Massive Quantum Electrodynamics in the Infinite-Momentum Frame*

Davison E. Soper†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 20 May 1971)

We extend an earlier canonical formulation of quantum electrodynamics in the infinite-momentum frame by replacing the photons with massive vector mesons. The structure of the theory remains nearly the same except that a new term appears in the infinite-momentum Hamiltonian describing the emission of helicity-zero vector mesons with an amplitude proportional to the meson mass.

I. INTRODUCTION

Recently a canonical formalism for quantum electrodynamics in the infinite-momentum frame was developed by Kogut and the present author.[1] Since then, discussions of current commutators on the light cone in a quark–vector-gluon model by Cornwall and Jackiw,[2] by Dicus, Jackiw, and Teplitz,[3] and by Gross and Treiman[4] have made it seem useful to extend the canonical formalism of Ref. 1 by replacing the photons with massive vector mesons. The object of this paper is to provide such an extension.

We find that the required generalization is quite simple if we consider in addition to the vector field A^μ a scalar field B in the manner of Stückelberg's 1938 paper on massive vector mesons (gluons).[5,6] One significant result is that the formal structure of current commutators on the light cone[2,3] is unchanged by the introduction of the scalar field B.

The notation used here is that of Ref. 1, with two minor changes[7] designed to facilitate calculations in perturbation theory.[8] In addition, we make free use of the results of Ref. 1 and devote most of our attention to the changes made necessary by going from massless to massive vector mesons. It may be useful to recall briefly the main features of the notation used in Ref. 1. The components a^μ of a four-vector in the infinite-momentum coordinate system are related to the components $\hat{a}^\mu$ of that vector in the usual coordinate system by the transformation $a^0=2^{-1/2}(\hat{a}^0+\hat{a}^3)$, $a^1=\hat{a}^1$, $a^2=\hat{a}^2$, $a^3=2^{-1/2}(\hat{a}^0-\hat{a}^3)$. In particular, the components of a position vector are $x^\mu=(\tau, x^1, x^2, \mathfrak{z})$, where $\tau=2^{-1/2}(t+z)$ and $\mathfrak{z}=2^{-1/2}(t-z)$. In the infinite-momentum system, the coordinate τ plays the role of "time." The new components of a momentum vector are $p^\mu=(\eta,p^1,p^2,H)$, where $\eta=2^{-1/2}(E+p_z)$ and $H=2^{-1/2}(E-p_z)$. Finally, we recall that in the infinite-momentum coordinate system, the vector components $a_\mu=g_{\mu\nu}a^\nu$ with a lower index are related to the components labeled with an upper index by $a_0=a^3$, $a_1=-a^1$, $a_2=-a^2$, $a_3=a^0$. Thus, for instance, $p_\mu x^\mu=H\tau+\eta\mathfrak{z}-p^1x^1-p^2x^2$ and $\partial^0=\partial_3=\partial/\partial\mathfrak{z}$.

II. EQUATIONS OF MOTION

The canonical theory of quantum electrodynamics in the infinite-momentum frame[1] was based on the Lagrangian

$$\mathcal{L}(x)_{\rm QED}=\bar{\Psi}[(\tfrac{1}{2}i\overleftrightarrow{\partial}_\mu-eA_\mu)\gamma^\mu-m]\Psi-\tfrac{1}{4}(\partial^\nu A^\mu-\partial^\mu A^\nu)(\partial_\nu A_\mu-\partial_\mu A_\nu),$$

where $A^\mu(x)$ is the real vector field of the massless vector mesons and Ψ is a four-component Dirac field. In order to introduce a meson mass $\kappa>0$ and allow for mesons with helicity zero while maintaining gauge invariance, we introduce a real scalar field $B(x)$ in addition to A_μ and Ψ. Then we begin with the modified Lagrangian

$$\mathcal{L}(x)=\bar{\Psi}[(\tfrac{1}{2}i\overleftrightarrow{\partial}_\mu-eA_\mu)\gamma^\mu-m]\Psi-\tfrac{1}{4}(\partial^\nu A^\mu-\partial^\mu A^\nu)(\partial_\nu A_\mu-\partial_\mu A_\nu)+\tfrac{1}{2}(\kappa A^\mu-\partial^\mu B)(\kappa A_\mu-\partial_\mu B). \quad (1)$$

Variation of the fields Ψ, $\bar{\Psi}$, A_μ, and B gives the equations of motion

$$(\partial_\nu\partial^\nu+\kappa^2)A^\mu-\partial^\mu(\partial_\nu A^\nu+\kappa B)=J^\mu, \quad (2)$$

$$\kappa\partial_\mu A^\mu-\partial_\mu\partial^\mu B=0, \quad (3)$$

$$[(i\partial_\mu-eA_\mu)\gamma^\mu-m]\Psi=0, \quad (4)$$

where we have defined $J^\mu=e\bar{\Psi}\gamma^\mu\Psi$. [Notice that $\partial_\mu J^\mu=0$ as a consequence of the Dirac equation (4). Thus Eq. (3) is merely the divergence of Eq. (2).]

The reason for introduction of the seemingly superfluous scalar field B is that the gauge invariance of quantum electrodynamics is thereby preserved. Indeed, the Lagrangian, and hence the equations of motion, is left invariant by the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x),$$
$$B(x) \to B(x) + \kappa\Lambda(x), \tag{5}$$
$$\Psi(x) \to e^{-ie\Lambda(x)}\Psi(x).$$

We could, if we wanted, use this gauge invariance to choose the "Lorentz gauge" $B=0$. In this gauge the equations of motion would take the familiar form (after some simplifications)

$$(\partial_\nu\partial^\nu + \kappa^2)A^\mu = J^\mu,$$
$$\partial_\mu A^\mu = 0,$$
$$[(i\partial_\mu - eA_\mu)\gamma^\mu - m]\Psi = 0.$$

However, it turns out that it is very difficult to quantize the theory in the infinite-momentum frame in this gauge.

Instead, we choose the "infinite-momentum gauge,"

$$A^0(x) = 0. \tag{6}$$

Then the $\mu=0$ component of the equation of motion (2) reads[9]

$$\partial_3(\partial_3 A^3 + \partial_k A^k + \kappa B) = -J^0.$$

This equation can be solved for A^3 as follows:

$$A^3 = -\frac{i}{\eta}(\partial_k A^k + \kappa B) + \frac{1}{\eta^2}J^0, \tag{7}$$

where $1/\eta$ and $1/\eta^2$ are the integral operators[10]

$$\left(\frac{1}{\eta}f\right)(x) = -\tfrac{1}{2}i\int d\xi\,\epsilon(x^3-\xi)f(x^0, x, \xi),$$
$$\left(\frac{1}{\eta^2}f\right)(x) = -\tfrac{1}{2}\int d\xi\,|x^3-\xi|\,f(x^0, x, \xi).$$

Thus if we regard A^1, A^2, and B as independent dynamical variables, then A^3 is reduced to the status of a dependent field since it is determined at any "time" x^0 by the other fields at that x^0 according to the constraint equation (7).[11]

The equations of motion for the independent fields A^k and B can now be simplified by substituting the expression (7) for A^3 back into the equations of motion (2) and (3). From (7) we have

$$\partial_\nu A^\nu = -\kappa B - \frac{i}{\eta}J^0. \tag{8}$$

If we substitute this into (3) and remember that $\partial_\mu J^\mu = 0$, we get the equation of motion for B,

$$(\partial_\mu\partial^\mu + \kappa^2)B = -i\kappa\frac{1}{\eta}J^0. \tag{9}$$

If we substitute (8) into Eq. (2) with $\mu=1$ or 2, we get the equation of motion for A,

$$(\partial_\mu\partial^\mu + \kappa^2)A^k = J^k - i\frac{1}{\eta}\partial^k J^0. \tag{10}$$

The equations for the Dirac field are changed very little from those developed in Ref. 1 for quantum electrodynamics. The two components $\Psi_+ = P_+\Psi = \frac{1}{2}\gamma^3\gamma^0\Psi$ are independent dynamical variables. The two components $\Psi_- = P_-\Psi = \frac{1}{2}\gamma^0\gamma^3\Psi$ are dependent variables, to be determined by the constraint equation

$$\Psi_- = \frac{1}{2\eta}\gamma^0[-(i\partial_k - eA_k)\gamma^k + m]\Psi_+, \tag{11}$$

which follows from the Dirac equation. The equation of motion for Ψ_+ is

$$i\partial_0\Psi_+ = eA^3\Psi_+ + \tfrac{1}{2}[(i\partial_k - eA_k)\gamma^k + m]\gamma^3\Psi_-. \tag{12}$$

The only difference between this equation of motion and the corresponding equation in quantum electrodynamics is that A^3 depends on B through the constraint equation (7).

III. EQUAL-τ COMMUTATION RELATIONS AND FOURIER EXPANSIONS OF THE FIELDS

In order to make quantum fields out of the independent fields Ψ_+, $\vec{A}$, B we must specify their commutation relations at equal τ. By analogy with Ref. 1, we choose

$$\begin{aligned}
&\sqrt{2}\{\Psi_+(x)_\alpha, \Psi_+^\dagger(0)_\beta\}_{\tau=0} = (P_+)_{\alpha\beta}\delta(\mathfrak{z})\delta(\vec{x}),\\
&[A^i(x), A^j(0)]_{\tau=0} = -\tfrac{1}{4}i\delta_{ij}\epsilon(\mathfrak{z})\delta(\vec{x}),\\
&[B(x), B(0)]_{\tau=0} = -\tfrac{1}{4}i\epsilon(\mathfrak{z})\delta(\vec{x}),\\
&[\vec{A}(x), B(0)]_{\tau=0} = [\vec{A}(x), \Psi_+(0)]_{\tau=0}\\
&\qquad = [B(x), \Psi_+(0)]_{\tau=0}\\
&\qquad = \{\Psi_+(x), \Psi_+(0)\}_{\tau=0} = 0.
\end{aligned} \tag{13}$$

Using these commutation relations we can derive the commutation relations among the creation and destruction operators appearing in the Fourier expansion of the fields. Furthermore, the transformation properties of the fields under space translations in the transverse and $\mathfrak{z}$ directions and under rotations in the (x^1, x^2) plane determine the momentum and "infinite-momentum helicity"[12] of the states created and destroyed by these operators. Since the calculation is elementary, we only state the results. Let $b^\dagger(\eta, \vec{p}, s)$ [$d^\dagger(\eta, \vec{p}, s)$] be creation operators for electrons [positrons] with momentum $(\eta, \vec{p})$ and helicity s ($s=\pm\frac{1}{2}$). Let $a^\dagger(\eta, \vec{p}, \lambda)$ be creation operators for mesons with momentum $(\eta, \vec{p})$ and helicity λ ($\lambda = -1, 0, +1$). These operators have covariant commutation relations[13]

$$\begin{aligned}
\{b(p, s), b^\dagger(p', s')\}_+ &= \{d(p, s), d^\dagger(p', s')\}_+\\
&= \delta_{ss'}(2\pi)^3 2\eta\,\delta(\eta-\eta')\delta^2(\vec{p}-\vec{p}\,'),
\end{aligned} \tag{14}$$
$$[a(p, \lambda), a^\dagger(p', \lambda')] = \delta_{\lambda\lambda'}(2\pi)^3 2\eta\,\delta(\eta-\eta')\delta^2(\vec{p}-\vec{p}\,').$$

The expansion of $\Psi_+(x)$ at $\tau=0$ in terms of $b(p, s)$ and $d^\dagger(p, s)$ is

$$2^{1/4}\Psi_+(x)=(2\pi)^{-3}\int d\vec{p}\int_0^\infty \frac{d\eta}{2\eta}\sum_{s=\pm 1/2}[\sqrt{2\eta}\, w(s)e^{-ip\cdot x}b(p, s)+\sqrt{2\eta}\, w(-s)e^{+ip\cdot x}d^\dagger(p, s)], \tag{15}$$

where the spinors $w(s)$ are

$$w(+\tfrac{1}{2})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad w(-\tfrac{1}{2})=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{16}$$

The expansion of $\vec{A}(x)$ at $\tau=0$ contains creation and destruction operators for mesons with helicity +1 and −1; the expansion of $B(x)$ at $\tau=0$ contains creation and destruction operators for mesons with helicity 0:

$$\vec{A}(x)=(2\pi)^{-3}\int d\vec{p}\int_0^\infty\frac{d\eta}{2\eta}\sum_{\lambda=\pm1}[\vec{\epsilon}(\lambda)e^{-ip\cdot x}a(p,\lambda)+\vec{\epsilon}(\lambda)^*e^{+ip\cdot x}a^\dagger(p,\lambda)], \tag{17}$$

$$B(x)=(2\pi)^{-3}\int d\vec{p}\int_0^\infty\frac{d\eta}{2\eta}[-ie^{-ip\cdot x}a(p,0)+ie^{+ip\cdot x}a^\dagger(p,0)]. \tag{18}$$

The vectors $\vec{\epsilon}(\lambda)$ appearing in (17) are

$$\vec{\epsilon}(+1)=-2^{-1/2}(1, i),\quad \vec{\epsilon}(-1)=+2^{-1/2}(1,-i). \tag{19}$$

We may notice here that the formal structure of current commutators at equal τ in this theory, as discussed in Refs. 2 and 3, is unchanged by the addition of the scalar field $B(x)$. The equal-τ commutators $[J^\mu(x), J^\nu(0)]_{\tau=0}$ are calculated by writing the current $J^\mu=e(\bar{\Psi}_++\bar{\Psi}_-)\gamma^\mu(\Psi_++\Psi_-)$ in terms of the independent fields Ψ_+, $\vec{A}$, B and using the canonical commutators (13) to compute the current commutators. But according to the constraint equation (7), the dependent field Ψ_- (and hence the current) depends only on Ψ_+ and $\vec{A}$ but not on B.

IV. HAMILTONIAN

The invariance of the Lagrangian under τ translations provides us, using Noether's theorem, with a conserved canonical Hamiltonian

$$H=\int d\vec{x}d\mathfrak{z}\,\mathcal{H}(\tau,\vec{x},\mathfrak{z}), \tag{20}$$

where

$$\mathcal{H}=\bar{\Psi}\tfrac{1}{2}i\overleftrightarrow{\partial_0}\gamma^0\Psi-(\partial_0A_\alpha)(\partial_3A^\alpha)+(\partial_0B)(\partial_3B)-\mathcal{L}. \tag{21}$$

The first three terms in (21) cancel the terms in the Lagrangian containing ∂_0, and we are left with

$$\begin{aligned}\mathcal{H}=&-\bar{\Psi}[(\tfrac{1}{2}i\overleftrightarrow{\partial_k}-eA_k)\gamma^k-m]\Psi-\bar{\Psi}\tfrac{1}{2}i\overleftrightarrow{\partial_3}\gamma^3\Psi+eA^3\bar{\Psi}\gamma^0\Psi\\&-\tfrac{1}{2}(\partial_3A^3)(\partial_3A^3)-(\partial_3A^k)(\partial_kA^3)+\tfrac{1}{2}(\partial^kA^l)(\partial_kA_l)\\&-\tfrac{1}{2}(\partial^kA^l)(\partial_lA_k)-\tfrac{1}{2}\kappa^2A^kA_k-\tfrac{1}{2}(\partial^kB)(\partial_kB)\\&+\kappa A^k(\partial_kB)+\kappa A^3(\partial_3B).\end{aligned} \tag{22}$$

It is apparent that this form for the Hamiltonian is not very useful. However, if we substitute the expressions for A^3 and Ψ_- given by the constraint equations (7) and (11) into (22), then integrate the resulting expression to form H, and finally integrate by parts freely, we obtain a useful expression:

$$\begin{aligned}H=\int d\vec{x}d\mathfrak{z}\,\Big(&\tfrac{1}{2}e^2\sqrt{2}\,\Psi_+^\dagger\Psi_+\frac{1}{\eta^2}\sqrt{2}\,\Psi_+^\dagger\Psi_++e\sqrt{2}\,\Psi_+^\dagger\Psi_+\frac{1}{\eta}(\vec{p}\cdot\vec{A}-i\kappa B)\\&+\sqrt{2}\,\Psi_+^\dagger[m-(\vec{p}-e\vec{A})\cdot\vec{\gamma}]\frac{1}{2\eta}[m+(\vec{p}-e\vec{A})\cdot\vec{\gamma}]\Psi_++\tfrac{1}{2}\sum_{k=1}^2A^k(\vec{p}^{\,2}+\kappa^2)A^k+\tfrac{1}{2}B(\vec{p}^{\,2}+\kappa^2)B\Big).\end{aligned} \tag{23}$$

Here $\vec{p}$ is the transverse part of the differential operator $p^k=i\partial^k$, and $\vec{\gamma}=(\gamma^1,\gamma^2)$.

By using the equal-τ commutation relations (13), one can verify that the canonical Hamilton (23) actually generates τ translations in the theory. One finds, indeed, that $[iH,\vec{A}]=\partial_0\vec{A}$, $[iH,B]=\partial_0B$, and $[iH,\Psi_+]=\partial_0\Psi_+$, where the τ-derivatives of $\vec{A}$, B, and Ψ_+ are given by the equations of motion (9), (10), and (12).

An examination of the Hamiltonian (23) shows that the theory is changed very little when the vector-meson mass is changed from $\kappa=0$ to $\kappa>0$. One must, of course, introduce a helicity-zero meson into the theory and adjust the free-meson Hamiltonian from $\vec{p}^{\,2}/2\eta$ to $(\vec{p}^{\,2}+\kappa^2)/2\eta$. But the interactions among the electrons and helicity-(±1) mesons are unchanged, and the helicity-zero mesons interact with the electrons only through the very simple coupling $-ie\kappa\sqrt{2}\,\Psi_+^\dagger\Psi_+(1/\eta)B$. As $\kappa\to0$ this coupling vanishes, so that the helicity-zero me-

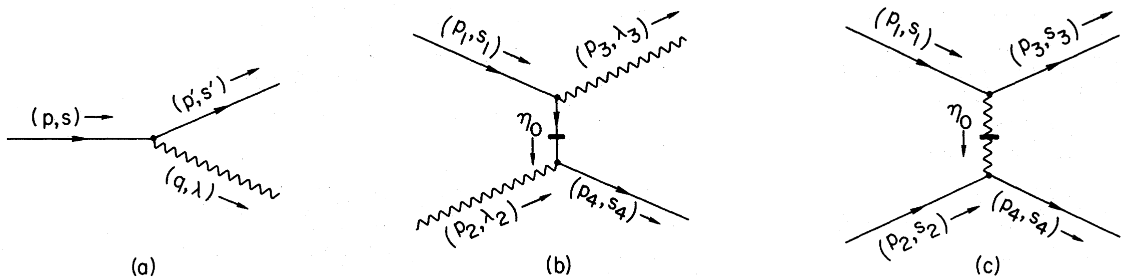

FIG. 1. Electron–vector-meson vertices.

sons are never produced.

We can illustrate the dynamics more vividly by writing out the rules for old-fashioned (τ-ordered) diagrams using the Hamiltonian (23).[14]

(1) A factor $(H_f - H + i\epsilon)^{-1}$ for each intermediate state.

(2) An over-all factor $-2\pi i\delta(H_f - H_i)$.

(3) For each internal line, a sum over spins and an integration

$$(2\pi)^{-3}\int d\vec{p}\int_0^\infty \frac{d\eta}{2\eta}.$$

(4) For each vertex, (a) a factor $(2\pi)^3\delta(\eta_{\rm out}-\eta_{\rm in})\times\delta^2(\vec{p}_{\rm out}-\vec{p}_{\rm in})$, (b) a factor $(2\eta)^{1/2}$ for each fermion line entering or leaving the vertex. [The factors $(2\eta)^{1/2}$ associated with each internal fermion line have the effect of removing the factor $1/2\eta$ from the phase-space integral.]

(5) Finally, a simple matrix element is associated with each vertex as a factor. There are three types of vertices, as shown in Fig. 1. The corresponding factors are (a) for single meson emission [Fig. 1(a)], a factor eM, where M is given by Table I, (b) for instantaneous electron exchange as shown in Fig. 1(b), a factor e^2/η_0 if all the particles are left-handed (otherwise, a factor zero), and (c) for the "Coulomb-force" vertex as shown in Fig. 1(c), a factor $e^2\eta_0^{-2}\delta_{s_1s_2}\delta_{s_3s_4}$.

V. FREE FIELDS

In this section and Sec. VI we will examine the question of whether the infinite-momentum formalism presented here is equivalent to the usual formalism for massive quantum electrodynamics developed in an ordinary reference frame. We begin with a short discussion of the free fields.

If the coupling constant e is zero, the equations of motion for the meson fields $\vec{A}$ and B are simply

$$(\partial_\mu\partial^\mu + \kappa^2)\vec{A}(x) = 0,$$
$$(\partial_\mu\partial^\mu + \kappa^2)B(x) = 0. \qquad (24)$$

These equations can be solved exactly, given initial conditions at $\tau = 0$. If (17) and (18) are the Fourier expansions of $\vec{A}(x)$ and $B(x)$ at time $\tau = 0$, then these same expansions will give $\vec{A}(x)$ and $B(x)$ for all τ if we put

$$p_0 = H(\eta, \vec{p}) = (\vec{p}^{\,2} + \kappa^2)/2\eta$$

in the exponentials $\exp(\pm ip\cdot x)$ inside the integrals.

With the solutions for $\vec{A}(x)$ and $B(x)$ in hand, we can write down $A^3(x)$ using the constraint equation (7). Finally, we recall that $A^0(x) = 0$. Thus we have the complete solution $(A^\mu(x), B(x))$ for the free vector-meson field in the infinite-momentum gauge. We can use the gauge transformation (5) to transform this solution back to the more familiar Lorentz gauge. To do this, we let

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x),$$
$$B'(x) = B(x) + \kappa\Lambda(x)$$

be the fields in the new gauge, and require that $B'(x) = 0$. Then

TABLE I. Matrix elements for meson emission. $p_\pm = 2^{-1/2}(p^1 \pm ip^2)$.

S	S'	λ	M
$\frac{1}{2}$	$\frac{1}{2}$	1	$(-q_-/\eta_q) + (p'_-/\eta')$
$\frac{1}{2}$	$\frac{1}{2}$	0	$-\kappa/\eta_q$
$\frac{1}{2}$	$\frac{1}{2}$	-1	$(+q_+/\eta_q) - (p_+/\eta)$
$\frac{1}{2}$	$-\frac{1}{2}$	1	$2^{-1/2}m\eta_q/\eta\eta'$
$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\frac{1}{2}$	$-\frac{1}{2}$	-1	0
$-\frac{1}{2}$	$\frac{1}{2}$	1	0
$-\frac{1}{2}$	$\frac{1}{2}$	0	0
$-\frac{1}{2}$	$\frac{1}{2}$	-1	$2^{-1/2}m\eta_q/\eta\eta'$
$-\frac{1}{2}$	$-\frac{1}{2}$	1	$(-q_-/\eta_q) + (p_-/\eta)$
$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\kappa/\eta_q$
$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$(+q_+/\eta_q) - (p'_+/\eta')$

$$A'_\mu(x) = A_\mu(x) - \kappa^{-1}\partial_\mu B(x). \tag{25}$$

(Note that this gauge transformation becomes singular in the limit $\kappa \to 0$.)

The free field $A'^\mu(x)$ which results from these operations can be written as

$$A'^\mu(x) = (2\pi)^{-3}\int d\vec{p}\int_0^\infty \frac{d\eta}{2\eta}\sum_{\lambda=-1}^{1}[e^\mu(p,\lambda)e^{-ip\cdot x}a(p,\lambda) + e^\mu(p,\lambda)^* e^{+ip\cdot x}a^\dagger(p,\lambda)], \tag{26}$$

where the polarization vectors $e^\mu(p,\lambda)$ are

$$\begin{aligned} e^\mu(p,1) &= -2^{-1/2}(0,1,i,(p^1+ip^2)/\eta),\\ e^\mu(p,-1) &= +2^{-1/2}(0,1,-i,(p^1-ip^2)/\eta),\\ e^\mu(p,0) &= \kappa^{-1}(\eta,p^1,p^2,H-\kappa^2/\eta)\\ &= \kappa^{-1}p^\mu - \delta_3^\mu\kappa/\eta. \end{aligned} \tag{27}$$

The field $A'_\mu(x)$ which we have obtained by canonical quantization in the infinite-momentum frame will be equal to the usual free vector field if the polarization vectors $e^\mu(p,\lambda)$ form an orthogonal set of spacelike unit vectors each orthogonal to p^μ:

$$\begin{aligned} e^\mu(p,\lambda)^* e_\mu(p,\lambda') &= -\delta_{\lambda\lambda'},\\ p^\mu e_\mu(p,\lambda) &= 0. \end{aligned} \tag{28}$$

A quick check shows that this is indeed the case.

One can also show, just as in Ref. 1, that the free Dirac field obtained in the infinite-momentum frame is equal to the usual Dirac field. We will not comment on this proof here except to note that the gauge change discussed above does not affect the Dirac field if $e=0$.

VI. SCATTERING THEORIES COMPARED

We have seen that massive quantum electrodynamics in the infinite-momentum frame is the same as ordinary massive quantum electrodynamics in the trivial case $e=0$. We cannot demonstrate that the two theories are the same for $e\neq 0$ since we are unable to solve for the exact interacting Heisenberg fields in either theory. However, it is possible to show that the perturbation expansions of the S matrix in the two theories are formally identical.

What we have to show is that the ordinary Feynman rules for massive quantum electrodynamics lead to the same expressions for scattering amplitudes as the rules for old-fashioned diagrams given in Sec. IV. Since the same demonstration has been given for quantum electrodynamics in Ref. 1, we will indicate here only how the argument can be modified to account for a nonzero meson mass and the contributions from helicity-zero mesons.

To that end, we examine the Feynman propagator for massive vector mesons

$$D_F(x)^{\mu\nu} = (2\pi)^{-4}\int d^4p\, e^{-ip\cdot x}\frac{-g^{\mu\nu}+p^\mu p^\nu/\kappa^2}{p^2-\kappa^2+i\epsilon}. \tag{29}$$

One can show (by simple computation if necessary) that

$$-g^{\mu\nu}+p^\mu p^\nu/\kappa^2 = \sum_{\lambda=\pm1}e^\mu(p,\lambda)e^\nu(p,\lambda)^* + \delta_3^\mu\delta_3^\nu\kappa^2/\eta^2 + \delta_3^\mu\delta_3^\nu(p^2-\kappa^2)/\eta^2 - (1/\eta)\delta_3^\mu p^\nu - (1/\eta)p^\mu\delta_3^\nu + p^\mu p^\nu/\kappa^2, \tag{30}$$

where the vectors $e(\eta,\vec{p},\lambda)$ are the polarization vectors for helicity ±1 defined in Eq. (27). If one used this expression in the numerator of the meson propagator, the last three terms will not contribute to any scattering process because of current conservation. Thus one is left with an effective propagator

$$D_F(x)^{\mu\nu} = (2\pi)^{-4}\int d^4p\, e^{-ip\cdot x}\Big[\sum_{\lambda=\pm1}e^\mu(p,\lambda)e^\nu(p,\lambda)^* + \delta_3^\mu\delta_3^\nu\frac{\kappa^2}{\eta^2}\Big](p^2-\kappa^2+i\epsilon)^{-1} + \delta_3^\mu\delta_3^\nu(2\pi)^{-4}\int d^4p\, e^{-ip\cdot x}\eta^{-2}\frac{p^2-\kappa^2}{p^2-\kappa^2+i\epsilon}. \tag{31}$$

The H integral in the first term can be done by contour integration as in Ref. 1. In the second term, $(p^2-\kappa^2)(p^2-\kappa^2+i\epsilon)^{-1}\to 1$ as $\epsilon\to 0$ so that the H integral gives a factor $\delta(\tau)$. Thus the meson propagator takes the form

$$D_F(x)^{\mu\nu} = -i(2\pi)^{-3}\int d\vec{p}\int_0^\infty\frac{d\eta}{2\eta}\Big[\sum_{\lambda=\pm1}e^\mu(p,\lambda)e^\nu(p,\lambda)^* + \delta_3^\mu\delta_3^\nu\frac{\kappa^2}{\eta^2}\Big][\Theta(\tau)e^{-ip\cdot x}+\Theta(-\tau)e^{+ip\cdot x}] + (2\pi)^{-3}\delta(\tau)\delta_3^\mu\delta_3^\nu\int d\vec{p}\int_{-\infty}^{\infty}d\eta\,\eta^{-2}e^{-i(\eta\mathfrak{z}-\vec{p}\cdot\vec{x})}, \tag{32}$$

where

$p_0 = H = (\vec{p}^{\,2} + \kappa^2)/2\eta$.

Note that this expression for the vector-meson propagator is nearly identical to the corresponding expression for the photon propagator derived in Ref. 1. In particular, the "Coulomb-force" term proportional to $\delta(\tau)$ remains unchanged.

There are only two changes in $D_F^{\mu\nu}$, which account for the corresponding changes in the perturbation theory rules of Sec. IV between $\kappa = 0$ and $\kappa > 0$. First, the free-meson Hamiltonian is changed from $H = \vec{p}^{\,2}/2\eta$ to $H = (\vec{p}^{\,2} + \kappa^2)/2\eta$. Second, a new term describing the propagation of helicity-zero mesons is added to $D_F^{\mu\nu}$, namely,

$$-i(2\pi)^{-3}\int d\vec{p}\int_0^\infty \frac{d\eta}{2\eta} e^\mu_{\text{eff}}(p,0) e^\nu_{\text{eff}}(p,0)^* [\Theta(\tau)e^{-ip\cdot x} + \Theta(-\tau)e^{+ip\cdot x}],$$

where the "effective polarization vector" for helicity-zero mesons is

$$e^\mu_{\text{eff}}(p,0) = -(\kappa/\eta)\delta^\mu_3 .$$

This is also the effective polarization vector for helicity-zero mesons in the initial and final states, since $e^\mu(p,0) = \kappa^{-1}p^\mu - (\kappa/\eta)\delta^\mu_3$, and the term $\kappa^{-1}p^\mu$ does not contribute to scattering amplitudes because of current conservation.

From here on, one can continue the argument just as in Ref. 1 to show that the covariant Feynman rules are equivalent to the rules for old-fashioned perturbation theory in the infinite-momentum frame given in Sec. IV.

ACKNOWLEDGMENTS

The author would like to thank R. Jackiw for his encouragement and for helpful discussions. I would also like to thank J. D. Bjorken and J. B. Kogut for their advice and criticisms.

---

*Supported by the U. S. Atomic Energy Commission.

†Address after September 1, 1971: Department of Physics, Princeton University, Princeton, N. J. 08540.

[1]J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970); see also R. A. Neville and F. Rohrlich, *ibid.* 3, 1692 (1971).

[2]J. M. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).

[3]D. A. Dicus, R. Jackiw, and V. L. Teplitz, this issue, Phys. Rev. D 4, 1733 (1971).

[4]D. J. Gross and S. B. Treiman, Phys. Rev. D 4, 1059 (1971).

[5]I am indebted to R. Jackiw for pointing this out.

[6]E. C. G. Stückelberg, Helv. Phys. Acta 11, 299 (1938).

[7]Specifically, we insert a factor $2(2\pi)^3$ in the normalization of states, $\langle p|p'\rangle = (2\pi)^3 2\eta\delta(\eta-\eta')\delta^2(\vec{p}-\vec{p}')$, and we use a circular-polarization basis for the vector mesons instead of a linear-polarization basis.

[8]Cf. J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D 3, 1382 (1971).

[9]We adopt the conventions that Latin indices are to be summed from 1 to 2 and that transverse vectors (a^1, a^2) are marked with an arrow $(\vec{a})$.

[10]The observant reader may notice that in Ref. 1, Eq. (7) was written as $A^3 = (-i/\eta)[\partial_k A^k + \kappa B + (i/\eta)J^0]$ and arguments were given for preferring this form. In this paper, in contrast to Ref. 1, we will not try to make such nice distinctions, nor will we worry about possible surface terms arising from integrations by parts.

[11]The current component J^0 occurring in the constraint equation (7) is $e\bar{\Psi}\gamma^0\Psi = 2^{1/2}e\Psi_+^\dagger\Psi_+$, where the two components Ψ_+ of Ψ are the independent dynamical variables for the Dirac field. Thus J^0 does not depend on the meson field.

[12]Cf. Ref. 1, Appendix B.

[13]These differ from Ref. 1 by a factor $2(2\pi)^3$.

[14]Cf. Ref. 6. Notice, however, that some of the matrix elements of the Hamiltonian, as given in Table I, differ by a phase from those given in Ref. 6. The difference is due to differing choices of phases for the states.