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Relaxation of Electron Velocity in a Rotating Neutron Superfluid: Application to the Relaxation of a Pulsar's Slowdown Rate*

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We estimate the relaxation time τ of an average electron velocity relative to a dilute array of vortex cores in a rotating, s-wave-paired neutron superfluid. At low temperatures, τ is found to vary exponentially with Δ^2/ϵ_F, the energy scale of vortex core excitations, where Δ is the gap parameter and ϵ_F is the neutron Fermi energy. For reasonable choices of Δ and ϵ_F, we find values of τ which include the values of a year and of several days observed, respectively, in the post-speedup relaxation of the Vela and Crab pulsars' slowdown rates.

I. INTRODUCTION

The slowdown rates of both the Vela and the Crab pulsar have been found, in post-speedup observation, to relax back to pre-speedup slowdown rates, with relaxation times of about a year and a few days, respectively.[1] This observation has led to the suggestion of a two-component model[2] of a neutron star, in which speedup occurs when the star's solid crust cracks slightly, reducing the star's moment of inertia by decreasing its ellipticity, and in which relaxation of the slowdown rate proceeds as the crust transfers its excess angular momentum to the star's liquid interior.

It is not possible to maintain the two-component picture, however, if the neutron liquid interior is taken to be normal; for in this case, its spinup would occur in a time very short compared to a day.[3] Thus, in order to explain the long relaxation times of the Vela and the Crab slowdown rates, their interiors are assumed to be neutron superfluid.

In a rotating superfluid, all circulation is confined to vortex cores; spinup of a superfluid occurs when angular momentum is transferred to the vortex cores, which may ultimately cause them to increase in number or to increase in length or to move toward the axis of rotation, depending on the details of their original distribution in space, their pinning, etc. In the present work we estimate the rate at which angular momentum is transferred to already existing vortices, and make no attempt to study their final configuration. In a neutron star, angular momentum is transferred from charged particles which reside in the liquid interior, but which, because of electromagnetic coupling, rotate with the crust. Of the charged particles, electrons are somewhat more effective than protons at interacting with the neutron liquid, even though electrons do not have strong, but only magnetic, interactions with neutrons. This is because the electrons are normal, while the protons are superconducting[4]; thus only a small fraction of the protons, those in superconducting vortex cores, interact at all with the neutrons; the magnitude of the effect of core-proton scattering off neutrons is, however, further reduced because the proton levels in the plane perpendicular to a vortex core are quantized.

This quantization has the effect that proton-neutron scattering events may be divided into two classes, inelastic events in which the proton changes its quantum state in the plane perpendicular to a vortex core axis, and quasielastic events in which the proton changes only its state of motion along a vortex core. The phase space for quasielastic events is evidently one-dimensional, and thus is of the order $(1/\xi_p^2)dk$, where ξ_p is the proton superfluid coherence length, and is of the order of the radius of a proton vortex core. The phase space for inelastic proton events is essentially the three-dimensional k^2dk; however, the inelastic cross section is weighted by a thermal activation probability, $e^{(-\Delta E/T)}$, where ΔE is the proton energy change. Thinking of a proton vortex as a cylinder of radius ξ_p containing normal protons, we see that the scale of proton-energy-level differences, corresponding to states of motion perpendicular to a core axis, is $\hbar^2/2M_p\xi_p^2 \equiv \pi^2\Delta_p^2/4\epsilon_{F,p}$, where M_p, Δ_p, and $\epsilon_{F,p}$ are the proton mass, superconducting gap, and Fermi energy, respectively. For a typical neutron-star temperature,[5] $T=10^8\,°\mathrm{K}\approx 0.01$ MeV, taking $(\Delta_p/\epsilon_{F,p})^2 \sim 5\times 10^{-3}$, and $\epsilon_{F,p}\sim 10$ MeV, we see that the thermal activation factor is rather small, on the order of 10^{-4}.

Taking into account both the phase space and thermal activation factors, we see that proton-neutron scattering is reduced from what it would be if the protons were normal ($\Delta_p=0$) by a factor on the order of

$$\frac{1}{k^2\xi_p^2}+\exp\left(\frac{-\pi^2\Delta_p^2}{4\epsilon_{F,p}T}\right)\approx\left(\frac{\pi\Delta_p}{2\epsilon_{F,p}}\right)^2+\exp\left(\frac{-\pi^2\Delta_p^2}{4\epsilon_{F,p}T}\right)$$
$$\sim 5\times 10^{-3}\,.$$

In making this estimate, we have taken the proton Fermi wave number, $k\sim k_{F,p}$, and have used the relation $\xi_p=\hbar k_{F,p}/\pi M_p\Delta_p$.

The proton-neutron coupling constant square is about 10^6 times that of the electron-neutron interaction, and the percentage of protons in cores is about 10^{-4}. Combining these numbers with the reduction in proton-neutron cross section, $\sim 5\times 10^{-3}$, we see that the effects of proton-neutron scattering are roughly comparable to or less than those of electron-neutron scattering. In the present calculation we deal only with the latter process.

This discussion of proton-neutron scattering has made use of an important feature of relaxation processes in a neutron star, which we will now apply in considering the effects of electron-neutron scattering. It is that neutron stars are very cold, and only processes which require very little thermal energy, therefore, can occur.

In principle, electrons can give up angular momentum to neutron vortices either directly, or by a two-step process wherein an electron creates or interacts with a bulk excitation of the neutron superfluid (a quasiparticle), and this excitation then transfers angular momentum to the vortices. But because the temperature of the neutron star is so low, there are very few excitations present in the bulk of the superfluid. If, for example, the gap parameter $\Delta\approx 2.5$ MeV and the temperature $T\approx 10^8\,°\mathrm{K}\approx 0.01$ MeV, then the probability of there being a neutron quasiparticle in bulk is proportional to $e^{-2.5/0.01}=10^{-109}$. The probability of an electron having enough energy to create a quasiparticle pair in the bulk is even less, $e^{-5/0.01}=10^{-217}$. Thus the two-step process of angular-momentum transfer is very unlikely to occur.

Again in the case of the neutrons, vortex cores may be thought of as cylindrical normal regions in the neutron superfluid, aligned along the axis of rotation of the star, and having a radius of the order of the neutron coherence length, $\xi=\hbar v_F/\pi\Delta$, where v_F is the neutron Fermi velocity. The scale of the separation of core energy levels is clearly on the order of $\hbar^2/2M\xi^2$, or $\pi^2\Delta^2/4\epsilon_F$, where M is the neutron mass and $\epsilon_F=\frac{1}{2}Mv_F^2$ is the neutron Fermi energy. Typically one might expect $\Delta/\epsilon_F \approx 0.05$. Thus the probability of the presence of a neutron core quasiparticle is proportional to $\exp(-\pi^2\Delta^2/4\epsilon_F T)\approx 10^{-14}$ at $T=0.01$ MeV. This factor evidently governs the probability of electron scattering from thermally excited core quasiparticles. Quasiparticle creation processes, e.g., electro-excitation of a pair of core quasiparticles, requires an electron energy of at least 2 times $\pi^2\Delta^2/4\epsilon_F$, and is therefore weighted by the factor $\exp(-\pi^2\Delta^2/2\epsilon_F T)\approx 10^{-27}$. We thus decide to restrict our attention to scattering of electrons by thermally excited neutron-core quasiparticles, the dominant relaxation process at low temperatures.

Using the variational approach to the linearized Boltzmann transport equation,[6] we estimate the relaxation time of an average electron velocity relative to a dilute array of neutron vortex cores. This relaxation time is to be identified with the post-starquake relaxation time of the pulsar slowdown rate. In Sec. II of this article we quote the formalism briefly. In Sec. III we describe our estimates of electron-neutron matrix elements and indicate how we carry out the calculation of the relaxation time τ. Finally in Sec. IV we present our results and conclusions.

We find that τ depends exponentially on the scale of core-excitation energy Δ^2/ϵ_F. Because the gap parameter is a sensitive function of neutron density, therefore, our model implies widely different values of τ for regions of the pulsar with different densities. The conclusion to be drawn from this fact is *not* that the minimum τ is to be identified with the observed τ; rather, it is that the frictional force on a vortex is a strong function of position in the pulsar. The calculation of an "average" τ to be compared with observation is a difficult hydrodynamic problem in which the vortex-vortex interaction as well as the position-dependent frictional force must be taken into account.

We do not attempt to solve this problem; we are content here to show that for reasonable values of Δ, ϵ_F, and T, we obtain a range of values of τ that includes the relaxation times of a year and a few days, respectively, of the Vela and the Crab pulsar slowdown rates.

II. FORMAL THEORY

We assume that immediately after a starquake, the pulsar's electron distribution is spatially uniform and characterized by an average velocity $\vec{v}$ relative to the neutron liquid. We neglect the curvature of electron orbits, due to electromagnetic fields, in calculating the electron-neutron scattering. These two assumptions imply that the streaming terms in the Boltzmann equation may be neglected, and we have simply

$$\frac{\partial f_e(\vec{p},r)}{\partial t}=\frac{\partial f_e}{\partial t}\bigg|_{\text{coll}}=\frac{N_v}{V}\int\frac{d^3p'}{(2\pi\hbar)^3}\bigg(f_e(\vec{p}',t)[1-f_e(\vec{p},t)]\sum_{n,n'}f_N(n')[1-f_N(n)]\frac{2\pi}{\hbar}W(\vec{p},n;\vec{p}',n')\delta(\epsilon_p-\epsilon_{p'}+\omega_n-\omega_{n'})$$
$$-f_e(\vec{p},t)[1-f_e(\vec{p}',t)]\sum_{n,n'}f_N(n)[1-f_N(n')]\frac{2\pi}{\hbar}W(\vec{p}',n';\vec{p},n)\delta(\epsilon_{p'}-\epsilon_p+\omega_{n'}-\omega_n)\bigg). \quad (1)$$

In Eq. (1), $f_e(\vec{p},t)$ is the single-electron distribution function, which relaxes slowly in time; $f_N(n)$ is the neutron-vortex-core-excitation distribution function, which is assumed by virtue of the strong neutron-neutron interaction to relax quickly, and which is therefore always an equilibrium distribution. The factors $(2\pi/\hbar)W$ times δ functions give the rate of electron-neutron scattering. $W(\vec{p},n;\vec{p}',n')$ is the matrix element squared of the magnetic dipole interaction between initial and final electron states of momentum $\vec{p}$ and $\vec{p}'$, and energy ϵ_p and $\epsilon_{p'}$, and initial and final neutron states characterized by indices n and n', and excitation energies ω_n and $\omega_{n'}$. The factor N_v is the number of vortices. Since the rotational velocity of a typical pulsar is 10^{-18} times the critical velocity at which vortices are close packed,[7] electrons scatter from single vortices, and the relaxation rate is N_v times that for scattering from an isolated vortex. The volume of the system V enters Eq. (2) as a normalization constant.

We linearize Eq. (1). Let

$$f_e(\vec{p},t)=f_e^{(0)}(\vec{p})+f_e^{(1)}(\vec{p},t), \quad (2)$$

where the equilibrium distribution is

$$f_e^{(0)}(\vec{p})=\{1+\exp(\beta[\epsilon(p)-E_F])\}^{-1}. \quad (3)$$

E_F is the electron Fermi energy and $\beta=1/kT$. Because of the sharpness of the Fermi surface, it is convenient to define $\varphi(\vec{p},t)$ by

$$f_e^{(1)}(\vec{p},t)=-\frac{df_e^0}{d\epsilon(\vec{p})}\varphi(\vec{p},t). \quad (4)$$

Substituting Eqs. (2) and (4) in Eq. (1), retaining only terms of first order in φ, using the fact that $W(\vec{p},n;\vec{p}',n')=W(\vec{p}',n';\vec{p},n)$, and Laplace-transforming φ according to

$$\varphi(p,t)=\int_0^\infty ds\, e^{-st}\varphi(p,s), \quad (5)$$

we obtain the eigenvalue equation for φ:

$$sf_e^{(0)}(\vec{p})[1-f_e^{(0)}(\vec{p})]\varphi(\vec{p},s)=\frac{N_v}{V}\int\frac{d^3p'}{(2\pi\hbar)^3}[\varphi(\vec{p},s)-\varphi(\vec{p}',s)]S(\vec{p},\vec{p}'), \quad (6)$$

where

$$S(\vec{p},\vec{p}')=f_e^{(0)}(\vec{p}')[1-f_e^{(0)}(\vec{p})]\sum_{n,n'}f_N(n')[1-f_N(n)]\frac{2\pi}{\hbar}W(\vec{p},n;\vec{p}',n')\delta(\epsilon_p-\epsilon_{p'}+\omega_n-\omega_{n'}). \quad (7)$$

It is important to recognize that $S(\vec{p},\vec{p}')$ is a symmetric function of its arguments, by virtue of the symmetry of the n and n' sums and the presence of the energy-conserving δ function.

According to Eq. (5), the minimum relaxation time for φ is determined by the largest possible s for which $\varphi(s)\neq 0$. We therefore seek the solution of Eq. (6) with the maximum eigenvalue s. Using the symmetry of $S(\vec{p},\vec{p}')$, we see that Eq. (6) corresponds to the variational equation

$$\delta s/\delta\varphi = 0\,, \tag{8}$$

where

$$s=\frac{N_v}{V}\int\frac{d^3p}{(2\pi\hbar)^3}\frac{d^3p'}{(2\pi\hbar)^3}\,\tfrac{1}{2}[\varphi(\vec{p})-\varphi(\vec{p}')]^2S(\vec{p},\vec{p}')\left(\int\frac{d^3p}{(2\pi\hbar)^3}f_e^{(0)}(\vec{p})[1-f_e^{(0)}(\vec{p})]\varphi^2(\vec{p})\right)^{-1}. \tag{9}$$

Since $S(\vec{p},\vec{p}')$ and $f_e^{(0)}(\vec{p})[1-f_e^{(0)}(\vec{p})]$ are positive for all values of their arguments, Eqs. (8) and (9) represent a maximum principle. Thus by substituting a trial $\varphi(\vec{p})$ in (9), we obtain an upper bound on s, and a lower bound, $1/s$, on τ, the relaxation time.[8]

In order to represent a spatially uniform initial electron distribution with average velocity $\vec{v}$, we must choose

$$\varphi(\vec{p})=\vec{p}\cdot\vec{v}\,g(\vec{p}^2)\,, \tag{10}$$

where $g(\vec{p}^2)$ is a trial variational function, for example, $g(\vec{p}^2)=1$.

III. EVALUATION OF THE LOWER BOUND ON τ

We estimate here the lower bound on τ given by the substitution of $\varphi(\vec{p})=\vec{p}\cdot\vec{v}$ into Eq. (9). With this substitution and the change of variables, $\vec{P}=\frac{1}{2}(\vec{p}+\vec{p}')$ and $\vec{k}=\vec{p}-\vec{p}'$ in the numerator, we obtain

$$\frac{1}{\tau}\leq\frac{N_v}{V}\int\frac{d^3k}{(2\pi\hbar)^3}\,\tfrac{1}{2}(\vec{k}\cdot\vec{v})^2\int\frac{d^3P}{(2\pi\hbar)^3}\,S(\vec{P}+\tfrac{1}{2}\vec{k},\,\vec{P}-\tfrac{1}{2}\vec{k})\left(\int\frac{d^3p}{(2\pi\hbar)^3}f_e^{(0)}(p)[1-f_e^{(0)}(p)](\vec{p}\cdot\vec{v})^2\right)^{-1}. \tag{11}$$

In order to derive an explicit expression for S, we must evaluate the electron-neutron scattering matrix element. The interaction responsible for electron-neutron scattering is the magnetic-dipole interaction, whose Hamiltonian is (in second quantized notation)

$$H_I=e\mu_N g_N\int d^3x\,d^3x'\{\overline{\psi}(x)\vec{\gamma}\psi(x)\}\cdot\left[\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\times\sum_\sigma\varphi_\sigma^\dagger(x')\,\vec{\sigma}\,\varphi_\sigma(x')\right]. \tag{12}$$

In Eqs. (12), the neutrons are taken to be nonrelativistic; $\varphi_\sigma^\dagger(x')$ is the neutron creation operator with spin index σ. The electrons are relativistic; their creation operators are the $\overline{\psi}(x)$. The components of $\vec{\gamma}$ are Dirac γ matrices. μ_N is the nuclear magneton, $e\hbar/2M_Nc$, while g_N is the neutron g factor, equal to -1.91.

The unperturbed superfluid is represented within the Hartree-Bogoliubov self-consistent field theory[9] by the Hamiltonian

$$H_N=\sum_{n,\sigma}\int\frac{dq}{2\pi\hbar}\,\omega_{qn}\alpha_{q,n,\sigma}^\dagger\alpha_{q,n,\sigma}\,, \tag{13}$$

where the α and $\alpha^\dagger$ are neutron quasiparticle annihilation and creation operators. For an infinitely long vortex in an otherwise uniform superfluid the quasiparticle-state quantum numbers are the momentum q along the direction of the vortex (the z direction), a radial and angular quantum number n, and a spin index σ. The $\varphi_\sigma(x)$, assuming s-wave pairing, are related to the α by

$$\begin{aligned}\varphi_\uparrow(\vec{x})&=\sum_n\int\frac{dq}{2\pi\hbar}\{\alpha_{qn\uparrow}e^{i(q/\hbar)z}u_{qn}(\vec{r})-\alpha_{qn\downarrow}^\dagger e^{-i(q/\hbar)z}v_{qn}^*(\vec{r})\}\,,\\ \varphi_\downarrow(\vec{x})&=\sum_n\int\frac{dq}{2\pi\hbar}\{\alpha_{qn\downarrow}e^{-i(q/\hbar)z}u_{qn}(\vec{r})+\alpha_{qn\uparrow}^\dagger e^{-i(q/\hbar)z}v_{qn}^*(\vec{r})\}\,,\end{aligned} \tag{14}$$

where $\vec{r}$ is the coordinate vector in the x-y plane. Explicit approximate expressions for the u_n and v_n, as well as the quasiparticle energies, ω_{qn}, have been given by Caroli and Matricon[10] for the system of interest, a single vortex in a Fermi superfluid. (We emphasize that our calculation is based on the assumption of s-wave pairing in the superfluid; for p-wave pairing the quantitative results will be different.)

We now calculate $\langle 0|c_{\vec{p}',s'}\alpha_{q'n'\sigma'}H_I\alpha_{qn\sigma}^\dagger c_{\vec{p},s}^\dagger|0\rangle$, the transition matrix element for electron-neutron scattering, in which $c_{\vec{p},s}^\dagger$ creates an electron of momentum $\vec{p}$ and z-component of spin s, and in which $|0\rangle$ is the superfluid neutron state with a vortex present, times the electron vacuum. The square of this matrix element, summed over final and averaged over initial states, is what we referred to as $W(\vec{p},n;\vec{p}',n')$ in our discussion of the Boltzmann equation [cf. Eq. (7). Note also that in what follows, the neutron state label n is replaced by the more appropriate (q,n).]. Performing the spin averaging by taking traces of γ matrices, we find

$$W(\vec{p},q,n;\vec{p}',q',n')=\left(\frac{2\pi\alpha g_n\hbar^3}{M}\right)^2 L\times 2\pi\hbar\delta(k_z-q+q')\kappa(\vec{p},\vec{p}')\mathcal{S}(k_\perp;q,n;q',n'), \tag{15}$$

where $\alpha=e^2/\hbar c$, and M and m are the neutron and electron masses, respectively. In this equation the electron contribution is

$$\kappa(\vec{p},\vec{p}')\equiv\frac{2m^2c^2}{(\vec{p}^2+m^2c^2)^{1/2}(\vec{p}'^2+m^2c^2)^{1/2}}\left[\frac{1}{k^2}\left(\frac{(\vec{p}^2+m^2c^2)^{1/2}(\vec{p}'^2+m^2c^2)^{1/2}}{m^2c^2}-1\right)-\frac{(\vec{k}\cdot\vec{p})(\vec{k}\cdot\vec{p}')}{m^2c^2k^4}\right] \tag{16}$$

while the neutron contribution is

$$\mathcal{S}(k_\perp;q,n;q',n')=\left|\int d^2r\, e^{(i/\hbar)(\vec{k}_\perp\cdot\vec{r})}[u^*_{qn}(r)u_{q'n'}(r)+v^*_{qn}(r)v_{q'n'}(r)]\right|^2. \tag{17}$$

The δ function in Eq. (15) expresses momentum conservation along the vortex core.[11]

In order to evaluate $1/\tau$ according to Eq. (11), we need

$$\int\frac{d^3P}{(2\pi\hbar)^3}S(\vec{P}+\tfrac{1}{2}\vec{k},\vec{P}-\tfrac{1}{2}\vec{k}).$$

S is given in terms of $W(\vec{p},q,n;\vec{p}',q',n')$ by Eq. (7). Substituting Eq. (15) in Eq. (7), we obtain

$$\int\frac{d^3P}{(2\pi\hbar)^3}S(\vec{P}+\tfrac{1}{2}\vec{k},\vec{P}-\tfrac{1}{2}\vec{k})=\frac{\alpha^2g_n^2\hbar}{2\pi M^2}\int_{-\infty}^{\infty}d\omega\frac{1}{1-e^{\beta\omega}}\frac{1}{1-e^{-\beta\omega}}\rho_e(\vec{k},\omega)\rho_N(\vec{k},\omega), \tag{18}$$

in which

$$\rho_N(\vec{k},\omega)\equiv\sum_{n,n'}\int dq[f_N(\omega_{q+\frac{1}{2}k_z,n'})-f_N(\omega_{q-\frac{1}{2}k_z,n})]\delta(\omega_{q+\frac{1}{2}k_z,n'}-\omega_{q-\frac{1}{2}k_z,n}+\omega)$$
$$\times\left|\int d^2r\, e^{(i/\hbar)(\vec{k}_\perp\cdot r)}(u^*_{q-\frac{1}{2}k_z,n}u_{q+\frac{1}{2}k_z,n'}+v^*_{q-\frac{1}{2}k_z,n}v_{q+\frac{1}{2}k_z,n'})\right|^2 \tag{19}$$

and

$$\rho_e(\vec{k},\omega)=\int d^3P[f_e^{(0)}(\epsilon_{\vec{P}+\frac{1}{2}\vec{k}})-f_e^{(0)}(\epsilon_{\vec{P}-\frac{1}{2}\vec{k}})]\kappa(\vec{P}+\tfrac{1}{2}\vec{k},\vec{P}-\tfrac{1}{2}\vec{k})\,\delta(\epsilon_{\vec{P}+\frac{1}{2}\vec{k}}-\epsilon_{\vec{P}-\frac{1}{2}\vec{k}}+\omega), \tag{20}$$

where we recall that both $F_e^{(0)}$ and F_N are equilibrium Fermi distributions of the form $F(\epsilon)=[1+e^{\beta(\epsilon-\epsilon_{\text{Fermi}})}]^{-1}$. In order to derive the formula, Eq. (18), we have used the identity

$$f(\epsilon_1)[1-f(\epsilon_2)]f(\epsilon_3)[1-f(\epsilon_4)]=[f(\epsilon_1)-f(\epsilon_2)][f(\epsilon_3)-f(\epsilon_4)]\frac{1}{1-e^{\beta(\epsilon_1-\epsilon_2)}}\frac{1}{1-e^{-\beta(\epsilon_1-\epsilon_2)}}, \tag{21}$$

which holds for $f(\epsilon)=(1+e^{\beta\epsilon})^{-1}$ when $\epsilon_1-\epsilon_2=\epsilon_4-\epsilon_3$.

Substituting Eq. (18) in Eq. (11), we obtain

$$\frac{1}{\tau}\leqslant\frac{N_v}{A}\frac{\alpha^2g_n^2\hbar}{2\pi M^2}\frac{\int d^3k\,\frac{1}{2}(v\cdot k)^2\int d\omega(1-e^{\beta\omega})^{-1}(1-e^{-\beta\omega})^{-1}\rho_e(\vec{k},\omega)\rho_N(\vec{k},\omega)}{\int d^3p\, f_e^{(1)}(\epsilon_p)[1-f_e^{(0)}(\epsilon_p)](v\cdot p)^2}, \tag{22}$$

in which $A\equiv V/L$ is the cross-sectional area of the star (N_v/A = number of vortices per unit area). It remains for us to evaluate ρ_e and ρ_N explicitly.

Since we are working in the small-temperature limit, ρ_e may be evaluated at $T=0$. Using Eqs. (3) and (16), and $\epsilon_p=(p^2c^2+m^2c^4)^{1/2}$, we find

$$\rho_e(\vec{k},\omega)\xrightarrow[T\to 0]{}\frac{\pi\omega}{c^2|\vec{k}|^3}(p_F^{\,2}+\tfrac{1}{4}k^2)\theta(4p_F^{\,2}-k^2), \tag{23}$$

where p_F is the electron Fermi momentum, related to the electron Fermi energy E_F by

$$p_F = (1/c)(E_F{}^2 - m^2c^4)^{1/2}, \tag{24}$$

and where

$$\theta(x) \equiv \begin{cases} 1, & x > 0 \\ 0, & x < 0 . \end{cases}$$

In order to evaluate ρ_N, we need information concerning neutron core excitations. Because of the cylindrical symmetry of a vortex, these excitations are characterized by a radial and an angular quantum number, and the momentum q, parallel to the vortex. In Ref. 10 it is shown that the radius of a vortex core is so small that only excitations with the lowest radial quantum number are bound to the core. Core excitations are found to exist for values of $q < k_F$, the neutron Fermi momentum, and for half-odd-integral values of the angular quantum number n, with excitation energies

$$\omega_{q,n} = \frac{n\hbar}{(k_F{}^2 - q^2)^{1/2}} \int_0^\infty dr \, \frac{\Delta(r)}{r} e^{-2\mathcal{K}(r)} \left(\int_0^\infty dr \, e^{-2\mathcal{K}(r)} \right)^{-1}, \tag{25}$$

where $\Delta(r)$ is the gap parameter as a function of radial coordinate in the unperturbed vortex core, and

$$\mathcal{K}(r) = \frac{M/\hbar}{(k_F{}^2 - q^2)^{1/2}} \int_0^r dr' \Delta(r') . \tag{26}$$

The functions u_{qn} and v_{qn} corresponding to these excitations are of the form $e^{in\theta}$ times a slowly varying envelope function of radial coordinate r which falls to zero for $r > \xi$, the coherence length, times a rapidly varying sinusoidal function of r.

At low temperatures, only states corresponding to the lowest excitation energies contribute to ρ_N, Eq. (19). Thus the n and n' sums are trivial; we retain only the term corresponding to $n = \frac{1}{2}$ and $n' = \frac{1}{2}$. Moreover, for $T \to 0$, only small momenta along the vortex are permitted. Thus, $q \ll k_F$ and

$$\frac{1}{(k_F{}^2 - q^2)^{1/2}} = \frac{1}{k_F} + \frac{1}{2}\frac{q^2}{k_F{}^3} + \cdots . \tag{27}$$

In the same spirit, we evaluate the matrix element

$$M_{\vec{K}}(q) \equiv \int d^2r \, e^{(i/\hbar)(\vec{k}_\perp \cdot \vec{r})} (u^*_{q-\frac{1}{2}k_z, \frac{1}{2}} u_{q+\frac{1}{2}k_z, \frac{1}{2}} + v^*_{q-\frac{1}{2}k_z, \frac{1}{2}} v_{q+\frac{1}{2}k_z, \frac{1}{2}}), \tag{28}$$

as though the change in it due to the fact that $q \neq 0$ was negligible. Taking account of the normalization condition[12]

$$\int d^2r \, (|u_{qn}|^2 + |v_{qn}|^2) = 1, \tag{29}$$

we thus obtain

$$M_{\vec{K}}(q) \approx M_{\vec{K}}(0) \approx \int_0^\infty dr \, J_0\!\left(\frac{|k_\perp| r}{\hbar}\right) e^{-2\mathcal{K}(r)} \left(\int_0^\infty dr \, e^{-2\mathcal{K}(r)} \right)^{-1}, \tag{30}$$

where J_0 is the zero-order Bessel function. Referring to Eq. (26), for $q^2 \ll k_F{}^2$, we have

$$\mathcal{K}(r) = \frac{1}{\hbar v_F} \int_0^r dr' \Delta(r') = \frac{1}{\pi\xi} \int_0^r dr' \frac{\Delta(r')}{\Delta(\infty)} . \tag{31}$$

Since we are interested in obtaining only a rough bound on τ, we do not attempt to evaluate $M_{\vec{K}}(0)$ very precisely. We are satisfied with a matrix element that has the correct asymptotic properties at $k_\perp \to 0$ and $k_\perp \to \infty$. At $k_\perp = 0$, it is obvious from Eq. (31) that $M_{\vec{K}}(0) \to 1$. As $k_\perp \to \infty$,

$$M_{\vec{K}}(0) \to \int_0^\infty dr \, J_0\!\left(\frac{k_\perp r}{\hbar}\right) \left(\int_0^\infty dr \, e^{-2\mathcal{K}(r)} \right)^{-1} = \frac{\hbar}{g_1 \xi k_\perp} . \tag{32}$$

To obtain the limiting form, Eq. (32), we have used [cf. Eq. (31)] $\mathcal{K}(0) = 0$, and the knowledge that $\mathcal{K}$ is a function that decreases slowly for $r/\xi \lesssim 1$ and exponentially for $r/\xi > 1$, which allows us to estimate

$$\int_0^\infty dr\, e^{-2\mathcal{K}(r)} \approx g_1 \xi\,, \tag{33}$$

where g_1 is a constant of order 1. Thus we arrive at the estimate

$$|M_{\vec{k}}(q)|^2 \approx \frac{1}{1+(g_1\xi k_\perp/\hbar)^2}\,, \tag{34}$$

a smooth function of $k_\perp$ with correct asymptotic properties. Similarly, we estimate

$$\omega_{q,\frac{1}{2}} \approx \frac{\hbar\Delta}{2k_F\xi}\left(1+\frac{q^2}{2k_F^{\,2}}+\cdots\right)g_2 = \frac{\pi\Delta^2}{4\epsilon_F}\left(1+\frac{q^2}{2k_F^{\,2}}+\cdots\right)g_2\,, \tag{35}$$

where g_2 [see Ref. (10)] is another constant of $O(1)$, and ϵ_F is the neutron Fermi energy, equal to $k_F^{\,2}/2M$.

Substitute Eqs. (34) and (35) in Eq. (19), remembering that only the term $n=n'=\frac{1}{2}$ contributes at small T. This yields

$$\rho_N(\vec{k},\omega) \approx \exp\left(-\frac{\beta\pi\Delta^2 g_2}{4\epsilon_F}\right)\int dq\left[\exp\left(-\frac{\beta\pi\Delta^2 g_2}{8\epsilon_F k_F^{\,2}}(q+\tfrac{1}{2}k_z)^2\right)-\exp\left(-\frac{\beta\pi\Delta^2 g_2}{8\epsilon_F k_F^{\,2}}(q-\tfrac{1}{2}k_z)^2\right)\right]$$
$$\times\frac{1}{1+(g_1\xi k_\perp/\hbar)^2}\,\delta\left(\omega+\frac{\pi\Delta^2 g_2}{4\epsilon_F}\,\frac{k_z q}{k_F^{\,2}}\right), \tag{36}$$

in which $\exp(-\pi\Delta^2\beta/4\epsilon_F)$ has been assumed to be small compared to 1. We carry out the q integral immediately using the δ function to obtain

$$\rho_N(k,\omega) \approx \frac{2k_F^{\,2}}{\gamma|k_z|}\left\{\exp\left[-\beta\gamma\left(1+\frac{k_z^{\,2}}{8\epsilon_F^{\,2}}\right)\right]\exp\left[-\frac{\beta k_\perp^{\,2}}{2\gamma}\left(\frac{\omega}{k_z}\right)^2\right]\sinh\frac{\beta\omega}{2}\right\}\frac{1}{1+(g_1\xi k_\perp^{\,2}/\hbar)^2}\,, \tag{37a}$$

where

$$\gamma \equiv \frac{\pi\Delta^2 g_2}{4\epsilon_F}\,. \tag{37b}$$

Substituting the results, Eqs. (23) and (37), in the inequality (22), we find

$$\frac{1}{\tau} \leqslant \frac{N_v}{A}\,\frac{\alpha^2 g_n^{\,2}\hbar}{c^2 M_N^{\,2}}\,\frac{e^{-\beta\gamma}}{\gamma D}\int d^3k\,\tfrac{1}{2}(\vec{v}\cdot\vec{k})^2\,\frac{p_F^{\,2}+\frac{1}{4}k^2}{k^3}\,\theta(4p_F^{\,2}-k^2)\,\frac{k_F^{\,2}}{|k_z|}\exp\left(-\frac{\beta\gamma k_z^{\,2}}{8k_F^{\,2}}\right)$$
$$\times\frac{1}{1+(g_1\xi k_\perp/\hbar)^2}\int d\omega\,\frac{\omega}{4\sinh\frac{1}{2}\beta\omega}\exp\left[-\frac{\beta k_F^{\,2}}{2\gamma}\left(\frac{\omega}{k_z}\right)^2\right], \tag{38}$$

where the denominator is

$$D=\int d^3p\, f_e^0(\vec{p})[1-f_e^0(\vec{p})](\vec{v}\cdot\vec{p})^2 = \frac{4\pi v^2}{3\beta c^2}\,p_F^{\,3}E_F\,. \tag{39}$$

The integral on ω cannot be done in closed form. An analytic expression which has the correct asymptotic behavior for both $k_z\to 0$ and $k_z\to\infty$ may, however, be constructed. We use the fact that

$$\int_{-\infty}^\infty z\,dz\, e^{-z^2}\operatorname{csch} az \to \begin{cases}\dfrac{\sqrt{\pi}}{a}\,, & a\to 0\\[2mm] \dfrac{2}{a^2}\,, & a\to\infty\end{cases} \tag{40}$$

to suggest the interpolation formula

$$\int\frac{\omega\,d\omega}{4\sinh\frac{1}{2}\beta\omega}\exp\left(-\frac{\beta k_F^{\,2}}{2\gamma k_z^{\,2}}\,\omega^2\right) \approx \left(\frac{\pi\gamma k_z^{\,2}/2\beta^3 k_F^{\,2}}{1+(\pi\beta\gamma k_z^{\,2}/8k_F^{\,2})}\right)^{1/2}. \tag{41}$$

Substitute Eq. (41) in Eq. (38). The factor $\exp(-\beta\gamma k_z^{\,2}/8k_F)$ in the remaining integral forces $k_z^{\,2}$ to be small. Therefore in the other factors, we replace k^2 by $k_\perp^{\,2}$. We thus obtain[13]

$$\frac{1}{\tau} \sim \frac{N_v}{A}\,\frac{\alpha^2 g_n^{\,2}\hbar k_F}{M_N^{\,2}c^2}\,\frac{e^{-\beta\gamma}}{2\gamma D}\int d^2k_\perp(\vec{v}\cdot\vec{k}_\perp)^2\,\frac{p_F^{\,2}+\frac{1}{4}k_\perp^{\,2}}{k_\perp^{\,3}}\,\theta(4p_F^{\,2}-k_\perp^{\,2})\,\frac{1}{1+(g_1\xi k_\perp/\hbar)^2}\left(\frac{\pi\gamma}{2\beta}\right)^{1/2}\int_{-\infty}^\infty dk_z\,\frac{\exp(-\beta\gamma k_z^{\,2}/8k_F)}{[1+(\pi\beta\gamma k_z^{\,2}/8k_F^{\,2})]^{1/2}}\,. \tag{42}$$

The direction of the pulsar's speedup velocity is perpendicular to its axis of rotation; thus $\vec{v}$ is taken to be perpendicular to the vortex core, and $(\vec{v}\cdot\vec{k}_\perp)^2=v^2k_\perp{}^2\cos^2\theta_{k_\perp}$. The integrals in Eq. (42) are straightforward. We find that

$$\frac{1}{\tau}\sim\frac{N_v}{A}\,\frac{\alpha^2g_n{}^2\hbar k_F}{M_N{}^2c^2}\,\frac{e^{-\beta\gamma}}{2\gamma D}\,v^2\left[\frac{\pi^2p_F{}^2}{2}\,\frac{\hbar}{g_1\xi}+O\left(\frac{\hbar}{p_Fg_1\xi}\right)\right]\frac{2k_F}{\beta^2}\,e^{-1/\sqrt{4\pi}}K_0\left(\frac{1}{\sqrt{4\pi}}\right), \tag{43}$$

where K_0 is the zero-order Bessel function of imaginary argument. Using Eqs. (39), (41a), and the relation $\xi=\hbar k_F/\pi M_N\Delta$, we obtain

$$\frac{1}{\tau}\sim\frac{N_v\pi\xi^2}{A}\,\frac{3\pi^2\alpha^2g_n{}^2}{4g_1g_2}\left(\frac{\epsilon_F}{E_F}\right)^2\frac{1}{\hbar\beta}\,\frac{\Delta}{\epsilon_F}\left(\frac{2Mc^2}{\epsilon_F}\right)^{1/2}e^{-1/\sqrt{4\pi}}K_0\left(\frac{1}{\sqrt{4\pi}}\right)\exp\left(-\frac{\beta\pi\Delta^2g_2}{4\epsilon_F}\right). \tag{44}$$

In (44), we note that the factor $N_v\pi\xi^2/A$ is the ratio of normal fluid (inside vortex cores) to superfluid. It equals Ω/Ω_{c_2}, the angular velocity of the pulsar divided by the critical velocity Ω_{c_2} at which the vortices become close packed.

IV. NUMERICAL ESTIMATE OF THE RELAXATION TIME; DISCUSSION

We identify the relaxation time of the post-speedup pulsar slowdown rate as the electron velocity relaxation time. According to Eq. (44),

$$\tau=\frac{\Omega_{c_2}}{\Omega}\,\frac{4g_1g_2}{3\pi^2\alpha^2g_n{}^2}\left(\frac{E_F{}^2}{\epsilon_F}\right)\frac{\epsilon_F}{\Delta}\left(\frac{\epsilon_F}{2Mc^2}\right)^{1/2}\beta\hbar\,\frac{e^{1/\sqrt{4\pi}}}{K_0(1/\sqrt{4\pi})}\exp\left(\frac{\beta\pi\Delta^2g_2}{4\epsilon_F}\right). \tag{45}$$

For convenience we recall that, in Eq. (45), Ω_{c_2}/Ω is the upper critical angular speed of the neutron liquid divided by its actual angular speed, $\alpha\equiv e^2/\hbar c$, g_n is the neutron g factor equal to -1.91, ϵ_F and E_F are the neutron and electron Fermi energies, respectively, M is the neutron mass, Δ is the superfluid gap parameter, g_1 and g_2 are constants of order 1, and K_0 is the zero-order Bessel function of imaginary argument.

Let us take $g_1\approx g_2\approx1$, $T=0.01$ MeV, and $\Omega_{c_2}/\Omega=10^{18}$, appropriately to the Crab pulsar.[14] We take $\epsilon_F=50$ MeV, corresponding to a neutron density of about 10^{14} g cm^{-3}, and $E_F=100$ MeV, corresponding to a number density of electrons one-tenth that of neutrons. Substituting in Eq. (45), we thus find

$$\tau\approx\frac{10^3}{\Delta}\times10^{0.8\Delta^2}\ \text{sec}, \tag{46}$$

where Δ is in MeV. τ is on the order of days for $\Delta\approx1.7$ MeV, on the order of a year for $\Delta\approx2.4$ MeV. These values are in the range of Δ's given by Hoffberg *et al.*[15] for s-wave pairing in neutron liquids whose densities lie between 10^{14} and 5×10^{13} g cm^{-3}. The remarkably strong dependence of τ on Δ makes it possible to fit a wide range of τ's with a reasonably narrow range of Δ's. However, it also complicates the problem of calculating an *average* τ for a pulsar. According to Ref. 15, Δ decreases rapidly as the density of neutrons increases from 0.4×10^{14} g cm^{-3} to 1.5×10^{14} g cm^{-3}. Thus τ is a very rapidly decreasing function of density, and in order to calculate an average τ, one will have to take into account the strong radial dependence of the frictional force on vortices as well as the vortex-vortex interaction.

For neutron liquid densities greater than 1.5×10^{14} g cm^{-3}, Ref. 15 indicates that neutron pairing will occur in relative p-waves rather than s-wave states. Our calculation bears only on the s-wave case, however. The study of vortices in p-wave superfluidity, to say nothing of electron-core scattering in this case, remains an important and open problem.

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Derivation of the Blackbody Radiation Spectrum by Classical Statistical Mechanics

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It is assumed that the fluctuating radiation energy density in a blackbody cavity is the sum of two stochastically independent terms: a zero-point energy density ρ_0 with Lorentz-invariant spectrum which persists at the absolute zero of temperature, and a temperature-dependent energy density ρ_T which satisfies the laws of statistical mechanics. The mean-square fluctuation $\langle(\delta\rho_T)^2\rangle$ of ρ_T is calculated from classical electromagnetic theory and is shown to depend explicitly on $\langle\rho_0\rangle$. Classical statistical mechanics leads then uniquely from $\langle(\delta\rho_T)^2\rangle$ to $\langle\rho_T\rangle$, which turns out to satisfy Planck's formula.

I. INTRODUCTION

Some fascinating new ideas concerning the physical meaning of the quantum theory have been developed in a series of papers by Boyer[1-4] and a related paper by Nelson.[5] In Boyer's work the main new concept is the existence, at the absolute zero of temperature, of a classical, fluctuating, electromagnetic background radiation which is, in some unknown fashion, equivalent to the ground state of the radiation field in quantum electrodynamics. Boyer demonstrates that incorporating his radiation background into classical statistical physics makes possible a classical derivation of Planck's blackbody spectrum. He also suggests that the universal background radiation might be the source of the random perturbations, postulated by Nelson, which transform continuous classical particle motion into an equivalent random-walk process. Since Nelson is able to derive Schrödinger's equation for particles from this classical random-walk model, we may well witness the emergence of an exciting, new interpretation of the quantum theory.

The present paper makes a small contribution to Boyer's work by deriving some of his results in a simple, axiomatic fashion. This procedure would be quite unconvincing without Boyer's penetrating analysis of classical statistical mechanics. However, once the foundations of the new theory have been established, an axiomatic approach has the virtue of conciseness, and may help to make the new ideas more readily accessible to a large audience.

Boyer has presented two different classical derivations of Planck's blackbody spectrum. One is the Einstein-Hopf derivation[6] of the Rayleigh-Jeans radiation law which leads to Planck's law if the classical radiation background is taken into account. This approach promotes valuable insights into the processes which establish dynamical equilibrium between radiation and matter. Unfortunately, the method is formally very cumbersome[7] and subject to doubts as to its general validity. The situation is such as if one derives Maxwell's velocity distribution from Boltzmann's statistical analysis of binary collisions between rigid spheres, and wonders what would happen if a more realistic model of molecules was used. The simple and universal approach to Maxwell's distribution is statistical mechanics, and that is the second road to the radiation law adopted by Boyer.

Following Einstein's pioneering work on energy fluctuations in the electromagnetic field, Boyer