critical feature is that $\Box_x$ eliminates the corresponding propagator; the analogous results for scalar and spinor fields are obtained by replacing $\Box_x$ by $(\Box_x + m^2)$ and $(i\not{\partial} + m)$, respectively.

[10]So named because in low orders they look like seagulls. Seagull terms have been studied in a different context by D. J. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969).

[11]Solutions to related problems and references are given by W. Zimmermann in *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis Summer Institute in Theoretical Physics (MIT Press, Cambridge, Mass., 1971).

[12]As in, e.g., R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix*, (Cambridge Univ. Press, Cambridge, England, 1966). These rules are also succinctly derived in W. Zimmermann, Commun. Math. Phys. 11, 1 (1968), in which it is shown that by modifying the "$i\epsilon$" prescription, Feynman integrals become Lebesgue integrals which converge to covariant distributions as $\epsilon \to 0$.

[13]Cases in which $F(u)$ is the indefinite integral of an inverse power (logarithmic terms) may be accommodated by an integration by parts, lumping the surface terms with "renorm. ET terms."

[14]In simple cases $\rho_{\mu\nu}$ is obtained from the Feynman parameters by a change of variables. $C_{\mu\nu}$ is causal and $C_{\mu\nu}$ has the support properties appropriate to a sum over positive-energy intermediate states provided $\rho_{\mu\nu}$ vanishes except for $M^2 > 0$ and Q lying within the intersection of the two half-cones $(x-P)^2 = 0$, $(x-P)_0 < 0$ and $(x+P)^2 = 0$, $(x+P)_0 > 0$. The restrictions on Q are always satisfied; positivity of the mass parameter does not hold in all cases.

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PHYSICAL REVIEW D VOLUME 4, NUMBER 6 15 SEPTEMBER 1971

Relativistic and Realistic Classical Mechanics of Two Interacting Point Particles*

Christian Fronsdal
University of California, Los Angeles, California 90024

(Received 17 May 1971)

Classical mechanics of two point particles interacting at a distance is given a Lorentz-covariant formulation without introducing unphysical degrees of freedom such as usually accompany the two-time formalism. The theory is then quantized and compared with quantum field theory to allow the determination of realistic potentials. Exact solutions are obtained for an inverse distance potential; classical orbits as well as quantum energy levels are determined.

I. INTRODUCTION

There exists no widely accepted formulation of relativistic classical mechanics of two or more interacting particles of finite mass. For some time it was believed that no satisfactory theory was possible, until this was refuted[1] by the actual construction of self-consistent models. Existence theorems have only limited interest, however. The nonrelativistic theory is useful only because of the fact that the potentials happen to be known to considerable accuracy, and a relativistic theory should include a prescription for the potential in order that it predict effects like the precession of the perihelion of Mercury. The only sure source of knowledge, from which accurate potentials can – at least in principle – be derived, is relativistic quantum field theory. (We do not mean to discount the theory of general relativity, but to simplify the perspective by treating it as a field theory in flat space.) Hence it would seem plausible that relevant models of classical relativistic mechanics must be obtained deductively from relativistic quantum field theory, rather than inductively from nonrelativistic mechanics.

A direct deduction of a classical relativistic mechanics from quantum field theory has been given recently, and now, with the incomparable advantage of hindsight, it is possible to proceed inductively and arrive at the same theory by naive arguments based on nonrelativistic mechanics and the requirement of Lorentz invariance.

The theory that had been obtained previously from quantum field theory is recovered in Sec. VII as an

example of the general framework prepared in Sec. III (kinematics) and Sec. V (dynamics). Since the formulation is essentially Hamiltonian, it may be of interest to explain how it avoids contradicting the theorem of Currie, Jordan, and Sudarshan,[2] according to which such a theory can have no interaction. Three-vector position variables exist that have vanishing mutual Poisson brackets, and in terms of them the theory is Hamiltonian in the usual sense; however, these position variables are not the space parts of four-vectors and consequently they do not satisfy all the equations assumed in the theorem.[2] By a simple change of variables the theory can be formulated in terms of four-vector position variables, but then the space parts do not have vanishing mutual Poisson brackets.

Quantization is carried out in Sec. IX. In Sec. X we review some properties of the relativistic wave equation and introduce spin. Sections XI and XII explore the connection with quantum field theory and the derivation of potentials. Finally, in Sec. XIII we summarize the results as far as they apply to classical relativistic mechanics. An actual practical application (to the advance of the perihelion of Mercury) has been carried out in collaboration with Huff and will be reported soon.

II. NONRELATIVISTIC KINEMATICS

We study two point particles and use the following notation:

$m_1, m_2,$ = masses, $m_+ \equiv m_1 + m_2$,

$\vec{x}_1, \vec{x}_2$ = position coordinates,

$\vec{p}_1, \vec{p}_2$ = momentum coordinates,

$E, \vec{p}$ = total energy and momentum.

The position and momentum coordinates are assumed to be canonical variables satisfying the Poisson relations

$$\{x_{1i}, p_{1j}\} = \delta_{ij} = \{x_{2i}, p_{2j}\}, \tag{2.1a}$$

$$\{x_{1i}, p_{2j}\} = 0 = \{x_{2i}, p_{1j}\}, \tag{2.1b}$$

$$\{x_{1i}, x_{1j}\} = \{x_{1i}, x_{2j}\} = \{x_{2i}, x_{2j}\} = 0, \tag{2.2a}$$

$$\{p_{1i}, p_{1j}\} = \{p_{1i}, p_{2j}\} = \{p_{2i}, p_{2j}\} = 0. \tag{2.2b}$$

Of the two familiar formulations of Hamiltonian mechanics, we prefer the one that treats the time t as a coordinate.[3] It has vanishing Poisson brackets with all the x's and p's, and is conjugate to the total energy:

$$\{E, t\} = 1. \tag{2.3}$$

From the start we demand that $\vec{p} = \vec{p}_1 + \vec{p}_2$, so that the total momentum is carried by the particles and no momentum is assigned to the "interaction." This is typical of action-at-a-distance theories, in which no separate degree of freedom is associated with an "interaction field." It is our intention to attempt to retain this feature when we make the transition to relativistic kinematics. It is probably superfluous to argue the suitability of $\vec{x}_1 - \vec{x}_2$ as an internal position coordinate – this choice has the effect of making the internal variables invariant under translations. The definitions of the relative momentum $\vec{q}$ and the "total position" $\vec{x}$ will be left somewhat arbitrary, subject only to the requirement that the transformation from $\vec{x}_1$, $\vec{x}_2$, $\vec{p}_1$, $\vec{p}_2$ to $\vec{x}$, $\vec{y}$, $\vec{p}$, $\vec{q}$, be linear and canonical. Thus, we insist that

$$\{x_i, p_j\} = \delta_{ij} = \{y_i, q_j\}, \tag{2.4a}$$

$$\{x_i, q_j\} = 0 = \{y_i, p_j\}, \tag{2.4b}$$

$$\{x_i, x_j\} = \{x_i, y_j\} = \{y_i, y_j\} = 0, \tag{2.5a}$$

$$\{p_i, p_j\} = \{p_i, q_j\} = \{q_i, q_j\} = 0. \tag{2.5b}$$

All these conditions are satisfied if

$$\vec{p} = \vec{p}_1 + \vec{p}_2, \quad \vec{q} = d\vec{p}_1 - (1-d)\vec{p}_2, \tag{2.6a}$$

$$\vec{x} = (1-d)\vec{x}_1 + d\vec{x}_2, \quad \vec{y} = \vec{x}_1 - \vec{x}_2, \tag{2.6b}$$

where d is an arbitrary constant.

Equations (2.5) and (2.6) are covariant with respect to Galilei transformations. Since it is our purpose to replace Galilei covariance by Lorentz covariance, we shall examine the Galilei transformations in greater detail than would otherwise be indicated.

The generators of infinitesimal Galilei transformations are

E and $\vec{p}$, translations in time and space (2.7a)

L_{ij}, rotations (2.7b)

L_i, accelerations. (2.7c)

In terms of the particle variables,

$$\vec{p} = \vec{p}_1 + \vec{p}_2, \tag{2.8a}$$

$$L_{ij} = (x_{1i}p_{1j} - x_{1j}p_{1i}) + (x_{2i}p_{2j} - x_{2j}p_{2i}), \tag{2.8b}$$

$$L_i = m_1 x_{1i} + m_2 x_{2i} - t p_{1i} - t p_{2i}. \tag{2.8c}$$

Each of the nine generators is just the sum of the generators for two free particles. In contrast, the tenth generator, of time translations, is assumed to be modified by the interaction. This does not prevent us from writing the total energy as a sum:

$$E = E_1 + E_2, \tag{2.9}$$

but we must keep in mind that the individual particle "energies" E_1 and E_2 have not been defined; thus the interaction energy may be included in E_1 or in E_2 or in both. It is natural, in view of the

closer association between energy and momentum to be introduced with Lorentz transformations, to complete (2.6a) and (2.9) by introducing the "relative energy"

$$\epsilon = dE_1 - (1-d)E_2. \tag{2.10}$$

Since E_1 and E_2 are undefined except for (2.9), the definition of ϵ remains arbitrary and may be chosen to suit our convenience. The time variable may be handled in the same manner; thus, (2.6b) may be completed with

$$t = (1-d)t_1 + dt_2, \quad \tau = t_1 - t_2. \tag{2.11}$$

In this way we have introduced a new set of variables, ϵ and τ. It is very important to avoid misinterpretation of this notation: The new variables do *not* correspond to another degree of freedom – they are *not* a canonical pair. Their definition remains incomplete and it would be entirely consistent to regard them as constant parameters that have no bearing on any measurements. In particular, we expect that it must be possible to fix $\tau = 0$ once and for all, so that

$$\{\tau, \text{ any observable}\} = 0. \tag{2.12}$$

In addition we must avoid introducing a new degree of freedom, and this suggests that

$$\{\tau, \epsilon\} = 0. \tag{2.13}$$

In terms of the variables (2.6),

$$L_{ij} = (x_i p_j - x_j p_i) + (y_i q_j - y_j q_i) \equiv L^x_{ij} + s_{ij}, \tag{2.14a}$$

$$L_i = (m_+ x_i - t p_i) + M_d y_i \equiv L^x_i + s_i, \tag{2.14b}$$

$$M_d \equiv d m_1 - (1-d) m_2 = d m_+ - m_2. \tag{2.14c}$$

Each of the six generators of "homogeneous Galilei transformations" is a sum of an external or "orbital" part and an internal or "spin" part. The four translations are purely external – a result of our definitions of external and internal variables. Every bracket $\{A, B\}$, where A is L^x_{ij}, L^x_i, $\vec{p}$, or E and B is s_{ij} or s_i, vanishes; hence the transformations generated by L^x_{ij}, L^x_i, $\vec{p}$, and E form a group of canonical transformations that will be referred to as the external (inhomogeneous) Galilei group, and the transformations generated by s_{ij} and s_i form a group of canonical transformations that we shall call the internal (homogeneous) Galilei group. It is very common to take $d = m_2/m_+$, so that $M_d = 0$; then $\vec{x}$ is the coordinate of the center of mass, and s_i vanishes. This choice is very unfortunate in the present context, since the corresponding Lorentz generators s_{i0} cannot vanish. Of course, it is possible to arrange that $\lim s_{i0} = 0$, but the job of guessing the correct relativistic generalization of Galilei-covariant mechanics is made unnecessarily difficult.

We are mainly concerned with the internal variables. The relevant bracket relations are

$$\{\text{Ext. generator or variable, Int. generator or variable}\} = 0, \tag{2.15a}$$

$$\{s_{ij}, q_k\} = \delta_{ik} q_j - \delta_{jk} q_i, \tag{2.15b}$$

$$\{s_{ij}, y_k\} = \delta_{ik} y_j - \delta_{jk} y_i, \tag{2.15c}$$

$$\{s_{ij}, s_k\} = \delta_{ik} s_j - \delta_{jk} s_i, \tag{2.15d}$$

$$\{s_i, s_j\} = 0, \tag{2.15e}$$

$$\{s_i, q_j\} = M_d \delta_{ij}, \quad \{s_i, y_j\} = 0. \tag{2.15f}$$

All these relations follow immediately from the expressions for s_{ij} and s_i given by (2.14),

$$s_{ij} = y_i q_j - y_j q_i, \tag{2.16a}$$

$$s_i = M_d y_i, \tag{2.16b}$$

and the bracket relations

$$\{\text{Ext. variable, Int. variable}\} = 0, \tag{2.17a}$$

$$\{q_i, q_j\} = 0 = \{y_i, y_j\}, \tag{2.17b}$$

$$\{y_i, q_j\} = \delta_{ij}. \tag{2.17c}$$

Our program is to guess the Lorentz-covariant analogs of Eqs. (2.15) and (2.16), and to determine basic brackets similar to (2.17b) and (2.17c) consistent with them.

III. RELATIVISTIC KINEMATICS

The generators of Lorentz transformations are

$$p_\mu = \{p_0, \vec{p}\}, \quad \text{translations in space-time} \tag{3.1a}$$

$$L_{\mu\nu} = \{L_{ij}, L_{i0}\}, \quad \text{rotations and accelerations.} \tag{3.1b}$$

Greek indices run from 0 to 3. The nonrelativistic limit is recovered as $L_{i0}/c \to L_i$, $(p_0 - c m_+)\, c \to E$.[4] The bracket relations

$$\{L_{\mu\nu}, p_\lambda\} = -g_{\mu\lambda} p_\nu + g_{\nu\lambda} p_\mu \tag{3.2}$$

are expressed in words by the statement that p_μ is a four-vector. Considered as a set of equations to determine $L_{\mu\nu}$, they have the solution

$$L_{\mu\nu} = L^x_{\mu\nu} + s_{\mu\nu}, \quad L^x_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{3.3}$$

where $s_{\mu\nu}$ is an "integration constant" that has vanishing brackets with p_μ, and x_ν is a set of coordinates conjugate to p_μ. The brackets between p_μ and x_ν,

$$\{p_\mu, x_\nu\} = g_{\mu\nu}, \tag{3.4a}$$

$$\{p_\mu, p_\nu\} = \{x_\mu, x_\nu\} = 0 \tag{3.4b}$$

reduce to (2.3), (2.4a), and (2.5a), (2.5b) in the nonrelativistic limit if $x_0 = ct$. These are invariant under Lorentz transformations provided that x_μ is a four-vector. Assuming that this is the case, we

notice that $L^x_{\mu\nu}$ satisfy the same bracket relations as $L_{\mu\nu}$, and that

$$\{s_{\mu\nu}, x_\lambda\}=0=\{s_{\mu\nu}, L^x_{\lambda\rho}\}.$$

From this we easily deduce that $s_{\mu\nu}$ satisfy the same bracket relations as the $L_{\mu\nu}$. Hence we have the same situation as in the case of the Galilei group: The p_μ and the $L^x_{\mu\nu}$ generate an external (inhomogeneous) Lorentz group and the $s_{\mu\nu}$ generate an internal (homogeneous) Lorentz group of canonical transformations.

Next, we postulate that internal variables y_μ, q_ν exist such that the internal generators $s_{\mu\nu}$ take the form

$$s_{\mu\nu}=y_\mu q_\nu - y_\nu q_\mu. \tag{3.5}$$

Apart from the analogy with Eq. (3.3), this is justified by the fact that when μ and ν are different from zero, (3.5) is the same as the nonrelativistic expression given by (2.16a). When $\nu=0$, the most general expression is $s_{i0}=y_i q_0 - y_0 q_i$, where q_0 and y_0 are to be determined. In the nonrelativistic limit, s_{i0}/c must tend to s_i; thus from (2.16b),

$$q_0/c \to M_d, \quad y_0/c \to 0. \tag{3.6}$$

If we remember that the internal variables ϵ and τ were left undefined in the discussion of Sec. II, we can enhance the analogy between internal and external variables by writing

$$(q_0 - cM_d)c=\epsilon, \quad y_0/c=\tau. \tag{3.7}$$

This defines ϵ in terms of the still undefined q_0 and restricts the "relative time" to vanish as $c\to\infty$. For the same reason we shall postulate that q_μ and y_μ transform like four-vectors,

$$\{s_{\mu\nu}, q_\lambda\}=-g_{\mu\lambda}q_\nu+g_{\nu\lambda}q_\mu, \tag{3.8a}$$

$$\{s_{\mu\nu}, y_\lambda\}=-g_{\mu\lambda}y_\nu+g_{\nu\lambda}y_\mu, \tag{3.8b}$$

and that

$$\{p_\mu \text{ or } x_\mu,\ q_\nu \text{ or } y_\nu\}=0. \tag{3.9}$$

In the nonrelativistic limit Eqs. (3.8) reduce to (2.15b) and(2.15c) when $\mu,\nu\neq 0$ and, by virtue of (3.6), to (2.15f) when $\nu=0$.

There remains to modify (2.17b) and (2.17c), so as to make them consistent with (3.8) and (3.9). Clearly,

$$A_{\mu\nu}\equiv\{q_\mu, q_\nu\},\ B_{\mu\nu}\equiv\{y_\mu, y_\nu\},\ C_{\mu\nu}\equiv\{q_\mu, y_\nu\} \tag{3.10}$$

must be second-rank tensors constructed from the available variables. The most general possibility is

$$A_{\mu\nu}=As_{\mu\nu}+A'\hat{s}_{\mu\nu},\quad B_{\mu\nu}=Bs_{\mu\nu}+B'\hat{s}_{\mu\nu}, \tag{3.11a}$$

$$C_{\mu\nu}=Cg_{\mu\nu}+C'q_\mu q_\nu+C''y_\mu y_\nu+D(q_\mu y_\nu+q_\nu y_\mu), \tag{3.11b}$$

where $\hat{s}_{\mu\nu}$ is the dual of $s_{\mu\nu}$ and $A, A', \ldots, D$ are (internal) scalars. Equations (3.8) require that $A'=B'=D=0$ and $C'=-B$, $C''=-A$; thus,

$$\{q_\mu, q_\nu\}=As_{\mu\nu},\ \{y_\mu, y_\nu\}=Bs_{\mu\nu}, \tag{3.12a}$$

$$\{q_\mu, y_\nu\}=g_{\mu\nu}-Bq_\mu q_\nu-Ay_\mu y_\nu. \tag{3.12b}$$

The dimension of A is L^{-2}, and the vanishing of $\{q_i, q_j\}$ as $c\to\infty$ gives $A\to 0$. A nonzero value of A would require the introduction of a fundamental length and is best avoided. There is no such problem with B, since its dimension is $(mc)^{-2}$. We shall take $A=0$.

Next, the magnitude of B is fixed by the requirement that the relativistic theory have the same number of degrees of freedom as the nonrelativistic theory. This means that the rank of the tensor (3.12b) cannot change as $c\to\infty$. Since the rank is three in the nonrelativistic theory, we have to choose B so as to make $\det(g_{\mu\nu}-Bq_\mu q_\nu)=0$ or $B=1/q^2$, where $q^2=g^{\mu\nu}q_\mu q_\nu$. Thus, our final choice of basic bracket relations is

$$\{q_\mu, y_\nu\}=g_{\mu\nu}-q_\mu q_\nu/q^2, \tag{3.13a}$$

$$\{q_\mu, q_\nu\}=0,\quad \{y_\mu, y_\nu\}=s_{\mu\nu}/q^2. \tag{3.13b}$$

Let us emphasize that we have just taken a crucial step. We have made certain that no insuperable difficulty of interpretation can arise, by fixing the number of independent canonical variables at the same number as in the nonrelativistic theory.

The degeneracy of (3.13a) means that the variables are subjected to one constraint. The quantity $q^2=g^{\mu\nu}q_\mu q_\nu$ has vanishing brackets with y_μ and with q_μ,

$$\{q^2, q_\mu\}=\{q^2, y_\mu\}=\{q^2, s_{\mu\nu}\}=0. \tag{3.14}$$

Later, dynamics will be introduced and the time development of the system will be represented as the unfolding of a family of canonical transformations. The canonical invariant q^2 will then be a constant. The actual value of q^2 is provided by the nonrelativistic limit (3.7); namely

$$q^2 \to q_0^2 \to c^2M_d^2. \tag{3.15}$$

Thus, if q^2 is fixed it must be

$$q^2=c^2M_d^2. \tag{3.16}$$

Comparison with (3.7) shows that

$$\epsilon=\vec{q}^2/2M_d. \tag{3.17}$$

If $d=1$, then the nonrelativistic $\vec{q}$ is $\vec{p}_1$, and $\epsilon=\vec{p}_1^2/2m_1$. If $\vec{x}$ is the position of the center of mass, then $M_d=0$, $q^2=0$, $q_0=|\vec{q}|$, and ϵ remains undefined.

The redundancy introduced by the Lorentz-covariant notation is thus removed by a constraint among the four-momentum variables. A similar constraint among the y_μ is not possible, since $\{q_\mu, y^2\}$

$\neq 0$. Instead, the reduction to three variables is brought about by the invariance of Eqs. (3.5), (3.8), and (3.13) under the "gauge transformation"

$$y_\mu \to y_\mu + \Lambda q_\mu, \tag{3.18}$$

where Λ is an invariant function of q_μ [and hence a constant, in view of (3.16)]. However, there is a preferred gauge. If Eq. (3.5) is solved for y_μ, the general solution is found to be

$$y_\mu = (1/q^2)s_{\mu\nu}q^\nu + aq_\mu, \tag{3.19}$$

where a is arbitrary. Substitution into (3.13) shows that a must be a constant; thus the arbitrariness is reduced to the choice of gauge. The gauge that we prefer is the one in which $a=0$; in that gauge $y_0 \to 0$ in the nonrelativistic limit and s_i has a vanishing bracket with y_i, which corresponds to our choice $\vec{y} = \vec{x}_1 - \vec{x}_2$. The nonrelativistic relative position coordinates are the space components $\vec{y}$ in the gauge in which the relative time vanishes. The covariant definition of that gauge is

$$y_\mu = (1/q^2)s_{\mu\nu}q^\nu. \tag{3.20}$$

Because of Eq. (3.5) this is equivalent to

$$q^\mu y_\mu = 0. \tag{3.21}$$

From (3.21) and (3.16) it follows that y_μ is spacelike:

$$y^2 = y_0{}^2 - \vec{y}^2 < 0, \tag{3.22}$$

which allows us to define the distance r between the two particles as

$$r = + \{\vec{y}^2 - y_0{}^2\}^{1/2} \tag{3.23}$$

Later, this will help us construct the interparticle potential.

The bracket $\{A, B\}$ of two arbitrary functions of the variables p_μ, x_μ, q_μ, y_μ is given by Eqs. (3.4) and (3.13):

$$\{A,B\} = \left(\frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x_\nu} - \frac{\partial B}{\partial p_\mu}\frac{\partial A}{\partial x_\nu}\right)g_{\mu\nu} + \left(\frac{\partial A}{\partial q_\mu}\frac{\partial B}{\partial y_\nu} - \frac{\partial B}{\partial q_\mu}\frac{\partial A}{\partial y_\nu}\right)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{1}{q^2}\frac{\partial A}{\partial y_\mu}\frac{\partial B}{\partial y_\nu}s_{\mu\nu}. \tag{3.24}$$

We may refer to the brackets (3.13) and their generalization (3.24) as canonical; this is justified as follows.

First, let us note that the number of independent internal variables is three coordinates and three momenta, since the constraints may be used to eliminate q_0 and y_0,[5]

$$q_0 = (\vec{q}^2 + M_d^2)^{1/2}, \tag{3.25a}$$

$$y_0 = q_0{}^{-1}\vec{y}\cdot\vec{q}. \tag{3.25b}$$

If A and B are expressed as functions of p_μ, x_μ, $\vec{q}$, and $\vec{y}$, then (3.24) takes the form

$$\{A,B\} = \left(\frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x_\nu} - \frac{\partial B}{\partial p_\mu}\frac{\partial A}{\partial x_\nu}\right)g_{\mu\nu} - \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial y_j} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial y_j}\right)\left(\delta_{ij} + \frac{q_i q_j}{q^2}\right) + \frac{1}{q^2}\frac{\partial A}{\partial y_i}\frac{\partial B}{\partial y_j}s_{ij}. \tag{3.26}$$

Next, one may easily verify that the Jacobi identity is satisfied:

$$\{\{A,B\},C\} + \{\{B,C\},A\} + \{\{C,A\},B\} = 0, \tag{3.27}$$

and from this follows that there exists a set of coordinates p_μ, x_μ, $Q_i(\vec{q},\vec{y})$, and $Y_i(\vec{q},\vec{y})$ in terms of which (3.26) takes the customary canonical form. Usually, one considers functions A and B that do not depend explicitly on the total energy p_0; then one of the terms in (3.26) vanishes, and there remains

$$\{A,B\} = -\left(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_j} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial x_j}\right)\delta_{ij} - \left(\frac{\partial A}{\partial Q_i}\frac{\partial B}{\partial Y_j} - \frac{\partial B}{\partial Q_i}\frac{\partial A}{\partial Y_j}\right)\delta_{ij}. \tag{3.28}$$

Since manifest Lorentz covariance is desirable, we shall prefer the equivalent form (3.24) and call that form "canonical".[6] A "canonical transformation" is one that preserves (3.13); if such a transformation is reexpressed in terms of the variables Q_i and Y_i, then it is canonical in the usual sense as well. Our use of the term "canonical transformation" is therefore completely conventional.

Probably the simplest transformation that brings (3.26) to the form (3.28) is

$$\vec{Y} = b\vec{y}, \quad \vec{Q} = b^{-1}\vec{q}, \tag{3.29}$$

with

$$b \equiv q_0/M_d = (1 + \vec{q}^2/M_d{}^2)^{1/2} \tag{3.30}$$

(This tends to unity in the nonrelativistic limit.) The $\vec{Q}$ defined by (3.29) is bounded in magnitude, since $|\vec{Q}| < M_d$. Notice again that the choice of center-of-mass coordinates, $M_d = 0$, must be avoided.

Finally, let us note that the theory possesses the usual integral invariants with respect to contact transformations. In particular, the theorem of Liouville states that the phase-space volume element

$$d^4x\, d^4p\, d^3Y\, d^3Q \tag{3.31}$$

is invariant. (We have inserted the factor $dx_0 dp_0$ in order to extend the invariance to include Lorentz transformations and other generalized contact transformations that transform the time x_0 and the energy p_0.) In terms of q_μ and y_μ the invariant (3.31) takes the form

$$d^4x\, d^4p\, d^4q\, d^4y\, \delta(q^2 - M_d^2)\, \delta(qy). \tag{3.32}$$

IV. NONRELATIVISTIC DYNAMICS

Equations of motion in nonrelativistic mechanics have the form

$$\dot{A} \equiv \frac{dA}{dt} = \frac{\partial A}{\partial t} - [H, A]. \tag{4.1}$$

Here $\dot{A}$ is the total time derivative of any function A of the canonical coordinates. The first term may also be written

$$\{E, A\} = \frac{\partial A}{\partial t} \tag{4.2}$$

in accordance with the convention (2.3). Thus we get the preferred form

$$\dot{A} = \{E - H, A\}. \tag{4.3}$$

Let us note in passing that $\dot{t} = 1$.

For simplicity consider an isolated system. The stationary states are parametrized by a real number assigned to E, and this number is defined by

$$E - H = 0. \tag{4.4}$$

That such a number exists follows from the fact that

$$d(E-H)/dt = \{E-H,\ E-H\} = 0. \tag{4.5}$$

In order to treat a specific physical system, it is necessary to know something about H. In a typical case one may have

$$H = H_0 + V, \tag{4.6}$$

where H_0 is a simple function of the momenta and V may be a function of the coordinates. In this case

$$\{y_i, V\} = 0, \quad \{q_i, V\} = F_i, \tag{4.7}$$

where the force F_i is likewise a function of the coordinates. The equations of motion take the form of a set of coupled first-order differential equations,

$$\dot{y}_i = f_i(\vec{y}, \vec{q}), \quad \dot{q}_i = g_i(\vec{y}, \vec{q}), \ldots, \tag{4.8}$$

the solution of which gives the variables as functions of the time and the initial conditions.

This theory retains its basic structure if we introduce one or more of the following generalizations:

1. It is not necessary to exhibit a set of variables that satisfy standard canonical commutation relations. We need only a general rule for evaluating all brackets of interest. Equation (3.24) fills that requirement.

2. The dependence of the function $E-H$ on E may be generalized, as will in fact be necessary in order to satisfy the requirements of Lorentz covariance.

3. The explicit specification of V as a function of the coordinates and the momenta may be dispensed with provided a sufficient number of brackets $\{A, V\}$ are known in terms of the coordinates, the momenta, and V itself. This situation would arise in our example if we were to formulate the theory in a general gauge. By fixing the gauge, we can avoid this particular complication.

All these generalizations have been encountered simultaneously.[7] Nevertheless there remains a well-defined, fairly narrow, and essentially Hamiltonian structure. The principal points are:

(a) There is only one "time"; it is conjugate to the total energy E, and $\{E, t\} = 1$. The equations of motion give all observables as functions of t and the initial conditions.

(b) A bracket $\{A, B\}$ that satisfies the usual conditions of a Poisson bracket is defined for any two functions A, B.

(c) There exists a function L' that depends on E and on the coordinates and the momenta and that generates time translations by means of the brackets, $dA/dt = \{L', A\}$. For any set of initial conditions, a value is assigned to E such that L' takes the value zero.

(d) Space translations and Lorentz transformations are also generated by brackets, and the equations of motion take the same form in any two reference frames that are related by a (inhomogeneous) Lorentz transformation.

We shall show that this framework is sufficiently general to allow the introduction of dynamics into the discussion of Sec. III.

V. RELATIVISTIC DYNAMICS

The dynamical postulate will be stated as a rule for calculating the ordinary time derivative $\dot{A} = dA/dt$ of an observable A. Formal Lorentz covariance will next be restored by replacing the time t by a linear combination $\lambda^\mu x_\mu$. Finally a more general time coordinate – such as is appropriate for the case of general relativistic covariance – will be introduced.

We postulate the existence of a function L' such that all the equations of motion can be written in the form

$$\dot{A} = dA/dt = \{L', A\}. \tag{5.1}$$

It is assumed that rules are given that allow the determination of all required brackets and that the usual conditions for Poisson brackets are satisfied. The function L' depends on a set of observables and on the total energy p_0. The possibility that A may have an explicit time dependence is taken into account by regarding p_0 and t as conjugate variables.[3] In particular,

$$\dot{t} = \{L', t\} = \frac{\partial L'}{\partial p_0}. \tag{5.2}$$

This quantity must be equal to unity. In the nonrelativistic theory $L' = E - H$, with $\partial H/\partial E = 0$, and $\dot{t} = 1$ identically. In the relativistic theory we must admit a more general p_0 dependence and $\dot{t}$ can reduce to unity only as a consequence of the equations that determine realizable motions.

Since L' is the analog of $E - H$ we add the postulate that, for any value of t, the values of the total energy p_0 and of the other variables in L' are related so that $L' = 0$. Naturally, this is not true identically, but only for realizable motions. Thus it is incorrect to substitute $L' \to 0$ inside a Poisson bracket. If we write

$$L' = L/I \tag{5.3}$$

and assume that $L' = 0$ imply $L = 0$, then $\dot{t} = 1$ for realizable motions if and only if

$$I = I_0, \quad I_\mu \equiv \frac{\partial L}{\partial p^\mu}. \tag{5.4}$$

Equations (5.3) and (5.4) will be regarded as identities, though it would perhaps be sufficient that the latter be true for realizable motions. We take L to be the fundamental dynamical function and I_0 to be defined by (5.4). Eliminating L' we have the following dynamical postulates:

$$\dot{A} = \{I^{-1}L, A\}, \tag{5.5a}$$

$$L = 0 \quad \text{for realizable motions.} \tag{5.5b}$$

As a particular case, $\dot{t} = 1$, which interprets the dot. We may also write

$$I_0 \dot{A} = \{L, A\}, \tag{5.6}$$

but this follows from (5.5) only for realizable motions (for which $L = 0$). Hence (5.6) cannot be used as a substitution rule inside a Poisson bracket.[8]

Invariance under Lorentz transformations places certain demands on the function L. To discover what they are we begin by generalizing our choice of time parameter. Instead of interpreting $\dot{A}$ as the derivative with respect to t, we introduce

$$s = \lambda^\mu x_\mu; \quad \lambda^2 = 1, \quad \lambda_0 > 0, \tag{5.7}$$

and interpret $\dot{A}$ as dA/ds. In this case $\dot{s} = 1$, and Eq. (5.4) is replaced by

$$I = \lambda^\mu I_\mu. \tag{5.8}$$

The dot operation is now formally invariant provided λ_μ transforms like a four-vector. Covariance of (5.5) is obviously guaranteed if the function L is invariant. Different frames are associated with different λ_μ, however, and what remains is to make sure that the direction of λ_μ be insignificant. This is most easily done by taking L to be independent of λ_μ, for then λ_μ enters only through the time parameter s.

Taking L invariant and independent of λ_μ, we can fix λ_μ, e.g., $\lambda_\mu = (1, 0, 0, 0)$, without losing Lorentz invariance. Our first formulation, with $s = t$, is therefore Lorentz covariant provided L is invariant. However, the invariance is not "explicit" and it may sometimes be advantageous to introduce another definition of s.

For any allowed motion it is, in principle, possible to determine I_0 as a function of t. For each allowed motion we may therefore evaluate the integral

$$s(t) = \int_{t_0}^{t} \frac{dt'}{I_0(t')}. \tag{5.9}$$

Assuming that $s(t)$ is an increasing function of t [which requires that $I_0 > 0$ – see comment following Eq. (9.14)], we can take $s = s(t)$ as time parameter. Instead of (5.6) we get

$$\dot{A} = \{L, A\}, \tag{5.10}$$

where the dot now stands for the derivative with respect to s. This equation can be adopted as dynamical postulate and is valid as a substitution rule inside Poisson brackets – in contrast with (5.6). In particular, we have

$$\{L, s\} = 1. \tag{5.11}$$

The new formulation is summarized by (5.10) and (5.5b).

As an illustration, let us consider the motion of a single particle in an external gravitational field. Then,

$$L = \frac{1}{2m}(g^{\mu\nu} p_\mu p_\nu - m^2), \tag{5.12}$$

$$\dot{x}^\mu = \{L, x^\mu\} = \frac{1}{m} g^{\mu\nu} p_\nu, \tag{5.13}$$

$$\ddot{x}^\mu = \frac{1}{2m^2}\{g^{\alpha\beta} p_\alpha p_\beta, g^{\mu\nu} p_\nu\}. \tag{5.14}$$

Since $g^{\alpha\beta}$ is a given function of x_μ, we have

$$\{p_\mu, g^{\alpha\beta}\} = g^{\alpha\beta}{}_{,\mu}, \tag{5.15}$$

and a simple calculation reduces (5.14) to the geo-

desic equation

$$\dot{x}^{\mu}+\Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}=0. \tag{5.16}$$

For allowed motions $L=0$, and (5.13) shows that this is the same as

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=1. \tag{5.17}$$

Consequently, ds is just the invariant line element:

$$ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{5.18}$$

As a second example consider a particle in an external scalar potential, with

$$L=\frac{1}{2m}(p^2-m^2)-V(x). \tag{5.19}$$

In this case we get, using the invariant time parameter,

$$\dot{x}_{\mu}=\frac{1}{m}p_{\mu}, \tag{5.20}$$

$$\ddot{x}_{\mu}=\frac{1}{m}\{p_{\mu},V(x)\}=\frac{1}{m}\frac{\partial V}{\partial x^{\mu}}\equiv\frac{1}{m}F_{\mu}(x). \tag{5.21}$$

If we use the noninvariant time t, we get instead

$$\dot{\vec{x}}=\vec{p}/p_0, \tag{5.22}$$

$$\ddot{x}_k=-\frac{m}{p_0^2}\left(\frac{\partial V}{\partial x_k}+\dot{x}_k\frac{\partial V}{\partial t}\right). \tag{5.23}$$

Finally, let us list the results for an external vector potential as well. With invariant time coordinate:

$$L=\frac{1}{2m}[(p-eA)^2-m^2], \tag{5.24}$$

$$\dot{x}_{\mu}=\frac{1}{m}(p-eA)_{\mu}, \tag{5.25}$$

$$\ddot{x}_{\mu}=\frac{e}{m}\dot{x}^{\nu}F_{\nu\mu}, \tag{5.26}$$

with $F_{\nu\mu}=A_{\nu,\mu}-A_{\mu,\nu}$. Using the time t we get instead:

$$\dot{\vec{x}}=(\vec{p}-e\vec{A})/(p_0-eA_0), \tag{5.27}$$

$$\ddot{x}_k=\frac{\dot{x}^{\mu}}{p_0-eA_0}\left(F_{\mu k}-F_{\mu 0}\frac{\dot{x}_k}{p_0-eA_0}\right). \tag{5.28}$$

The use of the invariant parameter seems to recommend itself.

It should be emphasized that there is nothing innovative in this discussion. What is new is the subsequent treatment of the two-body problem. The difficulties that have been encountered in the past, when the dynamics outlined above was combined with the more conventional two-body kinematics, are well known.[9] The main problem has always been the superabundance of independent degrees of freedom in the usual two-time formalism. This is just the feature that we have taken pains to avoid.

VI. TWO PARTICLES WITHOUT INTERACTION

In analogy with the nonrelativistic mechanics of two free point particles, we expect that the function L is of the form $\alpha p_1^2+\beta p_2^2+\gamma$. The coefficients α, β, and γ are all determined by the requirement that the usual form $E-\vec{p}_1^{\,2}/2m_1-\vec{p}_2^{\,2}/2m_2$ be obtained in the nonrelativistic limit, and the result is that L is just the sum of two one-particle L functions:

$$L=L_0\equiv\frac{1}{2m_1}(p_1^2-m_1^2)+\frac{1}{2m_2}(p_2^2-m_2^2). \tag{6.1}$$

In order to relate this to the formalism of Sec. III, we have to define p_{μ} and q_{μ} in terms of $p_{1\mu}$ and $p_{2\mu}$. Adopting the same relations as in the nonrelativistic case – Eqs. (2.6), (2.9), (2.10), and (2.11) – we write

$$p_{\mu}=p_{1\mu}+p_{2\mu},\quad q_{\mu}=dp_{1\mu}-(1-d)p_{2\mu}, \tag{6.2a}$$

$$x_{\mu}=(1-d)x_{1\mu}+dx_{2\mu},\quad y_{\mu}=x_{1\mu}-x_{2\mu}. \tag{6.2b}$$

We can solve these relations to express $p_{1\mu}$, $p_{2\mu}$, $x_{1\mu}$, and $x_{2\mu}$ in terms of p_{μ}, q_{μ}, x_{μ}, and y_{μ}, and then use the bracket relations (3.4) and (3.13); or we can make use of (3.24) and (6.2). The results are, using invariant time and Eq. (5.10),

$$\dot{x}_{1\mu}=\frac{p_{1\mu}}{m_1}+dq_{\mu}(q^2)^{-1}\left(\frac{p_2q}{m_2}-\frac{p_1q}{m_1}\right), \tag{6.3a}$$

$$\dot{x}_{2\mu}=\frac{p_{2\mu}}{m_2}+(1-d)q_{\mu}(q^2)^{-1}\left(\frac{p_1q}{m_1}-\frac{p_2q}{m_2}\right), \tag{6.3b}$$

and, of course, $\dot{p}_{1\mu}=\dot{p}_{2\mu}=0$ and $\ddot{x}_{1\mu}=\ddot{x}_{2\mu}=0$. Writing the same result in terms of ordinary time t, we just get an extra factor

$$I_0=(1-d)\frac{p_{10}}{m_1}+d\frac{p_{20}}{m_2} \tag{6.4}$$

on the left-hand side of (6.3).

The only physically relevant result here is that $\ddot{\vec{x}}_1=\ddot{\vec{x}}_2=0$. The motion is rectilinear and uniform, the velocities being given by (6.3). The motion is completely determined for all time by initial conditions for $\vec{x}_1$, $\vec{x}_2$ and $\dot{\vec{x}}_1$, $\dot{\vec{x}}_2$. The initial values of $\dot{\vec{x}}_1$, $\dot{\vec{x}}_2$ also fix all components of p_{μ} and q_{μ}, since we have two constraints[5]:

$$q^2=M_d^2=(m_+d-m_2)^2\quad\text{and}\quad L=0. \tag{6.5}$$

This in turn shows that $\dot{t}_1$ and $\dot{t}_2$ are determined. By "initial values" we may understand values of the dynamical variables at some value $\bar{t}$ of t, or, if we use invariant time, at some value $\bar{s}$ of s. The initial value of t is $\bar{t}$ in one case and irrelevant in the other case. The initial value of $\tau=y_0/c$ is fixed by the gauge condition $y^{\mu}q_{\mu}=0$. If we work in an unspecified gauge, $\tau(\bar{t})$ is irrelevant since the theory is gauge invariant. (L is independent of y_{μ}.)

Nevertheless, the theory has one unusual feature: The constraints (6.5) are not the same as, and not consistent with, $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$, except for a special choice of d. This is not really physically relevant since it has nothing to do with the particle motions. In the special cases $d=1$ and $d=0$, the expression for q_μ reduces to $p_{1\mu}$ and $p_{2\mu}$, respectively. In the first case the constraint on q^2 means that $p_1^2 = m_1^2$, while $L_0 = 0$ gives $p_2^2 = m_2^2$; in the second case the situation is reversed. Thus it is possible to recover the familiar mass-shell conditions at the price of giving up the symmetry between the two particles. We emphasize that so far no physically relevant criterion for choosing d has been found. Later on, advantages of taking $d=0$ or $d=1$ will emerge.

VII. TWO INTERACTING PARTICLES

The most interesting interaction is a Lorentz-covariant generalization of the $1/r$ potential. The straightforward procedure is to replace the L function (6.1) for two noninteracting particles by

$$L = L_0 - V(y), \tag{7.1}$$

where $V(y)$ depends only on the relative coordinates y_μ and has the form

$$V(y) = v(r), \tag{7.2}$$

r being the invariant defined by (3.23):

$$r = +\sqrt{-y^2}. \tag{7.3}$$

Since r must be real, this makes sense only if we impose the gauge condition (3.21), $q^\mu y_\mu = 0$. This can be avoided if we replace (7.3) by a set of bracket relations involving r, and this completely gauge-invariant formulation has already been developed,[7] but it is more straightforward to use (7.3) and put

$$q^\mu y_\mu = 0. \tag{7.4}$$

Equations (3.24) then give

$$\{r, q_\mu\} = y_\mu / r, \tag{7.5a}$$

$$\{r, y_\mu\} = r q_\mu / q^2, \tag{7.5b}$$

and thus, with $v' = \partial v / \partial r$,

$$\{V, q_\mu\} = y_\mu v' / r, \tag{7.6a}$$

$$\{V, y_\mu\} = r q_\mu v' / q^2. \tag{7.6b}$$

The equations of motion, in terms of invariant time, are

$$\dot{x}_{1\mu} = m_1^{-1} p_{1\mu} + d(q^2)^{-1}[p_2 q/m_2 - p_1 q/m_1 - r v'] q_\mu, \tag{7.7a}$$

$$\dot{x}_{2\mu} = m_2^{-1} p_{2\mu} - (1-d)(q^2)^{-1}[p_2 q/m_2 - p_1 q/m_1 - r v'] q_\mu, \tag{7.7b}$$

$$\dot{p}_{1\mu} = -\dot{p}_{2\mu} = -y_\mu v'/r. \tag{7.7c}$$

We shall examine these equations from several different points of view.

Initial-value problem. If for some value $\bar{s}$ of the invariant time parameter s, we assign arbitrary values to $\vec{x}_1$, $\vec{x}_2$, $\vec{p}_1$, and $\vec{p}_2$, then the future motion is completely determined: The initial value of q_0 is given by the constraint $q^2 = M_d^2$,[5] that of y_0 by $y^\mu q_\mu = 0$, while p_0 is determined for realizable motions by $L=0$. The initial value of x_0 relates the zero point of invariant time to the zero point of x_0 and is irrelevant. These initial values determine, through Eqs. (7.7), the first s-derivatives of $\vec{x}_1$, $\vec{x}_2$, $\vec{p}_1$, and $\vec{p}_2$.

Conservation laws. Translation invariance is expressed by

$$\dot{p}_\mu = \{L, p_\mu\} = 0 \tag{7.8}$$

and Lorentz invariance by

$$\dot{L}_{\mu\nu} = \{L, L_{\mu\nu}\} = 0. \tag{7.9}$$

In the frame $\vec{p}=0$ the space-space components of $L_{\mu\nu}$ are s_{ij}, and (7.9) gives

$$\dot{s}_{ij} = 0, \quad s_{ij} = y_i q_j - y_j q_i, \tag{7.10}$$

which shows that the motion is confined to a plane. The space-time components of $L_{\mu 0}$ in the frame $\vec{p}=0$ are $x_i p_0 + s_{i0}$, and (7.9) reduces to

$$p_0 \dot{x}_i + \dot{s}_{i0} = 0, \quad s_{i0} = y_i q_0 - y_0 q_i. \tag{7.11}$$

This is the relativistic analog of the constancy of the velocity of the center of mass. Another type of conservation law is Liouville's theorem of the constancy of the density of phase space. Since the time development of the system is determined by the unfolding of a family of canonical transformations, the phase-space volume element (3.32) is constant.[10]

Accelerations and forces. It is possible, with the help of the constraints on q^2 and yq, to eliminate $p_{1\mu}$, $p_{2\mu}$, x_{10}, and x_{20} from Eqs. (7.7), and thus to express the equations of motion in the form

$$\ddot{\vec{x}}_1(s) = \vec{f}_1[\vec{x}_1(s), \vec{x}_2(s), \dot{\vec{x}}_1(s), \dot{\vec{x}}_2(s), p_0], \tag{7.12a}$$

$$\ddot{\vec{x}}_2(s) = \vec{f}_2[\vec{x}_1(s), \vec{x}_2(s), \dot{\vec{x}}_1(s), \dot{\vec{x}}_2(s), p_0]. \tag{7.12b}$$

For allowed motions the dependence on p_0 can be eliminated by means of the further constraint $L=0$. The resulting equations have a form similar to those studied by Currie,[11] and by Hill,[12] but the difference in interpretation is important. Our "time coordinate" s is invariant; we may just as well use $t = x_0$ as time coordinate, but in any case it is the same time coordinate that appears in both Eqs. (7.12). Currie[11] and Hill[12] use two times; although their values are set equal in the equations

for the accelerations, they transform differently under Lorentz transformations, t_1 forming a four-vector with $\vec{x}_1$ and t_2 forming a four-vector with $\vec{x}_2$.

According to a celebrated theorem,[2] it is impossible to set up a canonical theory of two interacting particles in which the individual particle positions are the space parts of four-vectors. The present theory does not contradict the theorem since the primitive brackets between the $\vec{x}_1$, $\vec{x}_2$ are not canonical in the strict sense. Of course, we may write the equations in terms of $\vec{x}$ and $\vec{Y}$, whose mutual brackets are all zero, but $\vec{Y}$ is not the space part of a four-vector. (See the end of Sec. III.)

Solutions. In the special case when $v(r)$ is of the form $-\alpha/r$, it is easy to solve Eqs. (7.7) exactly. In that case the square brackets in (7.7a) and (7.7b) are constants by virtue of $L=0$. There then exists a linear combination of $\dot{x}_{1\mu}$ and $\dot{x}_{2\mu}$ that is a constant of the motion, and it is sufficient to solve for $\vec{y}$, say. In the frame $\vec{p}=0$ we may write

$$L = P_0 q_0 - P_4 M_d - v(r), \tag{7.13}$$

where P_0 and P_4 are constants. By direct calculation one finds that $P_0^2 - P_4^2$ is positive if

$$\left| m_+ - \frac{2m_1 m_2}{(1-d)^2 m_2 + d^2 m_1} \right| < p_0 < m_+ \tag{7.14}$$

and negative outside this region. It has already been pointed out that s_{ij} are constant, hence the motion is confined to a plane. In the special case considered now, the vector

$$A_i = (P_4 s_{i0} - P_0 M_d y_i) \, |P_0^2 - P_4^2|^{-1/2} \tag{7.15}$$

is also a constant of the motion. The vectors $\vec{A}$ and $\vec{B}$,

$$B_i = (P_0 s_{i0} - P_4 M_d y_i) |P_0^2 - P_4^2|^{-1/2}, \tag{7.16}$$

satisfy the following kinematical relation:

$$\frac{(\vec{A}\cdot\vec{B})^2}{n^2} + \frac{(\vec{A}\times\vec{B})^2}{l^2} = \vec{A}^2, \tag{7.17}$$

where $l^2 = \frac{1}{2} s_{ij} s_{ij}$ is the square of the angular momentum and (note that $r L_0 = -\alpha = \text{const}$)

$$n^2 = (r L_0)^2 (P_0^2 - P_4^2)^{-1}. \tag{7.18}$$

Equations (7.17) and (7.18) show that $\vec{B}$ moves on an ellipse when the inequalities (7.14) are satisfied and on a hyperbola when they are not. From the constancy of $\vec{A}$ follows that the same is true of $\vec{y}$, and thus also of $\vec{x}_1$ and $\vec{x}_2$. It is interesting that the escape energy is just $m_+ = m_1 + m_2$ independently of the value of d. To complete the determination of the motion, we derive a law of constant areal velocity. In fact, it follows immediately from Eqs. (7.7), when $v = -\alpha/r$, that the vector $\vec{y}\times\dot{\vec{y}}$ is constant, and that constant vectors $\vec{c}_1$ and $\vec{c}_2$ exist such that $(\vec{x}_1 - \vec{c}_1)\times\dot{\vec{x}}_1$ and $(\vec{x}_2 - \vec{c}_2)\times\dot{\vec{x}}_2$ are constants.[7]

In summary, it may be said that the equations of motion describe a system that is qualitatively nearly indistinguishable from its nonrelativistic analog. The choice of d remains undecided, except that the obvious and popularly preferred value $d = m_2/m_+$ is ruled out. We have noted the esthetic advantage of the extreme values, $d=0$ or $d=1$, which lead to the normal energy-momentum relations $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$ for free particles, but this is irrelevant in the present context.

VIII. RELEVANCE AND LIMITATIONS

The question of the most appropriate choice of the parameter d is only a specific example of the larger issue of relevance of the formalism. So far the guidelines have been Lorentz invariance and the correct nonrelativistic limit, which is not sufficient to arrive at a definite theory with predictive power. Classical mechanics makes no predictions about the motion of the planets until the correct potential is known, nor will the present theory account for the advance of the perihelion of Mercury before we correctly specify the potential $v(r)$.

In our opinion, constructive efforts such as the present are of little use unless they lead finally to definite models with power to make predictions about actual physical phenomena. Further, we believe that the additional input that is required must come from quantum field theory. For this reason we shall make no attempt to make physical applications at this stage, but proceed instead to make the transition to quantum theory.

So far, nothing has been said about the problem of three or more particles, and nothing has indicated that this would present very great difficulties. It might seem that one could solve the problem of two bodies without complicating the matter by introducing a third, but unfortunately this is not so. For a complete interpretation of the theory, it is necessary to introduce, if not a third particle, at least an external field. If the correct potential were known, one would expect to confirm the calculated orbits by means of actual observations. We shall assume that such observations are carried out with the help of electromagnetic interactions, although the following arguments could probably be adapted to other situations. An optical measurement of a world line is made by preparing an electromagnetic field $A_\mu(x)$ throughout space-time and noting the locus of arguments $(\vec{x}, t)$ for

which $A_\mu(x)$ is perturbed. It is *assumed* that this curve coincides with the world line of the particle. The importance of this assumption for us is that it amounts to postulating a local interaction between the particle and the field: The particle at point $(\vec{x}, t)$ senses the field at that point only.

It would be imprudent to assume that the orbits calculated above, $x_{1\mu}(s)$ and $x_{2\mu}(s)$, can be observed optically without first investigating the nature of electromagnetic interactions. In analogy with nonrelativistic mechanics, one expects to introduce such interactions by adding terms containing $A_\mu(x_1)$ and $A_\mu(x_2)$ to the L function. The first difficulty that appears is that of gauge invariance; the familiar substitutions do not give a gauge-invariant interaction because of the constraints and unusual bracket relations of the internal variables. An even greater objection is the fact that, when only one particle is charged, and no interaction between them is introduced, both particles are accelerated (except in an important special case, see below). By pursuing these ideas we shall come to a more precise delineation of the limits of our method.

Although it is difficult to write down an interaction between the field and each particle, there is no obstacle to a gauge-invariant interaction between the field and the two-particle system. The properties of the coordinates x_μ and p_μ are quite the usual ones, and the substitution

$$L_0(p_\mu, q_\mu) \to L = L_0[p_\mu - eA_\mu(x), q_\mu] \tag{8.1}$$

leads to a gauge-invariant theory. The only trouble is that the interaction occurs at the point $x = (1-d)x_1 + dx_2$; unless $d = 0$ or 1, there is no charge at this point. It is remotely possible that some physical relevance can be achieved by choosing d such that x is the center of charge, but we shall not place our bets on that. Instead, it may be argued that a definite indication of the most advantageous choice of d has finally appeared, namely, $d = 0$ or 1. With $d = 1$ (say) the substitution (8.1) achieves a local and gauge-invariant interaction of the field with particle 2, since $x = x_2$ in this case. One of the particles, at least, becomes visible. (In this special case, if there is no interaction between the particles, $\ddot{\vec{x}}_1$ turns out to vanish.) The other particle may be "seen" by switching to $d = 0$, or its orbit may be determined by symmetry arguments, but we cannot couple both to A_μ simultaneously.

Thus it is seen that the theory may give a good account of two particles with interaction between them and between one of them and an external field (or between one of them and one or more additional particles).

When d is set equal to 1, and the limit $m_1/m_2 \to \infty$ is taken, one obtains a theory of planetary orbits due to Thirring. For a discussion of this limit, and other theories closely resembling ours, see several papers by Anderson and von Baeyer.[13]

IX. QUANTIZATION

We seek a Hilbert space of functions $\psi(p, q)$ on which the kinematical variables act as self-adjoint operators satisfying the bracket relations of Sec. III, Poisson brackets being replaced by commutator brackets[4]:

$$\{A,B\} \to (i\hbar)^{-1}[A,B]. \tag{9.1}$$

The p_μ, x_μ brackets are just the familiar ones and it is clear that the operator x_μ is

$$x_\mu = -i\hbar \frac{\partial}{\partial p^\mu}. \tag{9.2}$$

The $s_{\mu\nu}$ is the generator of Lorentz transformations of q_μ. In the absence of additional spin degrees of freedom,

$$s_{\mu\nu} = i\hbar\left(q_\mu \frac{\partial}{\partial q^\nu} - q_\nu \frac{\partial}{\partial q^\mu}\right). \tag{9.3}$$

Comparison of this with the classical expression (3.5) suggests, of course, that y_μ be identified with $-i\hbar\partial/\partial q^\mu$; but this is inconsistent with the bracket relations (3.13). One might attempt to define y_μ by Eq. (3.19), choosing the parameter a such that the operator is Hermitian. If $s_{\mu\nu}$ and q_μ are Hermitian, then so is

$$\begin{aligned} y_\mu &= (1/2q^2)(s_{\mu\nu}q^\nu + q^\nu s_{\mu\nu}) \\ &= (1/q^2)(s_{\mu\nu}q^\nu - \tfrac{3}{2}i\hbar q_\mu). \end{aligned} \tag{9.4}$$

Hence a tends to zero with h. This allows us to define r^2 as $-y_\mu^2$, but the important operators r and r^{-1} remain ambiguous. For this reason we prefer to begin by introducing an operator form of r^{-1} and define y_μ by analogy with (7.5a):

$$y_\mu = (1/2i\hbar)(r[r, q_\mu] + [r, q_\mu]r). \tag{9.5}$$

Both (9.4) and (9.5) have the correct limits as either $h \to 0$ or $c \to \infty$; in fact, both definitions agree if we take the following definition of the operator r:

$$r^{-1}\psi(p, q) = -(M_d/\pi^2)\int \frac{(dq')}{(q-q')^2}\,\psi(p, q') \tag{9.6}$$

with

$$(dq) \equiv d^4q\,\delta(q^2 - M_d^2)\epsilon(q_0). \tag{9.7}$$

The proof of this statement, and some other results concerning these operators, are in the Appendix.

The integral operator (9.6) is almost a straightforward generalization of the conventional expression for r^{-1} in momentum space, to which it reduces

when $c \to \infty$ (effectively, because $q_0' \simeq cM_d$ tends to infinity). The δ function in (9.7) enables us to retain the constraint on q^2. The factor $\epsilon(q_0)$ could, without violating Lorentz invariance, be replaced by $1 + a\theta(-q_0)$ with arbitrary real a. The special choice $a = -2$ has been made for convenience.[14] The correspondence between operators and classical observables is many-to-one; therefore, we do not expect that the operator r should be uniquely determined by the correspondence principle. What freedom there is may be used to our best advantage. The principal reason for the choice made in (9.7) is that the case of the r^{-1} potential remains soluble in the quantum theory.

The inner product will be defined as

$$(\psi, \psi') = \int d^3p \int (dq)\psi^*(p,q)\psi'(p,q) \tag{9.8}$$

with the same volume element (9.7) as used in (9.6). In this metric r^{-1} is a positive definite self-adjoint operator. (See Appendix.)

In the classical theory dynamics was introduced in terms of a function L that corresponds to the nonrelativistic function $E - H$. For allowed motions $L = 0$. In nonrelativistic quantum mechanics, the function $E - H$ is the Lagrange operator, and the vanishing of $E - H$ is expressed as the wave equation. We shall therefore postulate the wave equation

$$L\psi = 0, \tag{9.9}$$

where the operator L is to be obtained from the classical L function by substitutions based on (9.2) and (9.6). The corresponding Lagrangian is

$$\int d^4p \int (dq)\psi^*(p,q)(L\psi)(p,q) \tag{9.10}$$

and this is real if L is Hermitian in the metric (9.8). If it seems that the transition from the classical to the quantum theory contains *ad hoc* elements, it should be emphasized that no ambiguities are involved in making the passage in the opposite direction. This is as it should be, since the classical theory needs quantum theory (in fact, quantum field theory) to select the relevant potential, while the opposite is not true. We require an unambiguous prescription to descend from quantum theory to classical mechanics. It will now be shown that the wave equation (9.9) leads to the equation of motion (5.6) as h tends to zero.[7] The procedure is a generalization of the proof of Ehrenfest's theorem.

We shall suppose that L is a polynomial L_p in p_μ, with operator coefficients. Let

$$I_\mu(p) = \frac{\partial L_p}{\partial p^\mu} \tag{9.11}$$

and define $I_\mu(p,p')$ symmetrically in p and p' such that $I_\mu(p,p) = I_\mu(p)$ and such that

$$(p - p')^\mu I_\mu(p,p') = L_p - L_{p'}. \tag{9.12}$$

Then the conserved canonical current is

$$J_\mu(p,p') = \int (dq)\psi^*(p) I_\mu(p,p')\psi(p'). \tag{9.13}$$

(The q arguments are suppressed.) The normalization condition for bound states is simply $J_0(p,p) = 1$, or[15]

$$\int (dq)\psi^* I_0 \psi = 1. \tag{9.14}$$

Notice that the positivity of I_0 required here was relevant in the classical theory as well, in connection with Eq. (5.9). The purpose of all this is to arrive at an expression for the expectation value $\langle A \rangle$ of the operator A; according to (9.14),

$$\langle A \rangle = \int (dq)\psi^* I_0 A \psi. \tag{9.15}$$

For the most part we shall use the same symbol to denote either the operator A or the corresponding classical observable, but for the present it is convenient to distinguish between them and we shall write $\tilde{A}$ for the classical quantity.

Define

$$\tilde{A} = \lim_{h \to 0} \langle A \rangle, \tag{9.16}$$

$$\{\tilde{A}, \tilde{B}\} = \lim_{h \to 0} \langle (i\hbar)^{-1}[A,B] \rangle, \tag{9.17}$$

and consider two states of definite total momenta p_μ and p'_μ, so that $L_p\psi(p) = L_{p'}\psi(p') = 0$. Then

$$\begin{aligned} 0 &= (i\hbar)^{-1} \int (dq)\psi^*(p) A L_{p'} \psi(p') \\ &= (i\hbar)^{-1} \int (dq)\psi^*(p)\{[A, L_{p'}] + (L_{p'} - L_p)A\}\psi(p'). \end{aligned} \tag{9.18}$$

Now let $\vec{p}' = \vec{p}$ and let $p_0' \to p_0$; then,

$$0 = (i\hbar)^{-1} \int (dq)\psi^*(p)\{-[L,A] + (p_0' - p_0) I_0 A\}\psi(p). \tag{9.19}$$

As $h \to 0$ this tends to

$$0 = -\tilde{I}_0^{-1}\{\tilde{L}, \tilde{A}\} + d\tilde{A}/dt, \tag{9.20}$$

which agrees with the classical dynamical postulate of Eq. (5.6). The classical dynamics is thus recovered.

X. WAVE EQUATIONS

We have been led to wave equations of the form[16]

$$[(p_1^2 - m_1^2)/2m_1 + (p_2^2 - m_2^2)/2m_2 - V]\psi \equiv L\psi = 0, \tag{10.1}$$

with

$$p_1=(1-d)p+q,\quad p_2=dp-q,\quad q^2=M_d{}^2. \tag{10.2}$$

The potential V will be taken to represent the mutual interaction between the particles, and to be an operator in q space. There is no reason why V cannot depend on p_μ, but for the present we consider only the simplest case when it does not. In terms of p, q,

$$-2m_1m_2L=2M_dp^\mu q_\mu-c_1p^2-c_2+2m_1m_2V, \tag{10.3}$$

where c_1 and c_2 are constants:

$$c_1=(1-d)^2m_2+d^2m_1,\quad c_2=m_+(M_d{}^2-m_1m_2). \tag{10.4}$$

The matrix elements of the conserved canonical current, between states of definite total four-momenta, are thus

$$\begin{aligned}J_\mu(p,p')&=\int\psi^*(p,q)I_\mu(p,p')\psi'(p',q)(dq)\\&=(-2m_1m_2)^{-1}\int\psi^*(p,q)[2M_dq_\mu-c_1(p+p')_\mu]\\&\quad\times\psi'(p',q)(dq).\end{aligned} \tag{10.5}$$

The two-particle system may be coupled to an external electromagnetic field via this current by making the substitution $p_\mu\to p_\mu-ieA_\mu(x)$, q_μ fixed, in (10.3). However, the physical meaning of such a coupling is clear only if $d=0$ or 1, in which case the field is coupled to only one of the two particles. In this respect quantization does nothing to change the conclusion reached in Sec. VIII.

As far as the choice of V is concerned, we know that $V=-\alpha/r$ gives a good account of the Coulomb problem, with the correct nonrelativistic limit. Fine structure, to the extent that it is a nonrelativistic phenomenon, may also be accounted for. But other modifications of the potential, such as would be needed to calculate the Lamb shift or the advance of the perihelion of Mercury, remain unknown until contact is made with quantum field theory.

Fermions. The form of (10.1) suggests that analogous equations for spin-$\frac{1}{2}$ particles be obtained by replacing one or both Klein-Gordon operators by Dirac operators. Thus, if only particle 2 is a Dirac particle:

$$[(p_1{}^2-m_1{}^2)/2m_1+(p_2\gamma-m_2)-V]\psi=0. \tag{10.6}$$

If both have spin $\frac{1}{2}$:

$$[(p_1\gamma^{(1)}-m_1)+(p_2\gamma^{(2)}-m_2)-V]\psi=0. \tag{10.7}$$

Like (10.1), Eqs. (10.6) and (10.7) have the correct nonrelativistic limits.

Approximations. In the static limit, e.g., $m_1\to 0$, with d arbitrary but fixed and nonzero, Eq. (10.1) turns into the Klein-Gordon equation for a single particle in an external potential. Similarly, Eq. (10.6) turns into the Dirac equation with a fixed external potential, and (10.7) allows inclusion of the effect of the spin of the source. Another interesting approximation consists of ignoring the coupling between the $q_0\geq M_d$ and $q_0\leq -M_d$ parts of the wave function. In this approximation we may put $q_0=+(\vec{q}^2+M_d{}^2)$. If $d=1$,

$$p_0\psi=[(m_1{}^2+\vec{p}_1{}^2)^{1/2}\pm(m_2{}^2+\vec{p}_2{}^2)^{1/2}+2m_2V']\psi. \tag{10.8}$$

Taking the upper of the two signs we obtain the Breit equation for two scalar particles. (The Breit potential V' differs from V by some terms arising from the fact that V does not commute with $\vec{p}_2{}^2$.) The same result is obtained if $d=0$, but not for any other value of d. In the same approximation, if $d=1$, Eq. (10.6) reduces to

$$p_0\psi=[(\vec{p}_1{}^2+m_1{}^2)^{1/2}+\vec{p}_2\cdot\vec{\alpha}-m_2\beta-\beta V]\psi, \tag{10.9}$$

which is the Breit equation for this case. Equation (10.7), on the other hand, does not reduce to the corresponding Breit equation – at any rate not so simply.

Exact solutions. Equation (10.1), with the "Coulomb" potential $V=-\alpha/r$, can be solved exactly.[16] Using the form (10.3) for L and setting $\vec{p}=0$, we get

$$-2m_1m_2L\psi=(P_0q_0-P_4M_d+\gamma/r)\psi=0, \tag{10.10}$$

with

$$P_0=2p_0M_d,\quad P_4=[c_1p_0{}^2+m_+(M_d{}^2-m_1m_2)]M_d{}^{-1}, \tag{10.11}$$

and $\gamma=-2m_1m_2\alpha$. More explicitly,

$$(P_0q_0-P_4M_d)\psi(q)=\gamma M_d\pi^{-2}\int\frac{(dq')}{(q-q')^2}\psi(q'). \tag{10.12}$$

Let the total energy p_0 be such that $P_0{}^2-P_4{}^2>0$; this is the condition for closed orbits and hence bound states. The condition on p_0 is $m_-{}^2<p_0{}^2<m_+{}^2$, with $m_+=m_1+m_2$ and

$$m_-=m_+-2m_1m_2/c_1. \tag{10.13}$$

To solve (10.12) we introduce Fock variables

$$\vec{u}=\vec{q}(P_0{}^2-P_4{}^2)^{1/2}(P_0q_0-P_4M_d)^{-1}, \tag{10.14a}$$

$$u_4=(1+\vec{u}^2)^{1/2}=(P_0M_d-P_4q_0)(P_0q_0-P_4M_d)^{-1}, \tag{10.14b}$$

$$\chi(u)\propto(P_0q_0-P_4M_d)^2\psi(q). \tag{10.14c}$$

Then (10.12) takes the four-dimensional form that reveals the O(4) symmetry:

$$\chi(u)=\frac{n}{\pi}\int\frac{d^4v\,\delta(v^2-1)}{(u-v)^2}\chi(v), \tag{10.15}$$

with

$$n\equiv-\gamma(P_0^2-P_4^2)^{-1/2}. \tag{10.16}$$

This is precisely the equation obtained by Fock for the nonrelativistic Coulomb problem. The eigenvalues are $n=1,2,3,\ldots$. Solving (10.16) for p_0, we get the spectrum

$$p_0^2=\frac{m_+^2+m_-^2}{2}+\frac{m_+^2-m_-^2}{2}\left(1-\frac{4m_1m_2}{m_+^2-m_-^2}\,\frac{\alpha^2}{n^2}\right)^{1/2}. \tag{10.17}$$

Consistency requires that all these energies lie in the interval $m_-^2<p_0^2<m_+^2$, and this is seen to be true so long as the radical is positive; whence the condition

$$\alpha^2<\frac{m_+^2-m_-^2}{4m_1m_2}=\left(\frac{M_d}{c_1}\right)^2. \tag{10.18}$$

Once again, center-of-mass coordinates (i.e., $M_d=0$) are ruled out, and so are very large values of d ($M_d/c_1\to 0$).

To obtain more stringent conditions on d, we expand (10.17) in powers of α and discover the very significant fact that the binding energy is independent of d to lowest order in α:

$$(p_0^2-m_+^2)/2m_+=-\frac{\mu\alpha^2}{2n^2}\{1-(c_1/2M_d)^2(\alpha/n)^2+\cdots\}. \tag{10.19}$$

Here $\mu=m_1m_2/m_+$. For hydrogen, c_1/M_d should not exceed 1 by an order of magnitude; if $d=0$ or $d=1$, the value is 1.

Unfortunately, it has not yet been possible to obtain exact solutions of Eq. (10.6).

XI. LIPPMANN-SCHWINGER EQUATION

In what sense can the wave equation

$$L\psi\equiv[L_0-V]\psi=0 \tag{11.1}$$

be considered an approximation to quantum field theory? More pertinently: How must the potential operator in the wave equation

$$\left[\frac{1}{2m_1}(p_1^2-m_1^2)+\frac{1}{2m_2}(p_2^2-m_2^2)-V\right]\psi=0 \tag{11.2}$$

be chosen in order that the solution have something to do with quantum field theory?[17] The Green's function of (11.2) is the operator i/L, and the scattering matrix is given by

$$T=-L_0\frac{1}{L}V=-V\frac{1}{L}L_0. \tag{11.3}$$

The equation for T is

$$T=-V+V\frac{1}{L_0}T. \tag{11.4}$$

In terms of the matrix elements $T(p_1p_2,p_1'p_2')$ this means

$$T(p_1p_2,p_1'p_2')=-V(p_1p_2,p_1'p_2')+\int(dq'')V(p_1p_2,p_1'',p_2'')\frac{1}{L_0(p_1'',p_2'')}T(p_1''p_2'',p_1'p_2'), \tag{11.5}$$

where $p_1+p_2=p_1'+p_2'=p_1''+p_2''$ and $q''=dp_1''-(1-d)p_2''$, or

$$T(p_1p_2,p_1'p_2')=-V(p_1p_2,p_1'p_2')-i\int d^4p_1''d^4p_2''\delta(p_1+p_2-p_1''-p_2'')V(p_1p_2,p_1''p_2'')G(p_1''p_2'')T(p_1''p_2'',p_1'p_2'). \tag{11.6}$$

This resembles the Bethe-Salpeter equation, except that

$$G(p_1p_2)=\frac{i\epsilon(q_0)\delta(q^2-M_d^2)}{(1/2m_1)(p_1^2-m_1^2)+(1/2m_2)(p_2^2-m_2^2)},$$

$$[=\mathcal{G}(p_1p_2)]. \tag{11.7}$$

Although all our functions have so far been restricted to $q^2=M_d^2$, we shall now suppose that V is extended to arbitrary values of its arguments (except that $p_1+p_2=p_1'+p_2'$). Equation (11.6) then defines $T(p_1p_2,p_1'p_2')$ for arbitrary p_1, p_2, and the analog of (11.6), obtained from the second form of (11.3). completes the definition of T for general values of all four momenta, subject only to $p_1+p_2=p_1'+p_2'$.

To avoid ambiguities let the extended functions be denoted by $\mathcal{U}$ and $\mathcal{T}$; then Eq. (11.6), with $V\to\mathcal{U}$ and $T\to\mathcal{T}$ can be abbreviated as follows:

$$\mathcal{T}=-\mathcal{U}-i\mathcal{U}\mathcal{G}\mathcal{T}. \tag{11.8}$$

Note the difference in meaning between the operator products in (11.3) and (11.8). In (11.3) an integration $\int(dq)$ is implied while in (11.8) we have suppressed $\int d^4p_1d^4p_2\delta(\cdots)$. We shall adopt the convention of using block letters in one case and script letters in the other.

Our question can now be formulated more succinctly: Given the expression (11.7) for $\mathcal{G}$, how must we choose $\mathcal{U}$ if $\mathcal{T}$ is to agree with the off-shell scattering matrix of some field theory? The

answer is, of course, to insert into Eq. (11.8) the $\mathcal{T}$ matrix that we wish to reproduce, and solve the equation for $\mathcal{V}$.

Normally, $\mathcal{T}$ is known in perturbation theory as a series in powers of the coupling constant:

$$\mathcal{T}=\mathcal{T}_1+\mathcal{T}_2+\cdots. \tag{11.9}$$

Expanding $\mathcal{V}$ similarly, inserting both expansions into (11.8) and equating terms of the same order, one obtains

$$\mathcal{V}_1=-\mathcal{T}_1,\quad \mathcal{V}_2=-\mathcal{T}_2-i\mathcal{V}_1\mathcal{G}\mathcal{V}_1,\ \ldots\ . \tag{11.10}$$

This allows one to evaluate $\mathcal{V}$ up to any desired order of perturbation theory.

If $\mathcal{V}$ is evaluated up to the nth order in the coupling constant, and the higher orders are neglected, then (11.8) will give a $\mathcal{T}_{\text{approx}}$ that, if expandable in powers of the coupling constant, is correct up to and including the nth order. However, this is not the reason one studies this type of equation. Rather, the hope is that $\mathcal{T}_{\text{approx}}$ will turn out to be a good approximation beyond the limits of applicability of perturbation theory.

The general procedure is then as follows: (1) Replace $\mathcal{V}$ by $\mathcal{V}_1+\cdots+\mathcal{V}_n$ (usually just a single term) in Eq. (11.8). (2) Restrict the momentum variables to $q^2=q'^2=M_d^2$ to obtain Eq. (11.5) with V replaced by the restriction $V_1+\cdots+V_n$ of $\mathcal{V}_1+\cdots+\mathcal{V}_n$. (3) Solve (11.5) by such standard methods as are available for ordinary Lippmann-Schwinger equations. (4) Substitute the solution into (11.8) to obtain the unrestricted $\mathcal{T}$. Some features of $\mathcal{T}$ can be obtained more simply; for example, the bound-state poles may be found by replacing V by $V_1+\cdots+V_n$ in (11.2) and solving the eigenvalue problem. Sometimes a complete set of solutions of (11.2) can be used to construct the solution of (11.5).[16]

A very substantial simplification occurs if $d=0$ or $d=1$. If $d=1$, say, then the restriction $q^2=M_d^2$ is just the restriction to the mass shell $p_1^2=m_1^2$. Thus, to find the on-shell scattering matrix, it is not necessary to carry out the fourth stop; instead, it suffices to pass to the limit $p_2^2=m_2^2$.

Lowest order. To illustrate, consider the case of two scalar fields interacting through the coupling

$$g_1\psi_1^*(x)\psi_1(x)A(x)+g_2\psi_2^*(x)\psi_2(x)A(x) \tag{11.11}$$

to a third scalar field $A(x)$ with mass m. The scattering of a particle of the field ψ_1 by a particle of the field ψ_2 is given in lowest order by[18]

$$\mathcal{V}_1=-\mathcal{T}_1=g_1g_2[(q-q')^2-m^2+i\epsilon]^{-1}(M_d/16m_1m_2\pi^3). \tag{11.12}$$

Substitution into (11.5) gives, in the c.m. frame,

$$T(q,p,q')=-g_1g_2[(q-q')^2-m^2]^{-1}(M_d/16m_1m_2\pi^3) -\int\frac{(dq'')}{(2\pi)^3}\frac{g_1g_2}{(q-q'')^2-m^2} \times\frac{M_d}{P_0q_0''-P_4M_d}T(q'',p,q'). \tag{11.13}$$

Here the $i\epsilon$ has been dropped since the constraint $q^2=q'^2=q''^2=M_d^2$ makes it redundant. Equation (11.13) is essentially an ordinary Lippmann-Schwinger equation.[19] In the case $m=0$ the exact solution can be obtained[16] in the form of a hypergeometric function of a complicated argument. To extend this function so as to relax the restriction to $q^2=q'^2=M_d^2$ with the help of (11.8) would appear to be difficult. It can probably be done though, and it might be of interest to carry out the rest of the program in order to determine how $\mathcal{T}$ depends on d. However, it is much simpler to put $d=1$. The final on-shell T matrix ($p_1^2=m_1^2$, $p_2^2=m_2^2$) is then easily obtained by restriction, the result being

$$T(s,t)=\frac{\gamma}{2m_2t}\pi^{-2}\frac{\Gamma[1-\nu(s)]}{\Gamma[1+\nu(s)]}\left(\frac{t}{\tilde{m}}\right)^{\nu(s)}. \tag{11.14}$$

Here $\nu(s)$ is the function obtained by expressing the right-hand side of (10.16) in terms of $s=p_0^2$, $\gamma=-g_1g_2/8\pi$, and $\tilde{m}$ is an arbitrary parameter that reflects the indeterminacy of the phase of Coulomb scattering [$\nu(s)$ is positive imaginary]. When $m=0$ the potential (11.12) is

$$-(g_1g_2/16m_1m_2\pi)r^{-1}. \tag{11.15}$$

The wave equation corresponding to (11.13) is therefore the same (when $m=0$) as (10.10), with

$$\gamma=-2m_1m_2\alpha=-g_1g_2/8\pi. \tag{11.16}$$

This result means that, in one case at least, the $\mathcal{T}$ matrix obtained with the approximation $\mathcal{V}\to\mathcal{V}_1$ has sensible physical properties outside the domain in which the first Born approximation dominates – in fact, outside the limits of applicability of perturbation theory. The question as to whether or not the approximation gives a faithful account of quantum field theory is taken up in Sec. XII.

XII. QUANTUM FIELD THEORY

Finally, we must inquire whether the $\mathcal{T}$ matrix found by the methods of Sec. XI is a good approximation to the scattering matrix of the quantum field theory from which the potential is taken. It would be impossible to answer this question but for the success of quantum electrodynamics. The experimental verification of the relativistic perturbation theory on one hand, and of the Schrödinger theory of hydrogen on the other, gives us confidence in a reduction of a quantum field theory to its static limit. The first step in that reduction

is to neglect, in the first approximation, all Feynman diagrams except generalized ladder graphs.

The next step is to approximate all the generalized ladder graphs of a given order by taking the static limit; i.e., by keeping the leading term in an expansion in inverse powers of the larger of the two particle masses. This leads to the Klein-Gordon or the Dirac theory of the lighter particle in a static field, for a certain class of field theories. (The scalar theory considered in Sec. XI has a static limit if g_1/m_1 is held fixed as $m_1 \to \infty$. The case of vector-meson exchange between scalar or spinor particles works well, as does a theory of spin-2 exchange. Nothing can be proved if the exchanged particle carries isotopic spin or other non-Abelian internal symmetries.) We return now to the question of how good is the $\mathcal{T}$ matrix of Sec. XI. It has been shown that the approximation $\mathcal{V} = \mathcal{V}_1$, in the limit $m_1 \to \infty$, agrees with that obtained from the generalized ladder graphs.[17,20] (Of course, this applies only to those cases when the limit of the generalized ladder graphs is known.) If quantum field theory is regarded as the ultimate source of knowledge, then our conclusion is that, under conditions approximating the static limit, the main part of the potential to use in the wave equations or in the Lippmann-Schwinger equations is just the Born term of the field theory. Corrections to this potential may be calculated in a straightforward manner, as indicated by Eq. (11.10), and these may be treated as small perturbations. In this manner the Lamb shift of (spinless) hydrogen has been recalculated. This leaves unsettled the precise limits of the domain in which the Born term provides the most essential part of the potential. It may easily be shown, for example, that a reasonable high-energy limit (eikonal formula) is obtained if the exchanged particle is a neutral vector meson.

XIII. CONCLUSIONS

It was our hope to formulate a classical relativistic mechanics with realistic physical applications. The first input into this attempt was our insistence that the relativistic theory have no more independent degrees of freedom than its nonrelativistic limit. This led to a set of basic Poisson relations for the relative coordinates that are a bit unusual, but yield an immediate and unexpected benefit: a natural relativistic definition of the r^{-1} potential. With this potential we could give a covariant account of Kepler motion. The second input was the idea that the correct relativistic potential should be obtainable from quantum field theory. We therefore quantized the theory, derived a covariant form of the Lippmann-Schwinger equation, and showed how the potential could be found. The main part of the potential – the r^{-1} part in the case of forces of infinite range – turned out to be simply related to the Born term of the relativistic-field-theoretic scattering amplitude. Corrections are associated with the higher terms and have been shown to be small. What remains is to calculate the most important correction to the potential and to use this to refine the covariant calculation of the classical orbits. The most obvious application, to the advance of the perihelion of Mercury, has already been done in collaboration with Huff and will be reported soon.

We should like to comment once more on some aspects of the proposed two-particle relativistic dynamics. (i) In spite of a well-known "no-go" theorem (but not in contradiction with it) the theory is in all essential respects Hamiltonian. (ii) We have stressed, in Sec. VIII, our opinion with regard to the measurability of the coordinates as particle positions. (iii) As a result of this discussion we arrived at strong arguments in favor of taking one of the extreme values 0 or 1 for the parameter d. Additional but basically equivalent reasons for preferring one of these values came up in Sec. XI. The choice of d that makes x_μ the center-of-mass coordinate reduces the theory to nonsense on all levels. The loss of symmetry between the two particles, that results from taking either of the two preferred values, is perhaps unaesthetic but of no practical importance.

ACKNOWLEDGMENTS

Thanks are due to Professor R. W. Huff and Professor R. Arens for a number of stimulating discussions. Conversations with Professor E. C. G. Sudarshan and Professor E. P. Wigner have also been helpful. Part of this work was carried out at the International Centre for Theoretical Physics at Trieste, and I want to express my thanks to Professor A. Salam and Professor P. Budini, as well as the IAEA, for their hospitality.

APPENDIX

In this paper we have attempted to eliminate all references to group-theoretical methods. After all, group theory can only help solve soluble problems and alternate methods can always be constructed. Nevertheless, we have not found another derivation, besides the following group-theoretical one, of some results needed in Sec. IX.[21]

Consider the space H of functions $f(z_0,z_1,z_2,z_3,z_4)$ on the half-cone $z^2 = z_0^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 = 0$, $z_0 > 0$, satisfying the normalization condition

$$\int |f(z)|^2 (dz) < \infty, \quad (dz) \equiv \delta(z^2)\theta(z_0) d^5 z . \tag{A1}$$

Let $z \to \Lambda z$ be a (1+4)-dimensional Lorentz transformation; then the operators $T_\Lambda : f(z) \to f(z\Lambda)$ form a unitary representation of SO(4,1). This representation may easily be reduced by fixing the degree of homogeneity of $f(z)$. We let

$$f(z) = z_4{}^N \psi(q_0, q_1, q_2, q_3), \tag{A2}$$

$$q_\mu = M_d z_4{}^{-1} z_\mu, \quad q^2 = M_d{}^2, \tag{A3}$$

[note in particular that $2zz' = -(z_4{}^2/M_d{}^2)(q-q')^2$] and find a unitary irreducible representation induced in the space of functions $\psi(q)$, defined on the double cone $q^2 = M_d{}^2$ and satisfying the normalization condition

$$\int |\psi(q)|^2 (dq) < \infty, \quad (dq) \equiv \delta(q^2 - M_d{}^2)\epsilon(q_0) d^4 q. \tag{A4}$$

The degree of homogeneity N must have real part equal to $-\frac{3}{2}$. Next we continue these representations analytically in N to the points -2 and -1, and find two equivalent unitary representations in spaces H_1 and H_2, say, with inner product

$$(\psi, \psi') = \int \psi^*(q)\psi'(q') \frac{(dq)(dq')}{(q-q')^2}. \tag{A5}$$

The action of the operators T_Λ on $\psi(q)$ can readily be deduced from (A2); the infinitesimal generators are $s_{\mu\nu}$ and $s_{\mu 4} \equiv M_d{}^{-1}(q^\nu s_{\mu\nu} + 2iq_\mu)$. The maps $\Gamma_A : f_1(z) = z_A f_2(z)$ between functions of degree -2 and functions of degree -1 define a set of operators from H_2 to H_1 that transform like the components of a five-vector under SO(4,1). Since the representations induced in H_1 and H_2 are equivalent, these operators can be made to act in H_2:

$$\Gamma_A{}^{-1} f_2(z) \propto z_A{}^{-1} \int f_2(z')(zz')^{-1}(dz'). \tag{A6}$$

The integral does not converge; nevertheless (A6) suggests that the operators defined by

$$\Gamma_A{}^{-1}\psi(q) \propto \frac{z_4}{z_A} \int \psi(q') \frac{(dq')}{(q-q')^2} \tag{A7}$$

obtained from (A6) by dropping a divergent integral over z_4', transform like the components of a five-vector under SO(4,1), and this is easily verified by direct test. Other correct results are also obtained by formal manipulation of (A6), and rigor can be supplied without too much trouble. The final result may be summarized as follows.

The operators $s_{\mu\nu}$, $s_{\mu 4}$, and $s_{A5} \equiv \Gamma_A$, $A = 0, 1, 2, 3, 4$, are the generators of a unitary irreducible representation of SO(4,2) in the space of functions $\psi(q)$ with the positive norm (A5). The precise form of the operators Γ_A are

$$\Gamma_4 = M_d r, \quad \Gamma_\mu = r q_\mu, \tag{A8}$$

where r is the operator defined by Eq. (9.6). Besides satisfying the commutation relations of SO(4,2)—which enables us to prove that (9.4) agrees with (9.5)—these operators satisfy the following identities ($\alpha, \beta, \ldots$ take values $0, 1, \ldots, 5$):

$$\epsilon^{\alpha\beta\gamma\delta\mu\nu} s_{\alpha\beta} s_{\gamma\delta} = 0, \tag{A9}$$

$$g^{\alpha\beta}[s_{\alpha\gamma}, s_{\beta\delta}]_+ = -2\hbar^2 g_{\gamma\delta}. \tag{A10}$$

In the limit $h \to 0$ the commutation relations lead to the Poisson brackets of Sec. III and the identities give known formulas, including, for example, $y_\mu{}^2 = -r^2$. The positivity of the operator r^{-1} in the metric (9.8) is the same as the positivity of the norm (A5).

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*Supported in part by the National Science Foundation.

[1] H. Van Dam and E. P. Wigner, Phys. Rev. 138, B1576 (1965); D. G. Currie and T. F. Jordan, in *Quantum Theory and Statistical Theory*, 1967 Boulder Lectures in Theoretical Physics, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA; J. G. Wray, NYO Report No. 3829-37, 1969 (unpublished).

[2] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).

[3] For a concise treatment of Hamiltonian dynamics, we prefer the book by C. W. Kilmister, *Hamiltonian Dynamics* (Wiley, New York, 1964). See, in particular, Chap. 5.

[4] Units such that $c = \hbar = 1$ will be used except in the few instances where we consider the limits $c \to \infty$ or $h \to 0$.

[5] Here and on some other occasions we ignore the negative root of $q_0{}^2 = \vec{q}^{\,2} + M_d{}^2$. This creates no problems in the context of classical mechanics, since all variables change continuously with time and no change of sign can occur.

[6] Compare the theory of H. C. Lee, as outlined by Kilmister, Ref. 3.

[7] C. Fronsdal, Phys. Rev. D 3, 1299 (1971).

[8] Equation (5.6) is inconsistent with the Jacobi identity unless $\dot{I}_0 = 0$.

[9] See, for example, P. Pearle, Phys. Rev. 168, 1429 (1968).

[10] I am grateful to Professor E. P. Wigner for raising this question.

[11] D. G. Currie, Phys. Rev. 142, 817 (1966).

[12] R. N. Hill, J. Math. Phys. 8, 1201 (1967).

[13] C. M. Andersen and H. C. von Baeyer, Ann. Phys. (N.Y.) 60, 67 (1970); *ibid.* 62, 120 (1971).

[14] Originally, the factor $\epsilon(q_0)$ appeared through the type of consideration that is here relegated to the Appendix. Lately, however, two new arguments for its inclusion in (9.6) have been discovered. First, this factor is needed to make $(r^{-1})^2$ well defined; hence (9.6) is the natural extension of the nonrelativistic formula only

if this factor is included. Second, the volume element (9.7) must be used in (9.8) as well – in order that r^{-1} be self-adjoint – and (9.8) has precisely the right form in the special case when the interaction is turned off. For this remark I am indebted to J. Espinoza.

[15]To avoid misunderstanding, some comments may be required. The normalization condition (9.14) is the condition satisfied by the Fourier component $\psi(p,q)$ for a definite value of p_μ. The case treated here is that of a bound state (i.e., an atom) with well-defined total energy and momentum. Such a state is not normalizable; nevertheless, (9.14) is the correct condition to be imposed on the Fourier component. It is seen that we are limiting the discussion to the case of an isolated system, although some of our applications refer to external potentials. The case of one particle moving in an external potential may be obtained as a limit of a two-particle interacting system as m_1 tends to infinity. We are also interested in the case when our two-particle system is affected by an external potential; the text does not adequately cover this, but the reader should have no trouble satisfying himself that there is no hidden difficulty.

[16]The following is mainly a review of the paper by C. Fronsdal and L. E. Lundberg, Phys. Rev. D 1, 3247 (1970). In that paper only the value $d=1$ was considered.

[17]The following is mainly a review of the paper by C. Fronsdal and R. W. Huff, Phys. Rev. D 3, 933 (1971). In that paper only the case $d=1$ was considered.

[18]The last factor is due to the fact that we have used a normalization convention that is different from what is customary in quantum field theory. Our $\mathcal{T}$ is $(M_d/16m_1m_2\pi^3)$ times the usual scattering matrix.

[19]The essential point is that the integration is three dimensional (being confined to the hyperboloids $q^2=M_d{}^2$) and that the kernel is nonsingular. In other words, ordinary methods of computation are applicable.

[20]F. Gross, Phys. Rev. 186, 1448 (1969).

[21]In some respects, additional details may be found in C. Fronsdal, Phys. Rev. 156, 1665 (1967). The most complete account is in a set of lecture notes, ICTP Report No. IC/70/34 (unpublished).