[20]S. Hori, Progr. Theoret. Phys. (Kyoto) 7, 578 (1952).
[21]The procedure we follow here is a variation on the one carried out for the function known as Lerch's transcendent [of which our $G_1(z)$ is a special case] in the Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, p. 27.
[22]A little thought shows that the inequality (4.2) may be generalized to Nth order so that the Wick-Hori series converges (at least) for $\sum_{1\le i<j\le N}|f^2\Delta_{ij}|<1$.

PHYSICAL REVIEW D VOLUME 4, NUMBER 6 15 SEPTEMBER 1971

Equal-Time Terms in Perturbation Theory and Convolutions of Tempered Distributions*

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(Received 15 December 1970; revised manuscript received 15 March 1971)

The matrix elements of commutators of local operators are expressed in terms of the corresponding time-ordered products to all orders of perturbation theory. An ambiguity in the definition of retarded commutator is resolved by demanding that two versions of the Lehmann-Symanzik-Zimmermann reduction formula agree. It is shown that when the equal-time commutator is sequence-independent, it agrees with the appropriately defined equal-time limit of the retarded commutator. The method enables one to rigorously obtain non-c-number Schwinger terms directly from Feynman diagrams, and to determine their significance in sum rules. A generalized convolution is introduced and a convolution theorem is proved which makes rigorous the correspondence between equal-time commutators and sum rules.

INTRODUCTION

A number of authors[1] have studied "Schwinger terms," both abstractly and in perturbation theory. Among recent results is the fact that every causal distribution has an equal-time (ET) limit defined on a nontrivial subspace of the space $\mathcal{S}(R^3)$. The nature of "Schwinger terms" is now quite well understood; the role they play in the study of scattering amplitudes has been less clear. In the present work is an attempt to clarify the relationship among various ET terms in perturbation theory. The first problem is to develop a method for calculating non-c-number Schwinger terms from Feynman diagrams.

In their basic paper on equal-time commutators (ETC's), Johnson and Low[2] gave a method for determining ETC's and non-c-number Schwinger terms directly from Feynman diagrams. Schroer and Stichel[3] have criticized this method on the grounds that T products are generally ill-defined, and have developed a new method. They use the Yang-Feldman equations to express the commutator in Jost-Lehmann-Dyson (JLD) representation, from which the relevent ET behavior is obtained. This method has the practical disadvantage of being difficult to apply in higher orders. The present approach is a synthesis of these methods.

The organization is as follows. In Sec. I, currents which are sources of fields are studied in perturbation theory. The matrix elements of current commutators are defined, and the ET terms which occur in the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas are specified. It is shown that to all orders of perturbation theory, the matrix elements of commutators may be expressed in spectral form using Feynman diagrams. Once this JLD representation is obtained, methods similar to those of Schroer and Stichel may be used to determine ET behavior. The procedure is simplified by the use of tables given by Bogoliubov and Parasiuk, and Bogoliubov and Shirkov.[4]

Section II A begins with a natural generalization of the Schwartz convolution[5] of tempered distributions. A corresponding generalization of pointwise multiplication of continuous functions is introduced and a convolution theorem is proved. Cases of particular interest in current algebra (products of distributions with θ and δ functions) are examined in some detail.

Section II B contains illustrative applications of the techniques of Sec. II A to procedures used in current algebra. It is shown that with one definition of the ET limit, an ambiguity discussed in Sec. I disappears.

Section III contains a brief summary and concluding remarks.

The following conventions are adopted. In Sec. I, the notation is that of, e.g., Bjorken and Drell.[6] $A^\dagger$ denotes the Hermitian adjoint of an operator A, and c^* denotes the complex conjugate of the complex number c.

In Sec. II, $\mathcal{S}(R^d)$ denotes the Schwartz space of smooth, rapidly decreasing functions on d-dimensional Euclidean space. The dual of $\mathcal{S}(R^d)$ is denoted $\mathcal{S}'(R^d)$. The "action" of a distribution $T \in \mathcal{S}'(R^d)$ on a test function $u \in \mathcal{S}(R^d)$ is indicated by placing T in square brackets and u in parenthesis; e.g.,

$$[f(x)](u(x)) = [f](u) = \int dx f^*(x)u(x).$$

The Fourier transform $\hat{u}$ of a testing function $u \in \mathcal{S}(R^d)$ is defined by

$$\hat{u}(k) = (2\pi)^{-d/2} \int dx\, e^{ikx} u(x),$$

and the Fourier transform $\hat{T}$ of a distribution $T \in \mathcal{S}'(R^d)$ is defined by $[\hat{T}](\hat{u}) = [T](u)$.

The symbol $(0, \infty)$ denotes the open interval of real numbers x: $0 < x$; $[0, \infty)$ denotes the closed interval $x: 0 \leq x$.

I. ET TERMS IN PERTURBATION THEORY

The course of the present investigation is indicated in Fig. 1. The idea is that the ambiguities one encounters following the vertical line should be resolved so that the diagram becomes commutative; the properties of scattering amplitudes calculated using the techniques of current algebra should agree with the properties calculated using the Feynman rules directly. In Sec. I A the various singular terms to be encountered along the vertical line are formally defined. Section I B contains more-precise definitions and techniques for calculating these objects in perturbation theory. Ambiguities are resolved by the means just described.

A. Formal Definitions of Objects to be Studied

1. Current Commutators and ET *Commutators*

In perturbation theory, inner products (S-matrix elements) of the form $\langle a_{\text{out}}|b_{\text{in}}\rangle$ and matrix elements of the form $\langle a_{\text{out}}|TA(x)B(y)|b_{\text{in}}\rangle$ are defined by the Gell-Mann–Low expansion and the Lehmann-Symanzik-Zimmermann reduction formula. Here

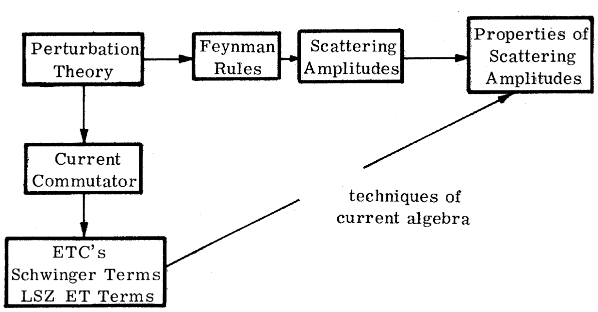

FIG. 1. Two methods for calculating properties of scattering amplitudes in perturbation theory. Consistency is obtained by equating (LSZ-T-ET terms) = seagull diagrams and (LSZ-Θ-ET terms) = (seagull diagrams + renorm. ET terms).

T denotes time-ordered products; $A(x)$ and $B(y)$ are Wick-ordered products of (derivatives of) fields. Accepting these quantities as defined, a reasonable formal definition of the commutator $[A(x), B(y)]$ is given by

Definition D1:

$$\langle a_{\text{out}}|[A(x), B(y)]|b_{\text{in}}\rangle = \sum_{n,m} \langle a_{\text{out}}|A(x)|n_{\text{in}}\rangle\langle m_{\text{out}}|n_{\text{in}}\rangle^* \langle m_{\text{out}}|B(y)|b_{\text{in}}\rangle - [A(x) \leftrightarrow B(y)]. \quad (1)$$

Definition D1 appears rather cumbersome. A less cumbersome definition is reasonable in some cases: We begin with the formal identity

$$[A(x), B(y)] = \epsilon(x_0 - y_0)[TA(x)B(y) - \bar{T}A(x)B(y)], \quad (2)$$

where $\bar{T}$ denotes the anti-time-ordered product. Next, using the relation

$$\langle a_{\text{out}}|\bar{T}A(x)B(y)|b_{\text{in}}\rangle = \langle b_{\text{in}}|TB^\dagger(y)A^\dagger(x)|a_{\text{out}}\rangle^* \quad (3)$$

we obtain

$$\langle a_{\text{out}}|[A(x), B(y)]|b_{\text{in}}\rangle = \epsilon(x_0 - y_0)[\langle a_{\text{out}}|TA(x)B(y)|b_{\text{in}}\rangle - \langle b_{\text{in}}|TB^\dagger(y)A^\dagger(x)|a_{\text{out}}\rangle^*]. \quad (4)$$

Finally, noting that when $\langle a|$ and $|b\rangle$ are stable single-particle states the distinction between in and out may be dropped, one obtains the following definition, valid for single-particle matrix elements.

Definition D2:

$$\langle a|[A(x), B(y)]|b\rangle = \epsilon(x_0 - y_0)[\langle a|TA(x)B(y)|b\rangle - \langle b|TA^\dagger(x)B^\dagger(y)|a\rangle^*]. \quad (5)$$

The individual matrix elements on the right-hand sides of D1 and D2 are well defined in renormalized perturbation theory; however both definitions suffer from ambiguity since the formal sums in D1 may diverge, and since one cannot unambiguously multiply arbitrary distributions by $\epsilon(x_0 - y_0)$. One expects that the sum in D1 should be defined so as to agree with D2 for single-particle matrix elements. Formal definitions, analogous to D1 and D2, may also be given for other (e.g., Wightman) products of fields.

These formal ambiguities may be entirely eliminated provided the matrix element of the time-ordered product can be written in the spectral form

$\langle\alpha|TA(x)B(y)|\beta\rangle$

$$=\int dM^2 C^{AB}(x-y,\, x+y;\, M^2,\, \alpha,\, \beta)\, i\Delta_F(x-y;\, M^2), \tag{6}$$

where the spectral function $C^{AB}(x-y, x+y; M^2, \alpha, \beta)$ has the support properties (in momentum space) appropriate to a JLD spectral function for the commutator of A and B. In this case the unordered product is simply obtained by replacing $i\Delta_F$ with Δ_+ in the integrand. [The replacement has no effect for spacelike $(x-y)$ where the T product and unordered product agree, and global validity follows from Wightman analyticity.] Thus once the spectral representation (6) is obtained, one has the following unambigous definition.

Definition D3:

$$\langle\alpha|[A(x), B(y)]|\beta\rangle = \int dM^2 C^{AB}\Delta(x-y;\, M^2). \tag{7}$$

In Sec. IB the relevant spectral functions C^{AB} are obtained directly from Feynman diagrams.

In order to define the ETC of two currents, we first define the commutator matrix elements as distributions in $\mathcal{S}'(R^8)$ using D3 and then take the ET limit of the resulting distribution. Thus we formally define

$$\langle a_{\text{out}}|[A(x), B(y)]_{x_0=y_0}|b_{\text{in}}\rangle = \lim_{x_0\to y_0} \langle a|[A(x), B(y)]|b\rangle. \tag{8}$$

It is well known that this definition is often ambiguous in practice; the limit depends on the precise mechanism by which the limit is taken. The point is that such ambiguities can be resolved by demanding consistency with the Feynman rules in practical applications: the limit on the right-hand side of (6) should be taken in such a way that the resulting object is really what appears in a sum rule.

2. *Schwinger Terms and* LSZ ET *Terms*

It frequently occurs that the ETC of the two currents is not well defined as a distribution in $\mathcal{S}'(R^7)$ but only as a linear functional on a proper subspace of $\mathcal{S}(R^7)$; in such a case we refer to the ETC as a real Schwinger term, in strict analogy to the case in quantum electrodynamics by Schwinger[7] and also by Brandt[8] and others. Suppose, for example, that one has

$$\langle a|[A(x), B(y)]_{x_0=y_0}|b\rangle = \alpha\delta(\vec{x}-\vec{y}) + \beta\partial_{\vec{k}}\delta(\vec{x}-\vec{y}). \tag{9}$$

We refer to the quantity $\beta\partial_{\vec{k}}\delta(\vec{x}-\vec{y})$ as a real Schwinger term only in the case that α is infinite.

When one formally performs the differentiation occurring in the LSZ reduction formulas, one encounters other ET terms, formally defined by

$$(\text{LSZ-}T\text{-ET}) = \Box_x T\phi(x)j(y) - T(\Box_x\phi(x)j(y)),$$
$$(\text{LSZ-}R\text{-ET}) = \Box_x\Theta(x_0-y_0)[\phi(x), j(y)] - \Theta(x_0-y_0)[\Box\phi(x), j(y)]. \tag{10}$$

Thus formally

$$(\text{LSZ-}T\text{-ET}) = (\text{LSZ-}R\text{-ET}) = 2\delta(x_0-y_0)\partial_{x_0}[\phi(x), j(y)] + \delta'(x_0-y_0)[\phi(x), j(y)]. \tag{11}$$

In various models in which Schwinger terms occur, the LSZ ET terms are infinite; in order to understand the physical significance of the former, the latter must also be understood.

Having formally defined the objects to be investigated, we now turn to the problem of finding corresponding precise definitions in perturbation theory.

B. Diagrams Corresponding to LSZ ET Terms and Commutators

1. *Diagramatic Identification of* LSZ-T-ET *Terms*

The LSZ ET terms have been formally defined as the difference between $\Box TA_\mu(x)j_\nu(y)$ and $Tj_\mu(x)j_\nu(y)$, where A_μ denotes an interpolating field and j_μ denotes a current expressed as a Wick-ordered product of fields, satisfying the Lagrange equations $\Box A_\mu = j_\mu$.[9] Each matrix element of $TA_\mu(x)j_\nu(y)$ corresponds to a set of diagrams. These diagrams fall into two classes: (i) the diagrams in which A_μ is directly connected to a point representing the interaction Lagrangian, and (ii) the seagull diagrams, in which A_μ is directly connected to the point representing $j_\nu(y)$. (The latter diagrams occur when the operator j_ν contains the field A explicitly, as in scalar electrodynamics.) The diagrams corresponding to $Tj_\mu(x)j_\nu(y)$ are in one-to-one correspondence with those of the first class; thus the desired identification is obtained upon identifying the LSZ-T-ET terms with seagull diagrams. The seagull[10] diagrams vanish for $x\neq y$ and thus have the support properties one expects from the formal definitions. It will be shown that these seagull diagrams also contribute to the retarded-commutator LSZ-R-ET terms.

2. *Diagrams for Current Commutators, Spectral Representations, and* LSZ-R-ET *Terms*

Diagrams may be given for current commutators corresponding to each of the definitions, D1, D2, D3. Thus to find the contributions of given order to the matrix element $\langle a_{\text{out}}|[j_\mu(x), j_\nu(y)]|b_{\text{in}}\rangle$

using definition D1, one first draws all diagrams for the corresponding T product. Pairs of cuts are then made in the diagrams, such that the out-external lines and the point representing $j_\mu(x)$ are separated by both cuts from the in-external lines and the point representing $j_\nu(y)$. The cut lines correspond to sums over the intermediate states $|n\rangle$ and $|m\rangle$ of (1), and on them the Feynman propagators are replaced by $\delta_+(p^2-m^2)$. On uncut lines lying between the cuts the Feynman propagators are replaced by their complex conjugates. The contribution to the commutator is obtained by summing over all such cut diagrams (each multiplied by the appropriate numerical factor), then subtracting the contributions of similar cut diagrams in which the roles of $j_\mu(x)$ and $j_\nu(y)$ are interchanged in making the cuts. Instead of dealing directly with these diagrams,[11] we attempt to satisfy the conditions of definition D3.

Feynman diagrams may be expressed in spectral form by completing the square of the relevant momentum variable. Let $\hat{T}_{\mu\nu}(k_1, k_2)$ denote the Fourier transform of a matrix element of a time-ordered product:

$$\hat{T}_{\mu\nu}(k_1, k_2)=\int dx dy\, e^{i(k_1x-k_2y)}\langle p_{\beta\,\mathrm{out}}|Tj_\mu(x)j_\nu(y)|p_{\gamma\,\mathrm{in}}\rangle . \tag{12}$$

Each Feynman diagram denotes a contribution to $\hat{T}_{\mu\nu}(k_1, k_2)$ of the form $\delta(q-k_2+k_1)\,\hat{I}_{\mu\nu}(p_\beta, p_\gamma, K)$, where $K=k_1+k_2$ and $q=\sum p_\beta - \sum p_\gamma$, and where

$$\hat{I}_{\mu\nu}=\lim_{\epsilon\to 0_+}\int dl_1\cdots l_r\,\frac{N_{\mu\nu}(K, p_\beta, p_\gamma, p_j)}{\prod(p_j^2-m_j^2+i\epsilon)}.$$

Introducing the Feynman α parameters and using standard techniques,[12] $\hat{I}_{\mu\nu}$ may be expressed as a sum of terms of the form

$$\hat{I}_{\mu\nu}=\int d\alpha_1\cdots\alpha_r\,\frac{\delta(1-\sum\alpha_i)\mathcal{P}_{\mu\nu}(K, p_\beta, p_\gamma, \alpha)F(J)}{d^2(\alpha)}, \tag{13}$$

where

$$J=A(\alpha)\{[K-Q(\alpha, p_\beta, p_\gamma)]^2-M^2(p_\beta, p_\gamma, \alpha)+i\epsilon C(\alpha)\}, \tag{14}$$

where $P=\sum p_\beta+\sum p_\gamma$ and where for convergent diagrams the function F is an inverse power. The functions d^2, A, C, M^2, Q, $\mathcal{P}_{\mu\nu}$ are obtained using the standard rules, then completing the square with respect to the variable K.

Now (14) may be expressed in spectral form by integrating over subsets of the α parameters on which the functions Q and M^2 are constant.[13] Thus

$$\hat{I}_{\mu\nu}=\int dQ\, dM^2\{\rho_{\mu\nu}^{(+)}(\partial_{M^2})^{n-1}[(K-Q)^2-M^2+i\epsilon]^{-1}+\rho_{\mu\nu}^{(-)}(\partial_{M^2})^{n-1}[(K-Q)^2-M^2-i\epsilon]^{-1}\}, \tag{15}$$

with

$$\rho_{\mu\nu}^{(\pm)}=\rho_{\mu\nu}^{(\pm)}(K, p_\beta, p_\gamma, M^2) =\int d\alpha_1\cdots\alpha_r\,\frac{\mathcal{P}_{\mu\nu}\delta(1-\sum\alpha_i)\Theta(\pm C)\delta(M^2-M^2(p_\beta, p_\gamma, \alpha))\delta(Q-Q(\alpha, p_\beta, p_\gamma))}{d^2(\alpha)[A(\alpha)]^n(n-1)!} . \tag{16}$$

Taking the Fourier transform of (15), one finds that each $I_{\mu\nu}$ corresponds to the contribution $I_{\mu\nu}(x, y)$ to the matrix element of the time-ordered product, with

$$I_{\mu\nu}(x, y)=e^{iq(x+y)/2}\int dM^2 dQ\, e^{iQ(x-y)}[\rho_{\mu\nu}^{(+)}(i\partial_x-i\partial_y, p_\beta, p_\gamma, M^2)\partial_{M^2}{}^{n-1}\Delta_F(x-y; M^2)-\rho_{\mu\nu}^{(-)}\partial_{M^2}{}^{n-1}\Delta_F^*(x-y, M^2)]. \tag{17}$$

Making the appropriate substitution of Green's functions, one obtains the contribution $C_{\mu\nu}(x, y)$ to the commutator matrix element $\langle p_\beta|[j_\mu(x), j_\nu(y)]|p_\gamma\rangle$,

$$C_{\mu\nu}(x, y)=e^{iq(x+y)/2}\int dM^2 dQ[(-\partial_{M^2})^{n-1}(\rho_{\mu\nu}^{(+)}-\rho_{\mu\nu}^{(-)})]\Delta(x-y; M^2). \tag{18}$$

[The fact that $\Delta(x; M^2)$ vanishes faster than any polynomial in M^2 as $M^2\to\infty$ has been used to eliminate surface terms in M^2. It should be noted that $\rho_{\mu\nu}$ has a polynomial dependence on the derivative (d_x-d_y).[14]]

The LSZ-R-ET terms may now be determined by the requirement that scattering amplitudes defined by the two reduction formulas agree. When $\rho_{\mu\nu}^{(-)}$ vanishes and products of distributions are defined such that the retarded commutator is obtained by replacing $\Delta(x-y; M^2)$ with $\Theta(x_0-y_0)\Delta(x-y; M^2)$ then, since the Fourier transforms of these distributions agree for positive frequencies, the desired consistency is obtained by equating the LSZ-R-ET terms with the LSZ-T-ET terms, the latter corresponding to seagull diagrams.

It has been assumed that the initial expression (17) is well defined, and in particular that $(M^2)^{-n}\rho(M^2)$ is integrable. This should occur in every theory in which the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization procedures are used to define the time-ordered matrix elements, for these procedures have been proved to yield tempered distributions corresponding to each Feynman diagram.[4,15,16] Schwinger terms should occur in the matrix elements of such renormalized theories only when they occur in the *integrand* of (17). However, for completeness we now examine the case in which $(M^2)^{-n}\rho(M^2)$ is not integrable, and (17) is ambiguous. The following method for resolving such ambiguities has been given by Steinman.[17]

Since the distribution Δ_F is a solution of the inhomogeneous Klein-Gordon equation, one has, for $b \neq m^2$,[18]

$$\Delta_F(x;\,M^2)=\sum_{n=0}^{n_0-1}\left(\frac{\Box+b}{b-m^2}\right)^n\frac{\delta(x)}{b-m^2}+\left(\frac{\Box+b}{b-m^2}\right)^{n_0}\Delta_F(x;\,m^2) \tag{19}$$

and thus, for $b<0$,

$$\int_0^\infty dM^2\rho(M^2)\Delta_F(x;\,M^2)=\sum_{n=0}^{n_0-1}a_n(\Box+b)^n\delta(x)+\int dM^2\rho(M^2)\left(\frac{\Box+b}{b-m^2}\right)^{n_0}\Delta_F(x;\,M^2). \tag{20}$$

In the case that $(m^2-b)^{-1}\rho$ is integrable, the coefficients a_n are well defined and are given by

$$a_n=\int dM^2\frac{\rho(M^2)}{(b-M^2)^{n+1}}.$$

When $(m^2-b)^{-1}\rho$ is not integrable, several of the coefficients are formally infinite, and must be specified by definition as part of a renormalization scheme. Applying a similar procedure in our slightly more complicated case, we arrive at an expression to replace (17),

$$I'_{\mu\nu}(x,\,y)=(\text{renorm. ET terms})+e^{iq(x+y)/2}\int dQdM^2e^{iQ(x-y)}\rho_{\mu\nu}\partial_{M^2}{}^{n-1}\left[\left(\frac{\Box+b}{b-M^2}\right)^{n_0}\Delta_F\right](x-y;\,M^2). \tag{21}$$

Here the renorm. ET terms all have support consisting of the origin $x-y=0$, and n_0 is the smallest nonnegative integer such that the integral is well defined. This does not change the definition (18) of the commutator matrix element. If the retarded commutator is defined by replacing the commutator function $\Delta(x-y;\,m^2)$ by the free-field retarded commutator in (18), then to obtain consistency one must equate

$$[\text{LSZ-}R\text{-ET terms}]=[\text{renorm. ET terms}]+[\text{LSZ-}T\text{-ET terms}]. \tag{22}$$

At this point, the program proposed in Fig. 1 has been completed for the case in which currents are the sources of fields. Current commutators and ET terms have been defined so that the retarded commutator LSZ reduction formula agrees with the Feynman rules. Thus any property of the scattering amplitude one deduces from properties of the current commutator will agree with the same properties obtained directly from the Feynman rules. In the Sec. II we take up the matter of deriving such properties; equal-time commutators are defined, and a convolution theorem is proved and used to derive sum rules.

II. CONVOLUTION OF DISTRIBUTIONS

A. Sequential Products and Convolutions

1. *The Schwartz Convolution*

The Schwartz convolution $[T_1 * T_2]$ of a pair of distributions T_1 and T_2 in $\mathcal{S}'(R^d)$ is defined by its action on a test function $u\in\mathcal{S}(R^d)$:

$$[T_1*T_2](u)=[T_1(k_1)\otimes T_2(k_2)](u(k_1+k_2)) \tag{23}$$

when (under conditions we now specify) the right-hand side of (23) is well defined. The direct product $T_1(k_1)\otimes T_2(k_2)$ is a well-defined element of $\mathcal{S}'(R^{2d})$; however, since $u(k_1+k_2)$ is not an element of $\mathcal{S}(R^{2d})$, (23) is not meaningful for arbitrary pairs of distributions. Suppose that the supports of T_1 and T_2 are such that there exists an infinitely differentiable function $v(k_1,\,k_2)$ taking the value 1 identically within the support of $T_1(k_1)\otimes T_2(k_2)$ and such that the product $v(k_1,\,k_2)u(k_1+k_2)$ is in $\mathcal{S}(R^{2d})$. Under such conditions, (23) is defined by

$$[T_1*T_2](u)=[T_1(k_1)\otimes T_2(k_2)](u(k_1+k_2)v(k_1-k_2)). \tag{24}$$

These conditions are met, for example, when $T_1,\ T_2\in\mathcal{S}'(R)$ and the supports of T_1 and T_2 are semibounded on the same side. Schwartz has also proved that the convolution (in this sense) of two distributions in $\mathcal{S}'(R^4)$ exists provided their supports lie in the forward light cone. One frequently encounters physical problems, the formal solution of which involves convolutions of distributions lying outside the domain of the Schwartz convolution. The following is a straightforward generalization of the Schwartz convolution.

2. *Sequential Convolutions and Products*

Let $\langle v_n\rangle$ be a sequence of test functions each in $\mathcal{S}(R^d)$ with the following properties: (i) $v_n(k)u(k)\to u(k)$ for each $u\in\mathcal{S}(R^d)$ and (ii) $\hat{v}_nh\to 0$ for each $h\in C_\infty(R^d)$ such that the support of the h excludes the origin; in each case convergence is in $\mathcal{S}(R^d)$.

We now define the convolution with respect to the sequence $\langle v_n\rangle$ of a pair of distributions T_1 and T_2 by

$$[T_1 * T_2]_{v_n}(u) \equiv \lim_{n\to\infty} [T_1(k_1)\otimes T_2(k_2)](u(k_1+k_2)v_n(\tfrac{1}{2}(k_1-k_2))). \tag{25}$$

The sequential convolution $(T_1 * T_2)_{v_n}$ is thus defined as a linear functional on a subspace of $\mathcal{S}(R^d)$; whenever the Schwartz convolution of T_1 and T_2 exists, the sequential convolution is well defined, and agrees with the Schwartz convolution (independent of sequence). On the other hand, it frequently happens that $[T_1 * T_2]_{v_n}$ is a sequence-independent element of $\mathcal{S}'(R^d)$ when the Schwartz convolution does not exist.

We now define a product of distributions which similarly generalizes the pointwise product of continuous functions. Let $\langle w_n\rangle$ be a sequence of functions in $\mathcal{S}(R^d)$ with the following properties: (i′) $w_n * u \to u$ in $\mathcal{S}(R^d)$ for each $u\in\mathcal{S}(R^d)$ and (ii′) $w_n h\to 0$ in $\mathcal{S}(R^d)$ for each $h\in C_\infty(R^d)$ such that the support of h excludes the origin. The product $[T_1\cdot T_2]$ of a pair T_1, T_2 of distributions in $\mathcal{S}'(R^d)$ with respect to the sequence $\langle w_n\rangle$ is defined by

$$[T_1\cdot T_2]_{w_n}(u) = \lim_{n\to\infty} [T_1(x)\otimes T_2(y)](u(\tfrac{1}{2}(x+y))w_n(x-y)). \tag{26}$$

$[T_1\cdot T_2]_{w_n}$ is again a linear functional on a subspace of $\mathcal{S}(R^d)$; when T_1 and T_2 are continuous functions, the product $[T_1\cdot T_2]_{w_n}$ agrees with the pointwise product.

Property (ii′) guarantees that the sequential product is "local" which we state precisely as

Lemma 1. Let T_1 and T_2 have finitely separated supports. Then $[T_1\cdot T_2]_{w_n}=0$.

Proof. The assumption of finite support separation implies the existence of a real $\epsilon>0$ such that support $[T_1(x)\otimes T_2(y)]\subset\{(x,y)\,|\,|x-y|>\epsilon\}$. Pick $h\in C_\infty(R^d)$ such that $h(z)$ vanishes for $|z|<\frac{1}{4}\epsilon$, and takes the value 1 identically for $|z|\geq\frac{1}{2}\epsilon$. Then

$$\begin{aligned}\lim_{n\to\infty}&[T_1(x)\otimes T_2(y)](w_n(x-y)u(\tfrac{1}{2}(x+y)))\\ &=\lim[T_1(x)\otimes T_2(y)](h(x-y)w_n(x-y)u(\tfrac{1}{2}(x+y)))\\ &=[T_1(x)\otimes T_2(y)](\lim_{n\to\infty} h(x-y)w_n(x-y)u(\tfrac{1}{2}(x+y)))=0.\end{aligned} \tag{27}$$

For each sequence v_n satisfying (i) and (ii), the corresponding sequence $\tilde{v}_n=(2\pi)^{-d/2}v_n$ satisfies (i′) and (ii′), and the following convolution theorem holds.

Theorem 1. With conventions as above,

$$([T_1 * T_2]_{v_n})^\wedge = (2\pi)^{d/2}[T_1\cdot T_2]_{\tilde{v}_n}.$$

It follows that when T_1 and T_2 are distributions of the form $T_i=S_i+u_i$, where the $\hat{S}_i$ have finitely separated supports and $u_i\in\mathcal{S}(R^d)$, then $[T_1 * T_2]_{v_n} = S_1 * u_2 + u_1 * u_2 + u_1 * S_2$; these conditions admit T_i which are unbounded functions with unbounded supports.

Dilation sequences are particularly useful sequences satisfying (i) and (ii). One easily verifies that for each $v\in\mathcal{S}(R^d)$ with $v(0)=1$, the corresponding sequence $v_n(x)=v(x/n)$ satisfies the requirements. (It should be noted that no support restrictions need be made; thus simple exponentials may be used.) For such dilation sequences the subscript n will be omitted from the corresponding products and convolutions.

3. *Products of Distributions in $\mathcal{S}'(R)$ with Θ Functions*

It is readily verified that, for any distribution T,

$$[1\cdot T]_{w_n} = [T\cdot 1]_{w_n} = T$$

and that

$$[(T_1+T_2)\cdot T]_{w_n} = [T_1\cdot T]_{w_n} + [T_2\cdot T]_{w_n}$$

whenever the latter both exist. Let $T^\pm\in\mathcal{S}'(R)$ have support contained in $[0,\pm\infty)$, support $w_n^\pm\subset(0,\pm\infty)$, and $\Theta_\pm(x)=\Theta(\pm x)$. Since $w_n^\pm(x-y)$ vanishes identically within the support of $[\Theta_\mp(x)\otimes T^\pm(y)]$, it follows that (cf. Fig. 2)

$$[\Theta_\mp\cdot T^\pm]_{w_n^\pm}=0, \tag{28}$$

and since $[1\cdot T^\pm]_{v_n^\pm}=T^\pm$, one obtains

$$[\Theta_\pm\cdot T^\pm]_{w_n^\pm}=T^\pm. \tag{29}$$

Similarly, locality of the product (·) implies that

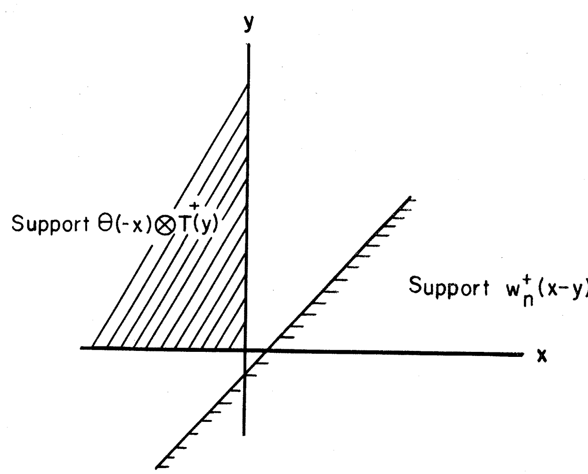

FIG. 2. $[\Theta_-\cdot T^+]_{w_n^+}=0$.

for every suitable sequence $\langle w_n\rangle$ and real $\epsilon > 0$,

$$[\Theta(\pm(x-\epsilon))\cdot T^{\pm}]_{w_n} = \lim_{\epsilon\to 0_+}[\Theta(\pm(x-\epsilon))\cdot T^{\pm}]_{w_n} = T^{\pm}, \tag{30}$$

and that for every function $u\in \mathcal{S}(R)$ which vanishes at the origin, $u=u^+ + u^-$, support $u^{\pm}\subset(0,\pm\infty)$, suitable sequence $\langle w_n\rangle$, and distribution T,

$$[\Theta_{\pm}\cdot T]_{w_n}(u) = [T](u_{\pm}). \tag{31}$$

Unfortunately, these results are the strongest which generally obtain; the sequential product of an arbitrary distribution with a Θ function is not generally sequence-independent (or even finite) at the origin. One may decompose any distribution into two parts $T=T^+ + T^-$, support $T^{\pm}\subset[0,\pm\infty)$. The operationally defined product with $\Theta_{\pm}$ of each part is given above; however, the decomposition is unique only up to a finite number of δ functions at the origin. In Sec. I, we have effectively made such decompositions and resolved the ambiguity at the origin using physical arguments.

It is worth noting that Fourier transform of Eq. (30) gives meaning to the Hilbert transform of an unbounded function.[5]

4. *Products of Distributions with δ Functions, and Distributions Evaluated at a Point*

Let $\langle w_n\rangle$ be a sequence of functions satisfying (i′) and (ii′). The value of a distribution T at the point $x=b$, defined with respect to the sequence $\langle w_n\rangle$, is given by[19]

$$T_{w_n}(b) = \lim_{n\to\infty} T(w_n(x-b)). \tag{32}$$

Property (ii′) guarantees that the definition is strictly local – it is not necessary that the functions w_n have compact support.

The value $T_{w_n}(0)$ of a distribution T at the origin is simply related to the product $[\delta(x)\cdot T(x)]_{w_n}$:

$$\begin{aligned}[\delta(x)\cdot T(x)]_{w_n}(u) &= \lim_{n\to\infty}[\delta(x)\otimes T(y)](u(\tfrac{1}{2}(x+y))w_n(x-y)) \\ &= \lim_{n\to\infty}[T(y)](u(\tfrac{1}{2}y)w_n(-y)) \\ &= [u(\tfrac{1}{2}y)T(y)]_{\tilde{w}_n}(0),\end{aligned} \tag{33}$$

where $\tilde{w}_n(x) = w_n(-x)$.

The proof of Theorem 1 of Ref. 18 is easily extended to give the following theorem.

Theorem 2. For any causal distribution T in $\mathcal{S}'(R^4)$, there exists a homogeneous polynomial P of finite degree in the variables (x_0, x_1, x_2, x_3) such that $[P(x)T]_{w_n}(0)$ is well defined and sequence-independent.

Having given rigorous operational definitions of the formal products and convolutions of distributions which occur in current algebra, we return to the topics of Sec. I.

B. Applications of Results of Sec. II A to Current Algebra

We stress again the necessity of the approach taken in Sec. I to define commutators in terms of specified T products of operators. One may give various interpretations to the formal product of a Θ function with a distribution – the difference between any two such interpretations is a distribution with support consisting of the origin. In Sec. I, a particular interpretation was chosen, and the ambiguity at the origin was resolved by demanding agreement between two versions of the LSZ reduction formula. One hopes that in all cases of physical interest (including those involving currents which are not sources of fields) that a similar natural resolution of the ambiguity is possible. (Alternatively one might hope that the relevent distributions are smooth enough that no ambiguity arises – this latter hope runs counter to simple examples in low orders of perturbation theory.)

In the following sections we review two methods which have been used to derive sum rules, relate them, and indicate how they may be generalized to apply in the case that real Schwinger terms are present.

1. *Sum Rules for Off-Mass-Shell Scattering Amplitudes*

Let $C^{ij}_{\mu\nu}(x) = \langle p|[j_\mu(\frac{1}{2}x), j_\nu(-\frac{1}{2}x)]|p'\rangle$ denote the matrix element of a commutator of two currents, between improper states of definite momentum, and let $\langle w_n\rangle$ be a sequence of testing functions in $\mathcal{S}(R)$ with the properties (i′) and (ii′). The equal-time limit with respect to the sequence $\langle w_n\rangle$ of $C^{ij}_{\mu\nu}(x)$ may be expressed in terms of its Fourier transform:

$$\begin{aligned}[\mathrm{ET}\ \lim C^{ij}_{\mu\nu}]_{w_n}(u(\vec{x})) &= \lim_{n\to\infty}[C^{ij}_{\mu\nu}](w_n(x_0)u(\vec{x})) \\ &= \lim_{n\to\infty}[C^{ij}_{\mu\nu}]^{\wedge}(\hat{w}_n(k_0)\hat{u}(\vec{k})).\end{aligned} \tag{34}$$

The right-hand side of (34) is formally the integral of the Fourier transform of the commutator over the energy variable, and corresponds to a sum over intermediate states. [The sequence w_n plays the role of a convergence factor – examples exist in which $\hat{C}_{\mu\nu}$ is not integrable, but for which the limit in (34) is sequence-independent.]

Let w_n^+ be a suitable sequence, with support $w_n^+\subset(0,\infty)$. The ET limit of the commutator agrees with the ET limit of the retarded commutator when taken with the sequence w_n^+; this agreement holds independent of any ambiguities in the definition of the retarded commutator. Thus the commutator on the right-hand side of (34) may be replaced by the retarded commutator to obtain a sum rule involving the off-mass-shell scattering amplitude, defined by deleting the LSZ ET terms from the retarded-commutator matrix elements.

The method above is a slight generalization of a method used by Schroer and Stichel.[3] An alternative rigorous procedure is obtained by defining the product $[\delta(x_0)\cdot C_{\mu\nu}^{ij}(x)]_{w_n}$, and using the convolution theorem just proved to derive corresponding sum rules. This technique has been used formally by Bollini and Giambiagi[20] and later by Amati, Jengo, and Remiddi.[21] The relationship between these methods is contained in Eq. (33).

2. *Schwinger Terms*

Real Schwinger terms (as defined in Sec. I, variously denoted super-Schwinger terms, etc., elsewhere in the literature) correspond to the nonfiniteness of the limits in Eqs. (34). The presence and nature of Schwinger terms in any model can be determined by expressing the current commutator in the spectral form given in Sec. I, and using tables given in Ref. 4 which relate spectral functions with ET behavior. Theorem 2 states that although $C_{\mu\nu}^{ij}$ may fail to have a finite ET limit, it may be multiplied by a polynomial in (x_0, x_1, x_2, x_3) such that the resulting distribution has an ET limit; sum rules analogous to Eqs. (34) hold in such situations and they involve derivatives of scattering amplitudes. This occurs in scalar electrodynamics when an additional ϕ^4 interaction is introduced.

III. SUMMARY AND CONCLUDING REMARKS

A method has been given whereby the matrix elements of current commutators and their ET behavior may be unambiguously determined to all finite orders of perturbation theory. An ambiguity in the definition of retarded commutator is resolved by the requirement that the LSZ reduction formula agree with the Feynman rules. It is shown that for an appropriately defined ET limit, the ET limit of the retarded commutator agrees with that of the commutator, independently of how such ambiguities are resolved.

Rigorous versions of sum rules are obtained, and generalized versions of them are shown to hold even in the presence of highly singular Schwinger terms.

Studies of current commutators in low orders of perturbation theory have produced many examples in which the assumptions of current algebra break down. The present investigation has been an attempt to clarify such examples and to extend the techniques of current algebra to cover such cases. The full program indicated in Fig. 1 has thus far been carried out only for the case in which currents are the sources of fields. The spectral-function technique should be useful in extending the program to more interesting cases, such as the σ model.

The apparent "disentangling" of the JLD spectral representation to give similar representations for unordered and time-ordered products which apparently occurs in perturbation theory might be expected to hold more generally. One would like to know how generally it does occur.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Joseph Sucher, Professor Richard Brandt, and Professor Claudio Orzalesi for many helpful conversations on these topics.

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*Supported by funds from the National Science Foundation.

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[10]So named because in low orders they look like seagulls. Seagull terms have been studied in a different context by D. J. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969).

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[12]As in, e.g., R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix*, (Cambridge Univ. Press, Cambridge, England, 1966). These rules are also succinctly derived in W. Zimmermann, Commun. Math. Phys. 11, 1 (1968), in which it is shown that by modifying the "$i\epsilon$" prescription, Feynman integrals become Lebesgue integrals which converge to covariant distributions as $\epsilon \to 0$.

[13]Cases in which $F(u)$ is the indefinite integral of an inverse power (logarithmic terms) may be accommodated by an integration by parts, lumping the surface terms with "renorm. ET terms."

[14]In simple cases $\rho_{\mu\nu}$ is obtained from the Feynman parameters by a change of variables. $C_{\mu\nu}$ is causal and $C_{\mu\nu}$ has the support properties appropriate to a sum over positive-energy intermediate states provided $\rho_{\mu\nu}$ vanishes except for $M^2 > 0$ and Q lying within the intersection of the two half-cones $(x-P)^2=0$, $(x-P)_0<0$ and $(x+P)^2=0$, $(x+P)_0>0$. The restrictions on Q are always satisfied; positivity of the mass parameter does not hold in all cases.

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PHYSICAL REVIEW D VOLUME 4, NUMBER 6 15 SEPTEMBER 1971

Relativistic and Realistic Classical Mechanics of Two Interacting Point Particles*

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Classical mechanics of two point particles interacting at a distance is given a Lorentz-covariant formulation without introducing unphysical degrees of freedom such as usually accompany the two-time formalism. The theory is then quantized and compared with quantum field theory to allow the determination of realistic potentials. Exact solutions are obtained for an inverse distance potential; classical orbits as well as quantum energy levels are determined.

I. INTRODUCTION

There exists no widely accepted formulation of relativistic classical mechanics of two or more interacting particles of finite mass. For some time it was believed that no satisfactory theory was possible, until this was refuted[1] by the actual construction of self-consistent models. Existence theorems have only limited interest, however. The nonrelativistic theory is useful only because of the fact that the potentials happen to be known to considerable accuracy, and a relativistic theory should include a prescription for the potential in order that it predict effects like the precession of the perihelion of Mercury. The only sure source of knowledge, from which accurate potentials can – at least in principle – be derived, is relativistic quantum field theory. (We do not mean to discount the theory of general relativity, but to simplify the perspective by treating it as a field theory in flat space.) Hence it would seem plausible that relevant models of classical relativistic mechanics must be obtained deductively from relativistic quantum field theory, rather than inductively from nonrelativistic mechanics.

A direct deduction of a classical relativistic mechanics from quantum field theory has been given recently, and now, with the incomparable advantage of hindsight, it is possible to proceed inductively and arrive at the same theory by naive arguments based on nonrelativistic mechanics and the requirement of Lorentz invariance.

The theory that had been obtained previously from quantum field theory is recovered in Sec. VII as an