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Multiplicity Distributions in Regge-Pole-Dominated Inclusive Reactions*

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Multiplicity distributions, the dependence on n of $\psi_n=\sigma_n/\sigma$, are discussed. Within the framework of the Amati-Fubini-Stanghellini model, a cluster expansion for the moments of ψ_n is derived. This same expansion is then derived as a consequence of asymptotic dominance of inclusive reactions by an isolated, factorizable Regge pole. Such an expansion furnishes a systematic way of describing the shape of ψ_n. It is argued that a Poisson distribution for multiple particle production can not be expected to occur, even for very high energies.

I. INTRODUCTION

The topic to be discussed in this paper is multiplicity distributions,[1] that is, the dependence on n of $\sigma_n/\sigma=\psi_n$ for a fixed large energy. σ_n is the production cross section for two particles to go into n particles, while σ is the total cross section. (Throughout this paper, only a single type of particle is considered. This is not necessary, but such an assumption simplifies the discussion.)

In Sec. II the model of Amati, Fubini, and Stanghellini[2] (AFS) will be used to derive a general expression [Eq.(2.11)] for the binomial moments in n of ψ_n. The approach used in deriving this equation is not unlike that used to obtain the cluster expansion in statistical mechanics. Equation (2.11) is in fact a cluster expansion.

In Sec. III the basic result, Eq. (2.11), of this paper is derived anew, this time without using the AFS model. In this derivation the ingredients are of a more general character, although they may well not be correct in the physical world. In particular, what is needed is that

$$\frac{1}{\sigma}\omega_{q_1}\omega_{q_2}\cdots\omega_{q_N}\frac{d\sigma}{d^3q_1d^3q_2\cdots d^3q_N},$$

the inclusive cross section for two particles to go into N particles with momenta $q_1, q_2, \ldots, q_N$, along with anything else, approach a constant for large energy, and that the dependence on, say, q_i factorize from all other variables when $|q_i - q_j|$ becomes large for all possible $j \neq i$. Such assumptions follow immediately if inclusive processes are dominated by an isolated and factorizable Regge pole in the appropriate asymptotic region.[3,4] If the leading angular momentum singularity, the Pomeranchuk singularity, is not an isolated pole, or nearly so, all that is derived in this paper would not apply to the physical world.[5,6]

In physical language, Eq. (2.11) is an asymptotic expression for the Nth moment,

$$\sum_{n=N}^{\infty} n(n-1)\cdots(n-N+1)\psi_n,$$

of ψ_n in a series of N terms of decreasing strength in $\ln s$. The first of these terms, $f_1{}^N$, which is proportional to $\ln^N s$ asymptotically, is a completely factorizable contribution expressing the Regge-pole factorizability of $\omega_{q_1}\cdots\omega_{q_N}(d\sigma/d^3q_1\cdots d^3q_N)$ when all the various $|q_i - q_j|$ are large. The second term, $\frac{1}{2}N(N-1)f_1{}^{N-2}f_2$, which is proportional to $\ln^{N-1}s$ asymptotically, expresses the correlation between particles i and j in $\omega_{q_1}\cdots\omega_{q_N}(d\sigma/d^3q_1\cdots d^3q_N)$ when $|q_i - q_j|$ is not large.[6] All lower-order terms in $\ln s$ express higher-order correlations among three or more of the q_i.

In Sec. IV a discussion of the general characteristics of ψ_n is given. It is emphasized, in this section, that in general one has no reason whatever to expect that ψ_n is a Poisson distribution. However, ψ_n is peaked in n about the value f_1, and the half-width of the distribution is on the order of, but not equal to, $\sqrt{f_1}$. An explicit counterexample to the Poisson nature of ψ_n is given.

In the Appendix, the multiplicity distribution for the Feynman-gas analog[6] is given. This is a special case of the general distributions of Sec. III in which two-body correlations are the highest correlations allowed.

The conclusions of this paper are partly negative, in the sense that a simple and appealing Poisson distribution is not found. However, I believe that the higher-correlation functions f_2, f_3, etc., which are a direct measure of the non-Poisson nature of multiple particle production, may be a convenient way to express this deviation from independent particle production. In particular, the expansion in f_i given by (2.11) is very similar to the expansion in inverse powers of the volume for a dilute gas.

II. MULTIPLICITY DISTRIBUTIONS IN THE AFS MODEL

The AFS model furnishes a particularly simple, and illuminating, framework in which to discuss multiplicity distributions. A general analysis of multiplicity distributions will be given, for this model, using a method which is a direct generalization of the method used by AFS for determining the average multiplicity.

In order to proceed to the main topic of this section, it may be helpful to remind the reader of a few simple facts about multiple particle production in the AFS model. The first observation is that σ_n, the production cross section for two particles to go into n particles, is given by $\sigma_n = (g^2)^n A_n$, where g is the coupling constant for the trilinear scalar vertex. A_n has no g dependence whatever. [In a Lagrangian formulation, the Lagrangian density $\mathcal{L}_I(x)$ is given by $\mu g\varphi^3(x)$, with φ the scalar field in the theory and μ a mass inserted to make g dimensionless.] The second important fact is that $A(s,t)$, the scattering amplitude for two particles to go into two particles, has a Regge-pole expansion at large s and fixed t. This expansion takes the form

$$A(s,t) \underset{s\to\infty}{\longrightarrow} s^{\alpha(t)}\beta(t) + s^{\tilde{\alpha}(t)}\tilde{\beta}(t) + \cdots, \tag{2.1}$$

where $\alpha(0) > \tilde{\alpha}(0)$. Lower-order poles are not explicitly written.

Given these two pieces of knowledge, let me now add a few definitions which will prove convenient in what follows. For a forward elastic reaction in which p_A and p_B are the momenta of the two incident, and outgoing, particles, denote $p_A \cdot p_B = \mu^2\cosh Y$. Henceforth, the energy dependence of σ_n will be parametrized by the variable Y. A normalized production cross section $\psi_n(Y) = \sigma_n(Y)/\sigma(Y)$ will be used throughout. The main topic to be discussed in this section is the n dependence of $\psi_n(Y)$ for a fixed large Y.

Introduce a generating function $\psi(Y,h)$ by

$$\psi(Y,h) = \sum_{n=2}^{\infty}\psi_n(Y)(1+h)^n = \frac{\sigma(Y, g^2(1+h))}{\sigma(Y,g^2)}. \tag{2.2}$$

[The notation $\sigma(Y,g^2)$ means that the total cross section is evaluated at $p_A \cdot p_B = \mu^2\cosh Y$ and with a coupling constant equal to g.] Then the Nth binomial moment of ψ_n obeys the equation

$$\sum_{n=N}^{\infty} n(n-1)(n-2)\cdots(n-N+1)\psi_n(Y) = \frac{\partial^N}{\partial h^N}\psi(Y,h)\bigg|_{h=0}. \tag{2.3}$$

Using (2.1) and (2.2), one can represent $\psi(Y,h)$ for large Y as

$$\psi(Y,h)=e^{f(Y,h)}+e^{-aY}(e^{g(Y,h)}-1)+\cdots, \tag{2.4}$$

with

$$f(Y,h)=\alpha(h)Y+\beta(h),\quad g(Y,h)=\tilde{\alpha}(h)Y+\tilde{\beta}(h), \tag{2.5}$$

and where

$$\alpha(0)=\beta(0)=\tilde{\alpha}(0)=\tilde{\beta}(0)=0,\quad a>0. \tag{2.6}$$

Equations (2.4)–(2.6) simply represent the Regge-pole asymptotic behavior of $\sigma(Y)$ along with the normalization of $\psi(Y,h)$.

Now, going back to (2.3), $\partial^N\psi(Y,h)/\partial h^N$ can be evaluated for large Y by using the first term in (2.4). Then

$$F_N(Y)\equiv\frac{\partial^N}{\partial h^N}\psi(Y,h)\Big|_{h=0}\approx\frac{\partial^N}{\partial h^N}e^{f(Y,h)}\Big|_{h=0}. \tag{2.7}$$

[Lower-lying Regge trajectories may be neglected if Y is sufficiently large compared to N. However, it is definitely possible that for a large *fixed* Y there may be an N_0 for which $\partial^N\psi(Y,h)/\partial h^N$ is not well approximated by $\partial^N e^{f(Y,h)}/\partial h^N$ for $N>N_0$.] In order to evaluate (2.7) write

$$f(Y,h)=\sum_{n=1}^{\infty}\frac{h^n}{n!}f_n(Y)\equiv\sum_{n=1}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial h^n}f(Y,h)\Big|_{h=0}. \tag{2.8}$$

From (2.5),

$$f_n(Y)=\alpha_nY+\beta_n, \tag{2.9}$$

with

$$(\alpha_n,\beta_n)=\frac{\partial^n}{\partial h^n}(\alpha(h),\beta(h))\Big|_{h=0}.$$

Then

$$e^{f(Y,h)}=\exp\left(\sum_{n=1}^{\infty}\frac{h^n}{n!}f_n\right)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{n=1}^{\infty}\frac{h^n}{n!}f_n(Y)\right)^m.$$

Define $(f_n/n!)h^n=X_n$. Then

$$e^{f(Y,h)}=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{n=1}^{\infty}X_n\right)^m$$

or

$$e^{f(Y,h)}=\sum_{n_1,n_2,\cdots}\frac{1}{n_1!}\binom{n_1}{n_2}\binom{n_2}{n_3}\cdots\binom{n_i}{n_{i+1}}\cdots X_1^{n_1-n_2}X_2^{n_2-n_3}\cdots X_i^{n_i-n_{i+1}}\cdots,$$

where the summation over n_1, n_2, etc. goes over all positive integers such that $n_1\geqslant n_2\geqslant n_3\geqslant\cdots n_i\geqslant n_{i+1}\geqslant\cdots$.

Substituting for X_n, the previous equation becomes

$$e^{f(Y,h)}=\sum_{n_1,n_2,\cdots}h^{n_1+n_2+\cdots+n_\infty}\frac{1}{n_1!}\binom{n_1}{n_2}\binom{n_2}{n_3}\cdots\binom{n_i}{n_{i+1}}\cdots(f_1)^{n_1-n_2}\left(\frac{f_2}{2!}\right)^{n_2-n_3}\cdots\left(\frac{f_i}{i!}\right)^{n_i-n_{i+1}}\cdots.$$

Finally,

$$F_N(Y)=\frac{\partial^N}{\partial h^N}e^{f(Y,h)}\Big|_{h=0}$$

$$=N!\sum_{n_1,n_2,\cdots}\frac{1}{n_1!}\binom{n_1}{n_2}\binom{n_2}{n_3}\cdots\binom{n_i}{n_{i+1}}\cdots(f_1)^{n_1-n_2}\left(\frac{f_2}{2!}\right)^{n_2-n_3}\cdots\left(\frac{f_i}{i!}\right)^{n_i-n_{i+1}}\cdots\delta(N-\sum_i n_i). \tag{2.10}$$

The summation in (2.10) goes over all positive integers n_i for which $\infty>n_1\geqslant n_2\geqslant n_3\geqslant\cdots n_i\geqslant n_{i+1}\geqslant\cdots$, while the δ function requires that $N=\sum_i n_i$. Thus the final formula for the asymptotic values of the Nth binomial moment of ψ_n is

$$F_N=\sum_{n=N}^{\infty}n(n-1)(n-2)\cdots(n-N+1)\psi_n(Y)$$

$$=N!\sum_{n_i}\delta(N-\sum_i n_i)\prod_{i=1}^{\infty}\left(\frac{f_i}{i!}\right)^{n_i-n_{i+1}}\frac{1}{(n_i-n_{i+1})!}. \tag{2.11}$$

A discussion of (2.11) will be postponed until the next section where a derivation of this equation will be given which does not make use of the AFS model.

III. AN INTERPRETATION AND GENERALIZATION OF Eq. (2.11)

The object of this section is to give an interpretation of Eq. (2.11), and to show that such an equation occurs independently of the model of the previous section. This discussion will be limited to the case of a world where only a single type of scalar boson exists. This limitation is only for convenience. In the inclusive reaction $p_A + p_B \to q_1 + q_2 + q_3 + \cdots q_N +$ anything, the following parametrization of p_A, p_B, and q_i is useful[7]:

$$\begin{aligned} p_A &= \mu(\cosh Y_A, 0, 0, \sinh Y_A), \\ p_B &= \mu(\cosh Y_B, 0, 0, \sinh Y_B), \\ q_i &= (\mu^2 + \vec{q}_i^{\,2})^{1/2}(\cosh y_i, \vec{q}_i/(\mu^2 + \vec{q}_i^{\,2})^{1/2}, \sinh y_i), \end{aligned} \tag{3.1}$$

where $\vec{q}_i$ is a two-vector. The differential cross section for the above inclusive reaction will be denoted by $d\sigma/d^3q_1 d^3q_2 \cdots d^3q_N$.

A simple counting argument allows one to write

$$\begin{aligned} \sum_{n=N}^{\infty} n(n-1)(n-2)\cdots(n-N+1)\psi_n(Y) &= \frac{1}{\sigma(Y)} \int d^3q_1 d^3q_2 \cdots d^3q_N \frac{d\sigma}{d^3q_1 d^3q_2 \cdots d^3q_N} \\ &= \frac{1}{\sigma(Y)} \int dy_1 dy_2 \cdots dy_N \frac{d\sigma}{dy_1 dy_2 \cdots dy_N} . \end{aligned} \tag{3.2}$$

Henceforth $[1/\sigma(Y)](d\sigma/dy_1 dy_2 \cdots dy_N)$ will be denoted by $d\psi/dy_1 dy_2 \cdots dy_N$.

Begin with the special case, $N=1$. Then (3.2) gives

$$\sum_{n=2}^{\infty} n\psi_n = \int dy \frac{d\sigma}{dy} .$$

Consider a straight line between Y_A and Y_B with y someplace on the line. When y is not close to either Y_A or Y_B the function $d\psi/dy$ is independent of y, if scaling occurs.[7] Furthermore, if the leading Regge singularity is an isolated, factorizable pole,

$$\frac{d\psi}{dy} \approx \alpha_1 + O(e^{-\Delta(Y_B - y)})$$

when y is moved away from the point Y_B. A similar equation holds for y in the region of Y_A. Thus,

$$\int dy \frac{d\psi}{dy} = \alpha_1 Y + \beta_1 = f_1(Y) . \tag{3.3}$$

Equation (3.3) serves as a definition of α_1 and β_1. The constant β_1 reflects the part of the integration where y is near either Y_A or Y_B.

For $N=2$ in (3.2),

$$\sum_{n=2}^{\infty} n(n-1)\psi_n = \int dy_1 dy_2 \frac{d\psi}{dy_1 dy_2} .$$

Now write

$$\frac{d\psi}{dy_1 dy_2} = \frac{d\psi}{dy_1}\frac{d\psi}{dy_2} + \frac{d\psi^{(2)}}{dy_1 dy_2} . \tag{3.4}$$

Note that $d\psi^{(2)}/dy_1 dy_2$ is not small only when $|y_1 - y_2|$ is not small.[6] As $|y_1 - y_2|$ increases from zero, the decrease of $d\psi^{(2)}/dy_1 dy_2$ is exponential in $|y_1 - y_2|$ so long as the leading Regge singularity is an isolated, factorizable pole in the J plane. Calling

$$\int dy_1 dy_2 \frac{d\psi^{(2)}}{dy_1 dy_2} = \alpha_2 Y + \beta_2 \equiv f_2(Y),$$

one obtains

$$\sum_{n=2}^{\infty} n(n-1)\psi_n(Y) = f_1^{\,2} + f_2 , \tag{3.5}$$

in agreement with (2.11).

For $N=3$, define $d\psi^{(3)}/dy_1 dy_2 dy_3$ by

$$\frac{d\psi}{dy_1 dy_2 dy_3} = \frac{d\psi}{dy_1}\frac{d\psi}{dy_2}\frac{d\psi}{dy_3} + \frac{d\psi^{(2)}}{dy_1 dy_2}\frac{d\psi}{dy_3} + \frac{d\psi^{(2)}}{dy_1 dy_3}\frac{d\psi}{dy_2} + \frac{d\psi^{(2)}}{dy_2 dy_3}\frac{d\psi}{dy_1} + \frac{d\psi^{(3)}}{dy_1 dy_2 dy_3} ,$$

or in an abbreviated notation,

$$\frac{d\psi}{dy_1 dy_2 dy_3} = \prod_{i=1}^{3} \frac{d\psi}{dy_i} + \sum_{\substack{i<j \\ i,j\neq k}} \frac{d\psi^{(2)}}{dy_i dy_j}\frac{d\psi}{dy_k} + \frac{d\psi^{(3)}}{dy_1 dy_2 dy_3} . \tag{3.6}$$

$d\psi^{(3)}/dy_1 dy_2 dy_3$, a three-particle correlation function, is not small only when all pairs $|y_i - y_j|$ are not large, as is evident from the definition (3.6). Calling

$$\int dy_1 dy_2 dy_3 \frac{d\psi^{(3)}}{dy_1 dy_2 dy_3} = \alpha_3 Y + \beta_3 \equiv f_3(Y), \tag{3.7}$$

the formula

$$\sum_{n=3}^{\infty} n(n-1)(n-2)\psi_n = f_1^{\,3} + 3 f_1 f_2 + f_3$$

is obtained. This equation also agrees with (2.11).

One last example, $N=4$, will be considered. After that it should be clear how one proceeds in the general case to verify (2.11) for any N. Define $d\psi^{(4)}/dy_1 dy_2 dy_3 dy_4$ by

$$\frac{d\psi}{dy_1 dy_2 dy_3 dy_4} = \prod_i \frac{d\psi}{dy_i} + \sum_{j<k} \frac{d\psi^{(2)}}{dy_j dy_k} \prod_{i\neq j,k} \frac{d\psi}{dy_i} + \tfrac{1}{2}\sum_{k<l} \frac{d\psi^{(2)}}{dy_k dy_l} \sum_{\substack{i<j \\ i,j\neq k,l}} \frac{d\psi^{(2)}}{dy_i dy_j} + \sum_{\substack{i<j<k \\ l\neq i,j,k}} \frac{d\psi^{(3)}}{dy_i dy_j dy_k}\frac{d\psi}{dy_l} + \frac{d\psi^{(4)}}{dy_1 dy_2 dy_3 dy_4} . \tag{3.8}$$

i, j, k, and l can take on the values 1, 2, 3, and 4. $d\psi^{(4)}/dy_1dy_2dy_3dy_4$ is a four-particle correlation function, and is not small only when all pairs $|y_i - y_j|$ are simultaneously not large. If one calls

$$\int dy_1dy_2dy_3dy_4 \frac{d\psi^{(4)}}{dy_1dy_2dy_3dy_4} = \alpha_4 Y + \beta_f = f_4(Y),$$

then

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)\psi_n = f_1^4 + 6f_1^2 f_2 + 3f_2^2 + 4f_1f_3 + f_4,$$

as demanded by (2.11). For the general case of any given N, one has to define correlation functions for up to N particles. With such appropriate correlation functions, (2.11) is reproduced. The essential requirement for (2.11) to hold is that only a single factorizable Regge pole be exchanged between particles i and j (when there is no y_k intermediate in value between y_i and y_j, and when $|y_i - y_j|$ is large) in a multiple $O(3,1)^4$ (multiple Regge pole) analysis of $d\psi/dy_1dy_2\cdots dy_N$.[8]

IV. DISCUSSION OF GENERAL CHARACTERISTICS OF MULTIPLICITY DISTRIBUTIONS

Now that (2.11) has been obtained from the AFS model, and in a somewhat more general context, it is appropriate to give a discussion of the general features of $\psi_n(Y)$. Firstly, is $\psi_n(Y)$ a Poisson distribution?

In the AFS model, when the coupling is very weak all the f_n are small compared to f_1. To be more explicit, suppose the φ^3 theory from which the AFS model is extracted has a coupling constant defined by $\mathcal{L}_I(x) = \mu g \varphi^3(x)$. Then, when g is small $f_n(Y)$ is proportional to g^{2n}. The only term in (2.11) which survives when $g \to 0$ is the term involving f_1 alone, so that

$$F_N(Y) \underset{g\to 0}{\longrightarrow} [f_1(Y)]^N .$$

Such a moment equation means that $\psi_n(Y)$ becomes a Poisson distribution as $g \to 0$. In general, however, there is no reason, either mathematical or physical, to believe that the weak-coupling limit is a reasonable approximation to the φ^3 theory or to the physical world.

When one looks closely at (2.11), an intriguing situation occurs. To observe this situation consider the first few terms of F_N [Eq. (2.11)]:

$$F_N(Y) = f_1^N + \tfrac{1}{2}N(N-1)f_1^{N-2}f_2 + \left[\tfrac{1}{2}\frac{N(N-1)}{2}\,\frac{(N-2)(N-3)}{2} f_1^{N-4} f_2^2 + \frac{N(N-1)(N-2)}{3!} f_1^{N-3} f_2\right] + \cdots . \quad (4.1)$$

As Y becomes large, each f_n increases linearly with Y. This means that, for a fixed N, F_N is ultimately dominated by f_1^N for sufficiently large Y. That is, all moments F_N, for N less than some fixed N_0, become the moments of a Poisson distribution when Y is greater than some Y_{N_0}. The question is, then, whether such an occurrence is enough to require that $\psi_n(Y)$ resemble a Poisson distribution for sufficiently large Y. To show that the answer to this question is in the negative, a counterexample will be given.

The example consists in a specification of $\psi(Y, h)$ in (2.4) as

$$\psi(Y, h) = \exp[(\alpha Y + \beta)h + \tfrac{1}{2}(\alpha Y + \beta)h^2] , \quad (4.2)$$

with α and β arbitrary parameters. [Note that $f_1(Y) = f_2(Y) = \alpha Y + \beta$.] Define $z = h + 1$. Then

$$\psi(Y, z) = e^{-f_1/2} e^{f_1 z^2/2} . \quad (4.3)$$

Equation (2.2) can be inverted by contour integration to give

$$\psi_n(Y) = \frac{1}{2\pi i}\int_C \psi(Y, z) z^{-n-1} dz , \quad (4.4)$$

where C is a closed contour encircling the origin in a counter-clockwise direction. Substituting (4.3) into (4.4) yields

$$\psi_n(Y) = \begin{cases} \dfrac{e^{-f_1/2}(\frac{1}{2}f_1)^{n/2}}{\Gamma(\frac{1}{2}n+1)} , & n \text{ even} \\ 0, & n \text{ odd} . \end{cases} \quad (4.5)$$

Equation (4.5) is clearly very far from being a Poisson distribution. The general case of arbitrary f_1 and f_2, the Feynman-gas analog, is given in the Appendix.

Although $\psi_n(Y)$ is not necessarily close to a Poisson distribution in general, the fact that $F_N(Y) \to f_1^N(Y)$ as $Y \to \infty$ does require that $\psi_n(Y)$ peak in n about a mean value $\bar{n}(Y) = f_1$. By "peak" I mean that the distribution must cut off sharply for $n \gg f_1$ and for $n \ll f_1$. More precisely, the half-width in n of the distribution $\psi_n(Y)$ about f_1 must be on the order of $\sqrt{Y}$. This result follows easily from (2.11).

Finally, let me remark that (2.11) can be inverted to give

$$f_n(Y) = n! \sum (-1)^{n_1 - 1} \frac{(n_1 - 1)!}{(n_1 - n_2)!(n_2 - n_3)!} \cdots \times \frac{1}{(n_i - n_{i+1})!} \cdots F_1^{n_1 - n_2} F_2^{n_2 - n_3} \cdots \times F_i^{n_i - n_{i+1}} \delta(n - \textstyle\sum n_i), \quad (4.6)$$

where, again, the summation is over all n_i satisfying $\infty > n_1 \geq n_2 \geq n_3 \geq \cdots n_i \geq n_{i+1} \cdots$. Equation (4.6) furnishes a way to relate the correlation functions f_i to the moments of the distribution ψ_n.

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APPENDIX: THE FEYNMAN-GAS ANALOG

The Feynman-gas analog can be defined by saying that all f_n except f_1 and f_2, in (2.11), are zero. In such a case one can solve for $\psi_n(Y)$ exactly in terms of known functions. The starting point is (2.11), which in this case reads

$$F_N(Y)=N!\sum_{n_1\geq n_2}\frac{\delta(N-n_1-n_2)}{(n_1-n_2)!n_2!}(f_1)^{n_1-n_2}\left(\frac{f_2}{2!}\right)^{n_2}. \tag{A.1}$$

A suitable generating function for F_N is

$$\psi(Y,h)=\exp(f_1h+\tfrac{1}{2}f_2h^2). \tag{A.2}$$

Using

$$\sum_{n=2}^{\infty}\psi_n(1+h)^n=\psi(Y,h),$$

one obtains

$$\psi_n(Y)=\frac{1}{n!}\frac{\partial^n}{\partial h^n}\psi(Y,h)\bigg|_{h=-1}. \tag{A.3}$$

Write $z=1+h$. Then

$$\psi(Y,z)=\exp\left(-\frac{f_1^{\,2}}{2f_2}\right)\exp\left[\tfrac{1}{2}f_2\left(z-\frac{f_2-f_1}{f_2}\right)^2\right].$$

Call $w=(\frac{1}{2}f_2)^{1/2}[z-(f_2-f_1)/f_2]$. Then

$$\psi_n(Y)=\frac{1}{n!}\exp\left(-\frac{f_1^{\,2}}{2f_2}\right)(\tfrac{1}{2}f_2)^{n/2} \times\frac{\partial^n}{\partial w^n}e^{w^2}\bigg|_{w=[(f_2-f_1)^2/2f_2]^{1/2}}. \tag{A.4}$$

Now $H_n(x)=(-1)^n e^{x^2}\partial^n e^{-x^2}/\partial x^n$, where $H_n(x)$ is the Hermite polynomial of nth order. Thus,

$$\psi_n(Y)=\frac{\exp(\frac{1}{2}f_2-f_1)}{n!}(\tfrac{1}{2}f_2)^{n/2} \times(-i)^nH_n(i[(f_2-f_1)^2/2f_2]^{1/2}). \tag{A.5}$$

Equation (A.5) is the most general solution to the Feynman-gas-analog model. This distribution approaches a Poisson distribution as $f_2/f_1\to 0$ but, in general, does not resemble a Poisson distribution for f_2 comparable to f_1.

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[1]A preliminary, and abbreviated, report on part of this work was presented at the *1971 Coral Gables Conference on Fundamental Interactions at High Energy* (to be published).

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[8]The question may arise as to the connection of Secs. II and III in this paper with (i) the standard virial expansion in statistical mechanics, and (ii) the expansion of the logarithm of the characteristic function of probability theory in terms of cumulants. [For (ii) see, for example, B. V. Gnedenko, *The Theory of Probability* (Chelsea, New York, 1962).] The expansion given in Sec. II is almost the cumulant expansion of probability theory. The difference is that the characteristic function in probability theory is defined via a Fourier transform, and one considers ordinary moments rather than the binomial moments which must be considered in the particle-physics context. In Sec. III this difference is crucial. In order to deal with binomial moments, one has to define the characteristic function via a Mellin transform rather than a Fourier transform. This Mellin transform may not exist in the context of Sec. III. For example,

$$\sum_{n=0}^{\infty}\frac{f_n}{n!}h_n,$$

the logarithm of the characteristic function, certainly is a convergent series, for sufficiently small h, in the context of the AFS model, but it may not exist in the more general context of Sec. III. Similar remarks hold for the connection to statistical mechanics, for in this case the "free energy" may not exist. All in all, I feel that too close an identification with the cumulant expansion, or with statistical mechanics, may be unwarranted until a characteristic function or free energy can be shown to exist.