),\, V_\epsilon(k_2, z_2)] = 0, \qquad (21)$$

unlike the vertices used so far in the literature. This commutation property is also obvious from Eqs. (8) and (20c). Due to this property the dual model we construct will be completely symmetric rather than just cyclically symmetric. [We remark that even in the SU(1, 1) case of Ref. 3, if one starts with the principal series rather than the analytic representations of SU(1, 1), then Eq. (21) will be true also in that case.]

Under a Lorentz transformation the vertex transforms as follows:

$$U(\Lambda)V_\epsilon(k, z)U^\dagger(\Lambda) = \exp[i\sqrt{\pi}k'\cdot\phi_\epsilon^\dagger(z_+')\,|cz_+ + d|^{2\epsilon}]\,\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon(z_-')\,|cz_- + d|^{2\epsilon}]\,e^{k^2/2\epsilon}, \qquad (22)$$

where

$$k_\mu' = \Lambda_{\mu\nu}k^\nu, \quad z' = \frac{az+b}{cz+d}. \qquad (23)$$

We see that the vertex does *not* transform like a representation unless $\epsilon = 0$. To take the limit $\epsilon\to 0$ we rewrite the right-hand side of Eq. (22) as

$$\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon^\dagger(z_+')(|cz_+ + d|^{2\epsilon} - 1)]\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon^\dagger(z_+')]$$
$$\times\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon(z_-')(|cz_- + d|^{2\epsilon} - 1)]\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon(z_-')]e^{k^2/2\epsilon}$$

and then commute the second and third terms to obtain

$$\exp[i\sqrt{\pi}k'\cdot\phi_\epsilon^\dagger(z_+')(|cz_+ + d|^{2\epsilon} - 1)]\ \exp[i\sqrt{\pi}k'\cdot\phi_\epsilon(z_-')(|cz_- + d|^{2\epsilon} - 1)]$$
$$\times\exp[\pi k^2 G_\epsilon(z_+', z_-')(|cz_- + d|^{2\epsilon} - 1)]V_\epsilon(k', z').$$

Now taking the limit $\epsilon\to 0$ and remembering that ϕ_ϵ diverges like $\epsilon^{-1/2}$ and G_ϵ like $(-\pi\epsilon)^{-1}$ [Eq. (7)], we obtain

$$\lim_{\epsilon\to 0} U(\Lambda)V_\epsilon(k, z)U^\dagger(\Lambda) = |cz + d|^{-2k^2}\lim_{\epsilon\to 0} V_\epsilon(k', z'). \qquad (24)$$

Thus as $\epsilon\to 0$ the vertex transforms like a field in the representation $2j_1 = 2j_2 = -k^2$. Notice that as promised the momentum and the z variable change simultaneously under a Lorentz transformation.

The amplitude will be constructed in the standard way by calculating the vacuum expectation value of a string of these vertices. Defining the vacuum as $\phi_\mu(z)\,|0\rangle = 0$, it is a trivial exercise to evaluate

$$\langle 0\,|\,V_\epsilon(k_1,z_1)V_\epsilon(k_2,z_2)\cdots V_\epsilon(k_n,z_n)\,|\,0\rangle = \prod_{i<j}^{n} \exp[-\pi k_i\cdot k_j\, G_\epsilon(z_i,z_j)]\, e^{nk^2/2\epsilon}. \tag{25}$$

In the limit of $\epsilon\to 0$, we use momentum conservation $\sum_{i=1}^{n} k_i = 0$, then the right-hand side of Eq. (25) becomes

$$\prod_{i<j}^{n} |z_i - z_j|^{2k_i\cdot k_j}. \tag{26}$$

To form an invariant we must multiply by a kernel and integrate over an invariant volume element. Equation (24) tells us what the kernel should be, namely,

$$R(\lambda_l)\prod_i |z_i - z_{i+1}|^{k^2},$$

where $R(\lambda_l)$ is an invariant function of the z_j and therefore depends only on the cross ratios of quadruples of z_j; that is, if $l=(i,j,m,n)$, then

$$\lambda_l = \frac{z_i - z_m}{z_j - z_m}\,\frac{z_j - z_n}{z_i - z_n}.$$

In this paper R will be taken equal to 1, but special forms of R may lead to different dual models.

The invariant volume element is

$$dV(z) = \frac{d^2z_1 d^2z_2\cdots d^2z_n}{|z_1-z_2|^2|z_2-z_3|^2\cdots|z_n-z_1|^2}(d\omega)^{-1},$$

where

$$d\omega = \frac{d^2z_a\, d^2z_b\, d^2z_c}{|z_a-z_b|^2|z_b-z_c|^2|z_c-z_a|^2}.$$

z_a, z_b, and z_c are any three of the n variables $z_1\cdots z_n$. We need to divide by $d\omega$ to avoid an infinite volume element.

Thus the amplitude becomes

$$A(k_1,\ldots,k_n) = \lim_{\epsilon\to 0}\left\langle 0\left|\int (d\omega)^{-1}\prod_i d^2z_i\,|z_i - z_{i+1}|^{k^2-2}\, V_\epsilon(k_i,z_i)\right|0\right\rangle \tag{27}$$

$$= \int (d\omega)^{-1}\left(\prod_i d^2z_i\,|z_i - z_{i+1}|^{k^2-2}\right)\prod_{i<j}^{n}|z_i - z_j|^{2k_i\cdot k_j}. \tag{28}$$

It is interesting to notice the symmetry and invariance properties of this model when $k^2=\alpha_0=2$. Under any permutation of the momenta not labeled by a, b, and c the amplitude is invariant, because we can freely change and rename the z variables and commute vertices [Eq. (21)]. If the permutation involves any or all the indices a, b, and c, then we can always apply an SL(2, C) transformation to remap the variables so that z_a, z_b, and z_c will be associated with k_a, k_b, and k_c, respectively, and we can rename the other variables and commute vertices freely. However, in this case when we apply the SL(2, C) transformation we have to *transform the momenta also*. Therefore, supposing we had made some permutation (p), we have

$$A(k_{p1},k_{p2},k_{p3},\ldots,k_{pn}) = A(k_1',k_2',\ldots,k_n'), \tag{29}$$

where k_i' are the transformed momenta. However, because of the commutation relation of Eq. (5) all momenta occur in the final form of the amplitude as Lorentz scalars. Hence, we can write

$$A(k_{p1},k_{p2},\ldots,k_{pn}) = A(k_1',k_2',\ldots,k_n') = A(k_1,k_2,\ldots,k_n). \tag{30}$$

Thus the invariance of the amplitude under any permutation is equivalent to its invariance under any Lorentz transformation. The factorization of this model has been studied by Yoshimura[6] and Del Giudice and Di Vecchia.[7]

III. CONCLUSION

We have constructed a model in which duality is closely connected to the Lorentz group of space-time transformations. In fact, in this model the internal invariance group associated with duality appears to be nothing but the homogeneous Lorentz group [Eq. (30)]. Although this does not explain duality it may give us a clue as to its meaning.

The methods used in this paper may be useful for the construction of new dual models. The interpretation of the SL(2, C) group of duality transformations as the Lorentz group of space-time transformations will hopefully lead to restrictions in the construction of models with spin and be useful for the analysis of the spectrum of dual models.

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APPENDIX: UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

We present a short summary of the so-called z basis,[5] in a new formalism which simplifies and clarifies the treatment of the unitary representations of the Lorentz group. The details of this formalism are given elsewhere.[2]

We label the states in the z basis by the two Casimir operators of the Lorentz group which have eigenvalues $j_1(j_1+1)$ and $j_2(j_2+1)$ and the Z and $\overline{Z}$ operators which have complex eigenvalues z and z^*. We denote these states by $|j_1 j_2 z\rangle$, such that

$$\{C_1, C_2, Z, \overline{Z}\}|j_1 j_2 z\rangle = \{j_1(j_1+1),\ j_2(j_2+1),\ z, z^*\}|j_1 j_2 z\rangle . \tag{A1}$$

The Z and $\overline{Z}$ operators are given in terms of the generators of the Lorentz group as follows. Let $J^{\alpha\beta}$ be the generators of Lorentz transformations; then the rotation generators J_i and boost generators K_i are given as

$$\begin{aligned}(J_1, J_2, J_3) &= (J^{23}, J^{31}, J^{12}) ,\\ (K_1, K_2, K_3) &= (J^{10}, J^{20}, J^{30}) .\end{aligned} \tag{A2}$$

For a unitary representation, J_i and K_i are Hermitian. From these we define the non-Hermitian operators

$$\vec{X} = \tfrac{1}{2}(\vec{J} + i\vec{K}), \quad \vec{X}^\dagger = \tfrac{1}{2}(\vec{J} - i\vec{K}) , \tag{A3}$$

which satisfy

$$\begin{aligned}&[X_i, X_j^\dagger] = 0 ,\\ &[X_i, X_j] = i\epsilon_{ijk} X_k ,\\ &[X_i^\dagger, X_j^\dagger] = i\epsilon_{ijk} X_k^\dagger ,\end{aligned} \tag{A4}$$

and the Casimir operators are given as

$$\begin{aligned}&\vec{X}\cdot\vec{X} = j_1(j_1+1) ,\\ &\vec{X}^\dagger\cdot\vec{X}^\dagger = j_2(j_2+1) ,\end{aligned} \tag{A5a}$$

$$j_1^*(j_1^*+1) = j_2(j_2+1) . \tag{A5b}$$

In Eq. (A5b) if $j_1^* = -j_2 - 1$ we are in the principal series; if $j_1 = j_1^* = j_2$ we are in the supplementary series or integer points. Then Z and $\overline{Z}$ are given within a representation (j_1, j_2) as

$$\begin{aligned}Z &= (j_1 + 1 + X_3)(X_1 + iX_2)^{-1} ,\\ \overline{Z} &= (-j_2 - 1 + X_3^\dagger)(X_1^\dagger - iX_2^\dagger)^{-1} .\end{aligned} \tag{A6}$$

They satisfy

$$[Z, \overline{Z}] = 0, \quad [Z, X_1 + iX_2] = 1, \quad [\overline{Z}, X_1^\dagger - iX_2^\dagger] = -1 , \tag{A7}$$

so that on functions of Z and $\overline{Z}$, $f(Z, \overline{Z})$, the operators $X_1 + iX_2$ and $X_1^\dagger - iX_2^\dagger$ act as differentials $-\partial/\partial Z$ and $\partial/\partial\overline{Z}$, respectively.

Under an SL(2, C) transformation defined by the 2×2 complex matrix

$$\Lambda = e^{-i\vec{\alpha}\cdot\vec{\sigma}/2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (ad - bc = 1) , \tag{A8}$$

Z and $\overline{Z}$ transform as

$$\begin{aligned}U^\dagger(\Lambda) Z U(\Lambda) &= \frac{aZ + b}{cZ + d} ,\\ U^\dagger(\Lambda)\overline{Z} U(\Lambda) &= \frac{a^*\overline{Z} + b^*}{c^*\overline{Z} + d^*} ,\end{aligned} \tag{A9}$$

where $U(\Lambda) = \exp[-i\vec{\alpha}\cdot\vec{X} - i\vec{\alpha}^*\cdot X^\dagger]$.

We also find that $\overline{Z}$ is related to the Hermitian conjugate of Z as follows:

$$Z^\dagger = G^{-1}\overline{Z} G, \quad \overline{Z}^\dagger = G^{-1} Z G , \tag{A10}$$

where

$$G = G^\dagger = (X_1 + iX_2)^{-\rho}\,(X_1^\dagger - iX_2^\dagger)^{-\rho} , \tag{A11}$$

with $\rho = j_1 + j_2^* + 1 = j_1^* + j_2 + 1$. The Hermitian and positive-definite operator G will be interpreted as a metric operator in Hilbert space.

We define *covariant* and *contravariant* states as follows:

Covariant ket state,

$$U(\Lambda)|j_1 j_2 z\rangle = (cz + d)^{2j_1}(c^*z^* + d^*)^{2j_2}\left|j_1 j_2 \frac{az + b}{cz + d}\right\rangle . \tag{A12}$$

Contravariant bra state,

$$\langle j_1 j_2 z | U^\dagger(\Lambda) = (cz+d)^{-2j_1-2}(c^*z^*+d^*)^{-2j_2-2}\left\langle j_1 j_2 \frac{az+b}{cz+d}\right| . \tag{A13}$$

They form a complete, orthogonal set,

$$\langle j_1 j_2 z | j_1 j_2 z'\rangle = \delta^{(2)}(z-z'), \qquad \int d^2z\, |j_1 j_2 z\rangle\langle j_1 j_2 z| = 1 . \tag{A14}$$

We remark that the ket and bra states transform differently, and because of the labeling of the states by non-Hermitian operators the bra state is *not* related to the ket state by Hermitian conjugation. This is an unfamiliar situation in physics. The extension of the Dirac bra and ket formalism through the distinction of covariant and contravariant states related by a metric operator will be found in Ref. 2. We find that the contravariant bra states are related to the covariant ket states through Hermitian conjugation plus the metric operator G^{-1} introduced in Eqs. (A10)–(A11),

$$\langle j_1 j_2 z| = (|j_1 j_2 z\rangle)^\dagger G^{-1}, \tag{A15a}$$

while the contravariant kets (covariant bras) are obtained from covariant kets (contravariant bras) through the application of G^{-1} (G):

$$G^{-1}|j_1 j_2 z\rangle = (\langle j_1 j_2 z|)^\dagger, \quad \langle j_1 j_2 z|G = (|j_1 j_2 z\rangle)^\dagger . \tag{A15b}$$

Here $(\langle j_1 j_2 z|)^\dagger$ is defined as a contravariant ket state and $(|j_1 j_2 z\rangle)^\dagger$ is defined as a covariant bra state. They transform according to $(-j_2^*-1, -j_1^*-1)$ and (j_2^*, j_1^*), respectively, as obtained from Eqs. (A12) and (A13) by Hermitian conjugation.[8] (The notation for these states is somewhat different in Ref. 2, and the reader who tries to compare them should not be confused at this point.) It can be shown that the application of G to the states is equivalent to the application of the integral operator

$$\begin{aligned}\langle j_1 j_2 z| &= (|j_1 j_2 z\rangle)^\dagger G^{-1} \\ &= \int d^2u\, G^{-1}{}_{j_1 j_2}(z,u)(|j_1 j_2 u\rangle)^\dagger , \\ (|j_1 j_2 z\rangle)^\dagger &= \langle j_1 j_2 z|G \\ &= \int d^2u\, G_{j_1 j_2}(z,u)\langle j_1 j_2 u| ,\end{aligned} \tag{A16}$$

where the integration is over all the complex plane, i.e., if $u = x+iy$, $d^2u = dx\,dy$, or $d^2u = \frac{1}{2} i\, du\, du^*$. Then

$$\begin{aligned}G_{j_1 j_2}(z,u) &= \langle j_1 j_2 z|G|j_1 j_2 u\rangle \\ &= \frac{\Gamma(1-\rho)}{\pi\Gamma(\rho)}|z-u|^{2\rho-2} , \\ G^{-1}{}_{j_1 j_2}(z,u) &= \langle j_1 j_2 z|G^{-1}|j_1 j_2 u\rangle \\ &= \frac{\Gamma(1+\rho)}{\pi\Gamma(-\rho)}|z-u|^{-2\rho-2} , \\ \delta^2(z-z') &= \int d^2u\, G_{j_1 j_2}(z,u) G^{-1}{}_{j_1 j_2}(u,z') .\end{aligned} \tag{A17}$$

For ρ = integer, these have well-defined limits in the sense of distributions as discussed in Ref. 2.

Thus the metric G turns out to be the same as the intertwining operator introduced by Gelfand *et al.*[5] except for the slightly different coefficients of Γ functions. Using these, we can rewrite the completeness relation (A14) in the following new forms:

$$\begin{aligned}1 &= \int d^2z_1 d^2z_2\, |j_1 j_2 z_1\rangle G^{-1}{}_{j_1 j_2}(z_1,z_2)(|j_1 j_2 z_2\rangle)^\dagger \\ &= \int d^2z_3 d^2z_4 (\langle j_1 j_2 z_3|)^\dagger G_{j_1 j_2}(z_3,z_4)\langle j_1 j_2 z_4| .\end{aligned} \tag{A18}$$

Now consider a bra vector $\langle f|$ with contravariant-basis bra vectors and covariant "components" $f(z)$, and similarly a ket vector $|g\rangle$ with contravariant-basis ket vectors and covariant "components" $g^*(z)$.

$$\langle f| = \int \langle j_1 j_2 z| f(z) d^2z , \qquad |g\rangle = \int (\langle j_1 j_2 z|)^\dagger g^*(z) d^2z . \tag{A19a}$$

These are in analogy to $\vec{f}^\dagger = \sum_n e^{\dagger n} f_n$ and $\vec{g} = \sum_m e^m g_m^*$ in the discrete case. The "components" are given by

$$f(z) = \langle f|j_1 j_2 z\rangle , \qquad g^*(z) = (|j_1 j_2 z\rangle)^\dagger|g\rangle = \langle g|j_1 j_2 z\rangle^* , \tag{A19b}$$

and they transform according to (j_1, j_2) and (j_2^*, j_1^*), respectively. A Hilbert space can be constructed in the space of these functions, with the following scalar product obtained from (A18):

$$(f,g) = \langle f|g\rangle = \int d^2z_1 d^2z_2 f(z_1) G^{-1}{}_{j_1 j_2}(z_1,z_2) g^*(z_2) \tag{A20}$$

in analogy with

$$\vec{f}^\dagger\cdot\vec{g} = \sum_{n,m} e^{\dagger n}\cdot e^m f_n g_m^* = \sum_{n,m} f_n G^{nm} g_m^*$$

in the discrete case.

Since G^{-1} is positive-definite [Eq. (A11)], this is a unified positive-definite scalar product for all the unitary representations of the Lorentz group, namely, the principal series, the supplementary series, and the integer-points cases. As discussed in Ref. 2 it reduces in each case to the special forms given by Gelfand *et al.*[5]

(1) Principal series ($\rho = 0$):

$$(f, g) = \int d^2z\, f(z) g^*(z).$$

(2) Supplementary series ($0 < |\rho| < 1$):

$$(f, g) = \frac{\Gamma(1+\rho)}{\pi\Gamma(-\rho)} \int d^2z_1 d^2z_2 f(z_1) \, |z_1 - z_2|^{-2\rho-2} g^*(z_2).$$

(3) Integer points ($\rho = n = 1, 2, 3, \ldots$):

$$(f, g) = (-1)^n \int d^2z\, f^{[n,n]}(z) g^*(z).$$

(4) Integer points ($\rho = n = -1, -2, \ldots$):

$$(f, g) = \frac{2(-1)^n}{\pi[\Gamma(-n)]^2} \int d^2z_1 d^2z_2 f(z_1) \times |z_1 - z_2|^{-2n-2} \ln|z_1 - z_2| g^*(z_2),$$

where

$$f^{[n,n]}(z) = \frac{\partial^{2n}}{\partial z^n \partial z^{*n}} f(z). \tag{A21}$$

For the integer cases 3 and 4 we must restrict ourselves to subspaces of functions which come out naturally in our formalism and which are identical to the ones found by Gelfand *et al.*[5]

In case 4 we must restrict ourselves to the subset of functions which satisfy

$$\int z^k z^{*l} f(z) d^2z = 0,$$

for $k, l \leq -\rho - 1$, when $\rho = -1, -2, \ldots$.

In this paper we are mainly concerned with case 3. Similarly to case 4, in order to avoid a degenerate scalar product we must restrict ourselves to a subspace of functions such that for $\rho = n$ the nth derivatives of the functions do not vanish, $f^{[n,n]}(z) \neq 0$. Then all polynomials of the form

$$p(z) = \sum_{k,l=0}^{n-1} a_{kl} z^k z^{*l} \tag{A22}$$

should be excluded. Note that for $\rho = n$ the set of such polynomials (E_n) forms an invariant subset under Lorentz transformations, thus forming a nonunitary finite-dimensional representation. The subspace of functions which form an infinite-dimensional unitary representation for $\rho = n = 1, 2, \ldots$ is the set of functions obtained from homogeneous functions and defined up to a polynomial in E_n (see Gelfand, Graev, and Vilenkin[5]). We denote this set by F_n. Thus, if $f(z) \in F_n$ and $p(z) \in E_n$, then $[f(z) + p(z)] \in F_n$. Let us denote the part of $f(z)$ which contributes to the scalar product in case 3 as $\tilde{f}(z)$. Then we can write

$$\tilde{f}(z) = \sum_{k,l=n}^{\infty} a_{kl} z^k z^{*l} \tag{A23}$$

and

$$f(z) = \tilde{f}(z) + p(z), \tag{A24}$$

where $p(z)$ is *any* polynomial in E_n. From the way it is defined, the only part which is "important" in $f(z)$ is obviously $\tilde{f}(z)$. Under a Lorentz transformation for which a function $g(z)$ transforms like

$$T_{\Lambda^{-1}} g(z) \equiv \langle g | U(\Lambda) | z \rangle = |cz + d|^{2\rho-2} g\left(\frac{az+b}{cz+d}\right), \tag{A25}$$

$p(z)$ goes into another polynomial, but $\tilde{f}(z)$ does not transform into a function of the form (A23). Therefore, to extract the "important" part of $T_{\Lambda^{-1}}\tilde{f}(z)$ we must subtract a polynomial. In other words, the set of functions of the form $\tilde{f}(z)$ as in (A23) is not an invariant set, and an additional "gauge" transformation (subtracting a polynomial) is needed to make it invariant. This is in analogy to the case of the transverse electromagnetic potential A_μ^T which is covariant only with an additional gauge transformation. Thus, in the case of the integer-point representations, it is possible to *choose* a gauge, just as we do in electrodynamics. If we choose the "gauge" such that we extract only $\tilde{f}(z)$ out of $f(z)$, then, as described above, we will need an additional gauge transformation to obtain a covariant $f(z)$. Instead, we could choose the "gauge" as follows.

Each term in Eq. (A22) is a linearly independent polynomial which is annihilated by the metric G^{-1} at integer points $\rho = 1, 2, \ldots$.

$$\int p(z') \delta^{[n,n]}(z' - z) d^2z' = p^{[n,n]}(z) = 0. \tag{A26}$$

Thus for $2j_1 + 1 = 2j_2 + 1 = \rho = n$, there are $n^2 = (2j_1 + 1) \times (2j_2 + 1)$ linearly independent polynomials which satisfy Eq. (A26). Using these polynomials we can *choose* the gauge so that we can write $f(z) \in F_n$ as

$$f(z) = \sum_{k,l=0}^{n-1} z^k z^{*l} \phi_{kl}(z), \tag{A27}$$

where each ϕ_{kl} can be written in the form

$$\phi_{kl} = \sum_{m,n=0}^{\infty} (b^{kl})_{mn} z^m z^{*n}.$$

Under a Lorentz transformation the form of $f(z)$ in (A27) will remain invariant, if the functions ϕ_{kl} transform into each other and acquire no extra factors of $(cz+d)$ and $(c^*z^*+d^*)$. In particular let us consider the case of the $(j_1=\frac{1}{2}, j_2=\frac{1}{2})$ or the $\rho=2$ representation. This is the vector representation and plays an important role in this paper. We can write

$$\langle f\,|\,\rho=2;\ z\rangle = f(z) = zz^*\phi_{11}(z) + z^*\phi_{10}(z) + z\phi_{01}(z) + \phi_{00}(z) \tag{A28}$$

or

$$f(z) = (z^*\ \ 1)\begin{pmatrix}\phi_{11}(z) & \phi_{10}(z)\\ \phi_{01}(z) & \phi_{00}(z)\end{pmatrix}\begin{pmatrix} z\\ 1\end{pmatrix}. \tag{A29}$$

Applying a Lorentz transformation we get

$$T_{\Lambda^{-1}} f(z) = \langle f\,|\,U(\Lambda)\,|\,z\rangle = (cz+d)(c^*z^*+d^*)f(z')\,, \tag{A30}$$

where

$$z' = \frac{az+b}{cz+d}.$$

Using Eq. (A29) we can rewrite Eq. (A30) as

$$T_{\Lambda^{-1}} f(z) = (z^*\ \ 1)\Lambda^\dagger\begin{pmatrix}\phi_{11}(z') & \phi_{10}(z')\\ \phi_{01}(z') & \phi_{00}(z')\end{pmatrix}\Lambda\begin{pmatrix} z\\ 1\end{pmatrix}. \tag{A31}$$

This is because

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} z\\ 1\end{pmatrix} = \begin{pmatrix}(az+b)/(cz+d)\\ 1\end{pmatrix}(cz+d).$$

Therefore, to describe the $(\frac{1}{2}, \frac{1}{2})$ representation we could just as well use the functions $\phi_{kl}(z)$ by specifying their Lorentz transformation as

$$(T_{\Lambda^{-1}}\phi_{ij})(z) = (\Lambda^\dagger)_{ik}\,\phi_{kl}(z')(\Lambda)_{lj}\,. \tag{A32}$$

Notice that we can write

$$\begin{pmatrix}\phi_{11}(z) & \phi_{10}(z)\\ \phi_{01}(z) & \phi_{00}(z)\end{pmatrix} = \phi_0(z)1 + \vec{\sigma}\cdot\vec{\phi}(z)\,, \tag{A33}$$

where σ_i are the usual 2×2 Pauli matrices. Then from Eq. (A32) we obtain the transformation properties of $\phi_\mu(z) = (\phi_0(z), \vec{\phi}(z))$, namely,

$$T_{\Lambda^{-1}}\phi_\mu(z) = (\Lambda^{-1})_\mu^\nu\,\phi_\nu\left(\frac{az+b}{cz+d}\right), \tag{A34}$$

where $\Lambda_{\mu\nu}$ is the 4×4 representation of the Lorentz transformation. Thus, the functions $\phi_\mu(z)$ describe the *unitary* integer-point representation $(\frac{1}{2}, \frac{1}{2})$ and transform like the direct product of two representations, namely, the finite-dimensional $(\frac{1}{2}, \frac{1}{2})$ representation and the infinite-dimensional unitary representation $j_1=j_2=0$, or $\rho=1$.

We can consider the functions $\phi_\mu(z)$ as limiting functions of the supplementary series as $\rho\to 1$, and then can write

$$\phi_\mu(z) = \lim_{\rho\to 1}\langle\phi_\mu|\,\rho, z\rangle\,. \tag{A35}$$

These are the functions which will be considered as operators in the text to construct a dual model.

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$$\lambda^*(\lambda^*+1) = \bar{\lambda}(\bar{\lambda}+1)\,.$$

Any one of the following sets satisfies this equation.

$(\lambda, \bar{\lambda})$: (j_1, j_2), $(-j_1-1, -j_2-1)$, (j_2^*, j_1^*), $(-j_2^*-1, -j_1^*-1)$.

These are the sets which label covariant and contravariant kets and bras.