Although this analysis is based upon a few additional assumptions, the numerical results obtained there are all consistent with a simple naive picture. Even if they changed by a small amount, our basic conclusions would not be affected.

[12]The matrix elements in (2.11) and (2.12) are taken from U(12) symmetry, but they are of course unique up to $(-t)^2$. B. Sakita and K. C. Wali, Phys. Rev. 139, B1355 (1965). The right-hand side of (2.11) satisfies conservation of the axial-vector current, and that of (2.12) is in agreement with mild t dependence of $\partial_\mu \mathfrak{F}^5_\mu$.

[13]See P. S. J. McNamee and F. Chilton, Rev. Mod. Phys. 36, 1005 (1964).

[14]This should be compared with the U(12)-symmetric value $G_V(0) = 1 + (m_\Delta + m_N)/m_p \simeq 3.85$.

[15]The present method of determining the pion coupling is quite different from that in the U(12) symmetry.

[16]Y. Nambu and M. Yoshimura, Phys. Rev. Letters 24, 25 (1970). We have readjusted their normalization to agree with that of Brene *et al.* in Ref. 14.

[17]Mass splittings do not affect any results to highest order in s, except in (3.7) where the masses explicitly appear.

[18]The form factors are real by time-reversal invariance and Hermiticity.

[19]We follow the conventions of V. Singh, Phys. Rev. 129, 1889 (1963).

[20]W. Rarita *et al.*, Phys. Rev. 165, 1615 (1968). The relation of their A' to Singh's (our) A and B is given in note 6 of their paper.

[21]If $\alpha(t) \equiv 1$ for at least one of the Regge poles, the cut produced by it has a branch point independent of t. Recall that $\alpha_{\rm cut}(t) = {\rm Max}[\alpha_1(t_1) + \alpha_2(t_2) - 1]$, where $2tt_1 + 2tt_2 + 2t_1t_2 - t^2 - t_1^2 - t_2^2 \geq 0$. We have taken $\alpha_P(t) \equiv 1$.

[22]If SU(3) symmetry were exact, $M_{\rm strong}(p\Sigma) = M_{\rm strong}(pp)$ in the high-energy limit. However, $m_\Sigma = m_N$ in this limit. In the charge-exchange process there is no obvious cancellation among different diagrams.

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Complex Regge Trajectories*

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Conditions are explored which require the presence of leading positive- or negative-signature trajectories of the form $\alpha(t) = 1 + {\rm const}\sqrt{t} + O(t)$. General forms of continued partial-wave amplitudes with such trajectories are given, and their implications for high-energy limits are evaluated. The restrictions on the character of the complex singular surfaces are discussed. Explicit examples are given for amplitudes with complex Regge trajectories.

I. INTRODUCTION

From a semiclassical point of view, diffraction scattering looks like one of the more straightforward and simple phenomena. Nevertheless, since the notions of dispersion theory and complex angular momenta appear to form a reasonable framework for the descriptions of fundamental particles, it is relevant to look for a description of diffraction scattering in terms of crossed-channel properties. The features that are relevant to the asymptotic expansion of an amplitude in the s channel are the singularities in the complex angular momentum plane for the t channel.

The Regge-pole trajectories associated with physical particles and resonances are usually assumed to be approximately linear functions in the neighborhood of $t = 0$, and so are the branch-point trajectories associated with these Regge poles. However, the singular surfaces directly related to diffraction scattering may well be of a quite different type. Independent of their character – i.e., whether they are poles, square-root branch points, etc. – we may ask whether there are reasons to think that these trajectories, as analytic functions of t, have a singular point at $t = 0$. Many years ago, we introduced two-valued singular surfaces of the form

$$\alpha(t) = \alpha(0) \pm {\rm const}\sqrt{t} + O(t), \tag{1.1}$$

where, of course, the continued partial-wave amplitude $F(t, \lambda)$ must not inherit the branch point at $t = 0$.[1] Trajectories of this type appear in Schrödinger theory with repulsive r^{-2} potentials, and in several field-theoretical calculations which are in principle related to the Schrödinger case.[2,3]

Since high-energy limits of amplitudes are often considered to be calculable by iteration schemes involving s-channel unitarity, we also have considered trajectories of the form

$$\alpha(t) = 1 \pm {\rm const}\sqrt{t} \tag{1.2}$$

in the neighborhood of $t = 0$, because they are a solution of the bootstrap-type condition[4,5]

$$\alpha(t)=\alpha_n(t)=n\alpha(t/n^2)-n+1. \tag{1.3}$$

In the s channel, with an appropriate choice of $F(t,\lambda)$, trajectories of the form (2) give rise to amplitudes which are rather directly written as a superposition of Bessel functions like

$$F(s,t)\sim s\int_0^1 d\xi\,\psi(\xi,s)J_0(\xi\sqrt{-at}\,\ln s), \tag{1.4}$$

with weight functions $\psi(\xi,s)$ having support for nonzero values of ξ in the limit $s\to\infty$.[4,6,7]

In a recent paper,[8] we have shown that amplitudes which give rise to constant and different asymptotic total cross sections for particle and antiparticle scattering must have leading negative- and positive[9]-signature trajectories with a square-root branch point at $t=0$ as in Eq. (2). In this paper, we discuss the more general conditions under which trajectories of the form (2) are required to be present. We write down general forms of continued partial-wave amplitudes with complex trajectories and evaluate their implications for the high-energy limits in the crossed channel. We study the implications of t- and s-channel unitarity and analyticity properties for the character of these singular surfaces. Finally, we give an explicit one-parameter family of amplitudes with different asymptotic cross sections for particle and antiparticle scattering, and we consider examples for amplitudes with rising cross sections.

II. CONDITIONS REQUIRING COMPLEX TRAJECTORIES

A Regge trajectory is a complex singular surface $\lambda=\alpha(t)$ of the continued partial-wave amplitude $F(t,\lambda)$ corresponding to the invariant scattering amplitude $F(s,t)$. At this point, we do not specify the character of this singular surface explicitly, but we will restrict it indirectly in the following. As is well known, the position of the singularities $\lambda=\alpha(t)$ in the complex angular momentum plane determines the power behavior of the corresponding contribution to the asymptotic expansion of the amplitude $F(s,t)$ for $s\to\infty$.

For a trajectory which is regular near $(t,\lambda)=(0,1)$, i.e.,

$$\alpha(t)=1+\alpha'(0)t+\cdots, \tag{2.1}$$

we know that we obtain a logarithmic shrinkage of the diffraction peak for $\alpha'(0)>0$:

$$\int_{-s}^0 dt|f(s,t)|^2\propto(\ln s)^{-1}, \tag{2.2}$$

where

$$f(s,t)\equiv F(s,t)/F(s,0). \tag{2.3}$$

However, the properties of the forward-scattering amplitude $F(s,0)$ may be such as to require a stronger shrinkage than in Eq. (2.2). We can easily derive a bound for the integral over the diffraction peak in terms of $F(s,0)$. Since, with our normalization,

$$\sigma_{tot}\sim(16\pi/s)\operatorname{Im}F(s,0),\qquad \sigma_{el}\sim\frac{16\pi}{s^2}\int_{-s}^0 dt|F(s,t)|^2, \tag{2.4}$$

we obtain from the condition $\sigma_{el}\le\sigma_{tot}$ the bound

$$\int_{-s}^0 dt|f(s,t)|^2\le\frac{s\operatorname{Im}F(s,0)}{|F(s,0)|^2}. \tag{2.5}$$

Furthermore, using the partial-wave expansion for $F(s,t)$ and recalling the Schwarz inequality, we find the familiar bound[10]

$$\sigma_{el}\ge\frac{16\pi}{s}\frac{|F(s,0)|^2}{\frac14 as(\ln s)^2}, \tag{2.6}$$

where the radius $\sqrt{a}$ is defined by the maximal orbital angular momentum $L=\frac12\sqrt{as}\,\ln s$ which is relevant for $s\to\infty$.[11] From Eq. (2.6), we deduce the lower bound

$$\int_{-s}^0 dt|f(s,t)|^2\ge\frac{4}{a}(\ln s)^{-2}. \tag{2.7}$$

The bounds (2.5) and (2.7) indicate that amplitudes with $F(s,0)$ restricted by

$$\frac{4}{a}(\ln s)^{-2}\le\frac{s\operatorname{Im}F(s,0)}{|F(s,0)|^2}\le O((\ln s)^{-1-\epsilon}), \tag{2.8}$$

$\epsilon>0$, require a leading Regge trajectory which has a branch point at $t=0$. A regular trajectory would generally not give sufficient shrinkage.

Let us analyze the bound in Eq. (2.7) in more detail. Splitting the amplitude F into its positive- and negative-signature parts

$$F_\pm(s,t)=F(s,t)\pm\bar F(s,t), \tag{2.9}$$

where $\bar F$ is the antiparticle amplitude corresponding to F, we assume the forward amplitudes $F_\pm(s,0)$ to be of the form[3]

$$F_+(s,0)\propto is(\ln s-\tfrac12 i\pi)^{\beta_+}=is(\ln s)^{\beta_+}+\tfrac12\pi\beta_+ s(\ln s)^{\beta_+-1}+\cdots \tag{2.10}$$

and

$$F_-(s,0)\propto-\frac{2}{\pi}\frac{1}{\beta_-+1}s(\ln s-\tfrac12 i\pi)^{\beta_-+1}=-\frac{2}{\pi}\frac{1}{\beta_-+1}s(\ln s)^{\beta_-+1}+is(\ln s)^{\beta_-}+\cdots. \tag{2.11}$$

We have the bounds[10]

$$\beta_+\le2\quad\text{and}\quad\beta_-\le\tfrac12\beta_+, \tag{2.12}$$

and we restrict ourselves to nondecreasing total cross sections, which implies $\beta_+ \geq 0$. The *Ansätze* (2.10) and (2.11) are consistent with the forward dispersion relations.[12] The negative-signature term (2.11) may be completely absent, corresponding to cases where $\alpha_-(0) < 1$.

In order to distinguish between positive- and negative-signature trajectories, it is convenient to consider in place of Eq. (2.5) the related inequalities

$$\int_{-s}^{0} dt \left(\frac{\operatorname{Im}F(s,t)}{\operatorname{Im}F(s,0)}\right)^2 \leq s[\operatorname{Im}F(s,0)]^{-1} \tag{2.13}$$

and

$$\int_{-s}^{0} dt \left(\frac{\operatorname{Re}F(s,t)}{\operatorname{Re}F(s,0)}\right)^2 \leq \frac{s\operatorname{Im}F(s,0)}{[\operatorname{Re}F(s,0)]^2}. \tag{2.14}$$

The right-hand side of Eq. (2.13) is proportional to

$$(\ln s)^{-\beta_+} \tag{2.15}$$

and, if $\beta_+ \geq 1+\epsilon$, $\epsilon > 0$, we see that the positive-signature trajectory requires a branch point at $t=0$ in order to provide sufficient shrinkage.

The right-hand side of Eq. (2.14) is relevant for us only if $\alpha_-(0)=1$. Then it is proportional to

$$(\ln s)^{-2\beta_- - 2 + \beta_+}, \tag{2.16}$$

provided $\beta_+ \leq 2+\beta_-$, and we have more than logarithmic shrinkage for

$$2\beta_- + 2 - \beta_+ \geq 1+\epsilon. \tag{2.17}$$

Under these circumstances, the negative-signature trajectory must provide the excess shrinkage and requires a branch point at $t=0$. It always dominates the asymptotic behavior of $\operatorname{Re}F(s,0)$ if the inequality (2.17) holds.

Let us now consider a leading trajectory of either signature which has a branch point at $t=0$. If $\alpha(t)$ is given by

$$\alpha(t) = 1 + \text{const}\, t^{1/\beta}, \quad \beta \leq 2 \tag{2.18}$$

it can contribute to the high-energy limit of the appropriate part of $F(s,t)$ a term which is proportional to

$$\operatorname{Re}(s^{1+\text{const}\, t^{1/\beta}}). \tag{2.19}$$

We assume that the branches of $\alpha(t)$ are chosen such that, either in Eq. (13) or (14), the integral is proportional to

$$(\ln s)^{-\beta}. \tag{2.20}$$

Then we get sufficient shrinkage for

$$\beta \geq \beta_+ \quad \text{or} \quad \beta \geq 2\beta_- + 2 - \beta_+, \tag{2.21}$$

respectively. We see that it is always sufficient to have $\beta = 2$, but in general the conditions (2.21) do not require this. Nevertheless, using the methods described in I, we can argue that trajectories of the form (2.18) are only consistent for $\beta=2$, unless we are willing to introduce special "hiding cuts."[13]

If the trajectory $\alpha(t)$ in Eq. (2.18) is considered as a singular surface of the continued partial-wave amplitude $F(t,\lambda)$, this amplitude will generally acquire a branch point at $t=0$ which is not allowed.[13] There are exceptions in cases where

$$1/\beta = m/n, \tag{2.22}$$

with m and n being integers. Under these circumstances we have a *finite* number of trajectories $\alpha_1 \cdots \alpha_n$ which cross at $t=0$. If all these branches of the function $\alpha(t)$ appear in $F(t,\lambda)$ in a completely symmetric fashion, we find that the branch point of $\alpha(t)$ is not inherited by $F(t,\lambda)$. However, we can exclude all ratios (2.22) except $m/n = \frac{1}{2}$ by using bounds in the s channel. The range of interest for β is $1+\epsilon \leq \beta < 2$, and hence we have $n \geq 3$, $m < n$. As we have described in I, there is then always a branch of $\alpha(t)$ such that the appropriate part of the amplitude $F(s,t)$ contains terms which, as far as the power law is concerned, behave like

$$s^{1+c|t|^{m/n}}, \tag{2.23}$$

with $c > 0$. Such terms violate bounds like

$$|F(s,t)| \leq \text{const}\, s(\ln s)^2, \tag{2.24}$$

and also the simple unitarity requirement

$$\operatorname{Im}F(s,t) \leq \operatorname{Im}F(s,0), \tag{2.25}$$

for $t \leq 0$. Only through the introduction of a very special hiding cut in $F(t,\lambda)$ can we arrange to have the offending branches of $\alpha(t)$ removed from the physical sheet of the λ plane while preserving the symmetry which prevents $F(t,\lambda)$ from having a branch point at $t=0$ for $\lambda \neq 1$.[13]

Summing up, we conclude that leading Regge trajectories for a positive- or negative-signature amplitude are expected to have a square-root branch point at $t=0$ under the following circumstances:

(a) $\alpha_+(t)$ is of the form (2.1) if $\beta_+ \geq 1+\epsilon$; also if $\alpha_-(0)=1$, $\beta_+ = \beta_- = 0$. The latter case has been discussed in I.

(b) $\alpha_-(t)$ has such a branch point if $\alpha_-(0)=1$ and $\beta_- \geq \frac{1}{2}(\beta_+ - 1) + \epsilon$.

In order to arrive at these conclusions, we have assumed that the shrinkage must be generated by the properties of the appropriate trajectory function. This is generally the case for singular pole and branch-point surfaces. However, if we allow appropriate essential singularities in the complex angular momentum plane, then there can be some other possibilities, at least in situations where the shrinkage is less than $(\ln s)^2$. But essential singularities are considerably restricted by constraints in the s and t channels. In particular, they violate unitarity in the t channel unless they have a very special t-dependent character.[3] In this paper, we do not intend to explore the details of these essential singularities. As has been mentioned before, we have also assumed that there are no hiding cuts.[13]

For some of the cases mentioned in (a) and (b), the requirement of complex trajectories can also be derived by using only s-channel constraints.[9,8,14] For example, s-channel methods are often sufficient for $\beta_- = \frac{1}{2}\beta_+$ or $\beta_+ = 2$, where the bounds (2.13) and (2.14) already require $(\ln s)^2$ shrinkage by themselves.

We have not considered here possible $\ln\ln s$ factors in the asymptotic expansion of $F(s, t)$. These factors depend upon the character of the singular surfaces $\alpha_\pm(t)$ and are not related to the position of the singularities in the complex angular momentum plane. At present, they are relevant only in cases where they would lead to a violation of a rigorous bound.

Finally, we remark that Regge trajectories with square-root branch points at $t=0$ may well be present in other cases where they are not directly required by general principles.

III. PARTIAL-WAVE AMPLITUDES AND HIGH-ENERGY LIMITS

For later use in the construction of explicit models, we consider in this section a certain class of continued partial-wave amplitudes $F(t, \lambda)$ which, in the neighborhood of $(t, \lambda) = (0, 1)$, are mainly determined by complex singular surfaces of the form (1.2). We write for (t, λ) near $(0, 1)$[6,7]

$$F(t,\lambda) \propto \int_0^1 d\xi \, \frac{\rho_\pm(\xi, t)}{[(\lambda-1)^2 - \xi^2 at]^{1/2}} + \cdots . \tag{3.1}$$

There are several restrictions for the real weight functions $\rho_\pm(\xi, t)$ which follow from s- and t-channel constraints; but within these restrictions our *Ansatz* (3.1) is rather general, in particular, if we allow $\rho_\pm$ to be generalized functions. As an example of t-channel restrictions, we mention the important relation

$$\int_0^1 d\xi \, \xi^{-1} \rho_-(\xi, 0) = 0, \tag{3.2}$$

which prevents unwanted singularities of the P-wave amplitude $F_1(t) = F_-(t, \lambda=1)$ at $t=0$.

The contribution of the singular terms (3.1) to the high-energy limit in the s channel are obtained from the contour integral[3,7]

$$F_\pm(s,t) \sim \frac{1}{2\pi i} \int_C d\lambda \, s^\lambda S_\pm(\lambda) K(t,\lambda) F_\pm(t,\lambda), \tag{3.3}$$

where C is an integration path enclosing the singularities of the integrand,

$$S_\pm(\lambda) = \frac{\mp 1 - e^{-i\pi\lambda}}{\sin \pi\lambda} \tag{3.4}$$

is the signature factor, and the function $K(t, \lambda)$ is given by

$$K(t,\lambda) = 2\sqrt{\pi}\, q^{-2\lambda}(t) \frac{\Gamma(\lambda + \frac{3}{2})}{\Gamma(\lambda+1)}. \tag{3.5}$$

For simplicity, we ignore in the following the function $K^{-1}(t, \lambda)$ and other possible λ-dependent factors of $F_\pm(t, \lambda)$ which are regular in the neighborhood of $(t, \lambda) = (0, 1)$.

For the positive-signature amplitude, the signature factor (3.4) is regular near $\lambda = 1$, and we write it in the form

$$S_+(\lambda) = i + \tan \tfrac{1}{2}\pi(\lambda - 1). \tag{3.6}$$

The high-energy limit of $F_+(s, t)$ is then given by

$$F_+(s,t) \sim is \int_0^1 d\xi \, \rho_+(\xi, t) J_0(\xi\sqrt{-at}\, \ln s) + s\int_0^1 d\xi \, \rho_+(\xi, t) \Phi_+(\xi\sqrt{-at}\, \ln s, t), \tag{3.7}$$

with

$$\Phi_+(\xi\sqrt{-at}\, \ln s, t) = -\frac{1}{\pi}\int_{-1}^{+1} dx \, \frac{\sin(x\xi\sqrt{-at}\, \ln s)}{(1-x^2)^{1/2}} \tanh(\tfrac{1}{2}\pi x \xi \sqrt{-at}). \tag{3.8}$$

For small values of at, this function may be expanded:

$$\Phi_+ = -\tfrac{1}{2}\pi\xi\sqrt{-at}\, J_1(\xi\sqrt{-at}\, \ln s) + O(t^2). \tag{3.9}$$

Note that we have expanded in powers of at, and not of $at(\ln s)^2$.

For $t = -\tau/a(\ln s)^2$ and fixed values of $\tau \geq 0$, the first and imaginary term in Eq. (3.7) is dominant.[15] But for fixed $t < -\epsilon$, $\epsilon > 0$, this is generally not the case, since the Bessel functions J_0 and J_1 have analogous asymptotic expansions for $\ln s \to \infty$.

As a special example for $\rho_+(\xi, t)$, we mention the case

$$\rho_+(\xi, t) \propto -\frac{1}{at}\delta'(\xi - 1 + \epsilon), \tag{3.10}$$

for which

$$F_+(t,\lambda) \propto [(\lambda-1)^2 - at]^{-3/2}. \tag{3.11}$$

In the s channel, this gives the asymptotic expression

$$F_+(s,t)\sim is(\ln s)^2\frac{J_1(\sqrt{-at}\,\ln s)}{\sqrt{-at}\,\ln s}+\tfrac{1}{2}\pi s(\ln s)\frac{1}{\pi}\int_{-1}^{+1}dx\,\frac{x^2}{(1-x^2)^{1/2}}\cos(x\sqrt{-at}\,\ln s)\frac{\tanh(\frac{1}{2}\pi x\sqrt{-at})}{\frac{1}{2}\pi x\sqrt{-at}}+\tfrac{1}{2}\pi s\frac{1}{\pi}\int_{-1}^{+1}dx\,\frac{x^2}{(1-x^2)^{1/2}}\frac{\sin(x\sqrt{-at}\,\ln s)}{\sqrt{-at}\,\ln s}\frac{1}{\cosh^2(\frac{1}{2}\pi x\sqrt{-at})}. \tag{3.12}$$

For small values of t, the two real terms in this equation can be expanded with the result

$$\tfrac{1}{2}\pi s(\ln s)J_0(\sqrt{-at}\,\ln s)+O(t). \tag{3.13}$$

Again we see that the imaginary part dominates only for fixed values of $\tau=-at(\ln s)^2\geq 0$, but not for fixed t.

For negative signature, the factor $S_-(\lambda)$ in Eq. (3.3) has a pole at $\lambda=1$, and it is convenient to write it in the form

$$S_-(\lambda)=i-\frac{2}{\pi}\frac{1}{\lambda-1}+C(\lambda-1), \tag{3.14}$$

with

$$C(\lambda-1)=\frac{2}{\pi}\frac{1}{\lambda-1}-\cot\tfrac{1}{2}\pi(\lambda-1)=\tfrac{1}{6}\pi(\lambda-1)+O[(\lambda-1)^3].$$

We obtain then

$$F_-(s,t)\sim is\int_0^1 d\xi\,\rho_-(\xi,t)J_0(\xi\sqrt{-at}\,\ln s)+\frac{2}{\pi}s(\ln s)\int_0^1 d\xi\left(\int_0^\xi dx\,\frac{\rho_-(x,t)}{x}\right)J_0(\xi\sqrt{-at}\,\ln s)-\frac{2}{\pi}s(\ln s)\left(\int_0^1 d\xi\,\frac{\rho_-(\xi,t)}{\xi}\right)\int_0^1 dx\,J_0(x\sqrt{-at}\,\ln s)+s\int_0^1 d\xi\,\rho_-(\xi,t)\Phi_-(\xi\sqrt{-at}\,\ln s,t), \tag{3.15}$$

with

$$\Phi_-=\frac{1}{\pi}\int_{-1}^{+1}dx\,\frac{1}{(1-x^2)^{1/2}}\sin(x\xi\sqrt{-at}\,\ln s)\left(\frac{1}{\frac{1}{2}\pi x\xi\sqrt{-at}}-\coth(\tfrac{1}{2}\pi x\xi\sqrt{-at})\right). \tag{3.16}$$

For small values of t, we can expand the real part of Eq. (3.15). Using the constraint (3.2) we obtain

$$\int_0^1 d\xi\,\frac{\rho_-(\xi,t)}{\xi}=t\int_0^1 d\xi\,\xi^{-1}\left(\frac{\partial}{\partial\xi}\rho_-(\xi,t)\right)_{t=0}+O(t^2) \tag{3.17}$$

and

$$\Phi_-=\tfrac{1}{6}\pi\xi\sqrt{-at}\,J_1(\xi\sqrt{-at}\,\ln s)+O(t^2). \tag{3.18}$$

In contrast to the positive-signature case considered before, in Eq. (3.15) the real part of the amplitude dominates by a factor of $\ln s$ for fixed $t\leq 0$, and not only for fixed $\tau=-at(\ln s)^2$. The leading asymptotic expression is given by the second (and the third) term. This feature is a consequence of the pole at $\lambda=1$ in the signature factor (3.14).

Since we started with an *Ansatz* for the continued partial-wave amplitudes, the weight functions $\rho_\pm$ could be assumed to have some dependence upon t (or upon λ). Let us now, however, take only the characteristic singular part of Eq. (3.1) and restrict the functions $\rho_\pm$ to their dependence upon the parameter ξ. Then the expressions for $F_\pm(s,t)$ in Eqs. (3.7) and (3.15) can be related to the general impact-parameter representation, which, in turn, is related in the high-energy limit to the partial-wave expansion.[16] We write this representation in the form

$$F(s,t)\sim\tfrac{1}{2}s\int_0^{\sqrt{a}\,\ln s}db\,b\,\frac{\eta(s,b)e^{2i\delta(s,b)}-1}{2i\rho(s)}J_0(b\sqrt{-at})=s\int_0^1 d\xi\,\psi(\xi,s)J_0(\xi\sqrt{-at}\,\ln s), \tag{3.19}$$

where we have the identification

$$\mathrm{Re}\,\psi(\xi,s)=(\ln s)^2\xi\tfrac{1}{4}a\eta\sin 2\delta,\qquad \mathrm{Im}\,\psi(\xi,s)=(\ln s)^2\xi\tfrac{1}{4}a(1-\eta\cos 2\delta). \tag{3.20}$$

Here

$$\rho(s)=\frac{s-4m^2}{s}\approx 1,\quad b=\xi\sqrt{a}\,\ln s,$$

and δ as well as η are real functions of s and of $b=\xi\sqrt{a}\,\ln s$, with $0\leq\eta\leq 1$.

Comparing Eqs. (3.7) and (3.15) with the representation (3.19) for t-independent weight functions, we have the exact relation

$$\mathrm{Im}\,\psi_\pm(\xi,s)=\rho_\pm(\xi) \tag{3.21}$$

and the approximate correspondence

$$\begin{aligned} \mathrm{Re}\psi_-(\xi, s) &\sim \frac{2}{\pi}(\ln s)\int_0^\xi dx\, \frac{\rho_-(x)}{x}, \\ \mathrm{Re}\psi_+(\xi, s) &\sim -\frac{\pi}{2}(\ln s)^{-1}\left(\frac{d}{d\xi}[\xi\rho_+(\xi)\Theta(\xi-1)]\right), \end{aligned} \tag{3.22}$$

which gives only the leading terms for $\ln s \to \infty$.

The special example (3.11) for the amplitude $F_+(t,\lambda)$ has a t-dependent factor in the weight function $\rho_+(\xi, t)$, but we can make an identification with $\psi_+(\xi, s)$ on the basis of Eqs. (3.12) and (3.13), which gives

$$\mathrm{Im}\psi_+(\xi, s) \propto \xi(\ln s)^2$$

and the approximate connection

$$\mathrm{Re}\psi_+(\xi, s) \propto \tfrac{1}{2}\pi(\ln s)\delta(\xi-1).$$

By writing partial-wave amplitudes in (3.1) as superpositions of square-root terms, we have selected a form which gives rise to Bessel-function representations for the high-energy limits which have some similarity to familiar classical forms for the description of diffraction scattering.[17] In order to illustrate this feature, we conclude this section with some well-known examples. From our equations we see that for certain cases the leading terms of the amplitude $F(s, t)$ are completely determined for $s\to\infty$ and $\tau = -at(\ln s)^2$ fixed.[15,18,19] This happens if we demand that the inelastic cross section approaches the limit

$$\sigma_{\mathrm{inel}} \sim \pi a(\ln s)^2, \tag{3.23}$$

which is the maximal value of the *inelastic* cross section which is compatible with the bound $L \le \frac{1}{2}\sqrt{as}\,\ln s$ for relevant orbital angular momenta in the s-channel partial-wave expansion. Since

$$\sigma_{\mathrm{inel}} \sim 2\pi\int_0^{\sqrt{a}\ln s} db\, b[1-\eta^2(s,b)], \tag{3.24}$$

we see that Eq. (3.24) implies $\eta = 0$, and from Eqs. (3.20), we find then for the leading terms

$$\mathrm{Re}\psi \sim 0, \quad \mathrm{Im}\psi \sim \tfrac{1}{8}a\xi(\ln s)^2, \tag{3.25}$$

so that

$$F\left(s, t=\frac{-\tau}{a(\ln s)^2}\right) \sim i\tfrac{1}{8}as(\ln s)\frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}}, \tag{3.26}$$

which corresponds essentially to a fully absorbing disk with the radius $R(s) = \sqrt{a}\ln s$, with $\sigma_{\mathrm{tot}} \sim 2\sigma_{\mathrm{el}} \sim 2\sigma_{\mathrm{inel}}$.

Complete saturation of the Froissart bound for σ_{tot} implies

$$\sigma_{\mathrm{tot}} \sim 4\pi a(\ln s)^2. \tag{3.27}$$

It corresponds to $\eta\to 1$, $\delta \to \frac{1}{2}\pi$ for $0\le\xi\le 1$. We have the same expressions as in Eqs. (3.25) and (3.26) except for a factor 2, and $\sigma_{\mathrm{tot}} \sim \sigma_{\mathrm{el}}$.

We have used the form (3.4) for the signature factor because it gives a simple representation of the absorptive parts in terms of the weight functions $\rho_\pm$. Instead, we could write, for example,

$$S_+(\lambda) = ie^{-(i\pi/2)(\lambda-1)}(\sin\tfrac{1}{2}\pi\lambda)^{-1},$$

absorb the factor $\sin(\frac{1}{2}\pi\lambda)$ into $K(t,\lambda)$, and obtain a high-energy limit of the form

$$F_+(s, t) \sim is\int_0^1 d\xi\,\rho_+(\xi, t)J_0(\xi\sqrt{-at}(\ln s - \tfrac{1}{2}i\pi)).$$

Analogous expressions emerge for negative-signature amplitudes, where we may use

$$S_-(\lambda) = e^{-(i\pi/2)(\lambda-1)}\frac{-2}{\pi}\frac{1}{\lambda-1}+\cdots$$

as an expansion of $S_-(\lambda)$ near $\lambda = 1$.

IV. CHARACTER OF SINGULAR SURFACES

As we have seen in Secs. II and III, the required high-energy behavior and the unitarity constraints in the s channel imply certain restrictions for the character of the singular surfaces (1.2). Another important constraint is the continued unitarity condition in the t channel, together with the analytic properties of the partial-wave amplitudes $F(t,\lambda)$.[20] Of course, the unitarity condition applies *a priori* for real, positive values of $t \ge t_0$, where t_0 is a two-particle threshold. But even if we write an expansion of $F(t,\lambda)$ which exhibits the relevant terms in the neighborhood of $(t,\lambda) = (0,1)$, we should take care that the singularities of this function are either compatible with the unitarity constraints if continued to $t\ge t_0$, or at least that they can reasonably be expected to be made compatible by correction terms which are not important for the calculation of the high-energy limit of $F(s,t)$.

As an example, we assume that $F(t,\lambda)$ satisfies the elastic unitarity condition for $t_0 \le t < t_i$, and we consider the implications of this condition for a singular surface $\lambda = \alpha(t)$ which has fixed (i.e., t-independent) character.[21] Suppose that

$$\lim_{\lambda\to\alpha(t)} F(t,\lambda) = \infty. \tag{4.1}$$

Then we find that either[22]

(a) the function $\alpha(t)$ has a branch point at $t = t_0$ with a cut for real $t\ge t_0$ so that $\alpha^{\mathrm{II}}(t) \ne \alpha(t)$, where $\alpha^{\mathrm{II}}(t)$ is the continuation of $\alpha(t)$ through this branch cut, or

(b) the function $F(t,\lambda)$ has specific, square-root type branch-point surfaces $\lambda = \alpha_S(t)$ such that $\alpha(t_0) = \alpha_S(t_0)$.

In order to see how the requirements (a) and (b) come about, it is convenient to define the function

$$\varphi^{-1}(t,\lambda) = F^{-1}(t,\lambda) + i\rho(t), \tag{4.2}$$

with $\rho(t)=[(t-t_0)/t]^{1/2}$. Since the continuation of $F(t,\lambda)$ through the branch cut $t_0 \leq t < t_i$ into the second sheet is given by

$$[F^{\mathrm{II}}(t,\lambda)]^{-1}=F^{-1}(t,\lambda)+2i\rho(t), \tag{4.3}$$

we find that

$$\varphi^{\mathrm{II}}(t,\lambda)=\varphi(t,\lambda). \tag{4.4}$$

Hence the function $\psi(t,\lambda)$ does *not* have a branch point at $t=t_0$. But from Eqs. (4.1) and (4.2) it follows that

$$\varphi^{-1}(t,\lambda=\alpha(t))=i\rho(t). \tag{4.5}$$

We see that either $\alpha(t)$ has a branch point at $t=t_0$ in order to generate the corresponding branch point at the right-hand side of Eq. (4.5), or $\varphi(t,\lambda)$, and hence also $F(t,\lambda)$, has a shielding cut. This shielding cut is a singular surface $\lambda=\alpha_S(t)$ or $t=t_S(\lambda)$ with $t_S(\lambda=\alpha(t))=t_0+\cdots$, which generates the branch point in Eq. (4.5) in the manner[23]

$$[t_s(\lambda)-t]^{1/2}\to(t_0-t)^{1/2}$$

for $\lambda\to\alpha(t)$. The limit $\lambda\to\alpha$ and the continuation $F\to F^{\mathrm{II}}$ are then no longer interchangeable.

In order to give an example, we suppose that $F(t,\lambda)$ has a logarithmic branch point at $\lambda=\alpha_L(t)$ so that the expansion of F near $\lambda=\alpha_L$ is dominated by the term

$$F(t,\lambda)\sim\ln[\lambda-\alpha_L(t)]. \tag{4.6}$$

From Eq. (4.3), we find that

$$F^{\mathrm{II}}(t,\lambda=\alpha_L(t))=1/2i\rho(t), \tag{4.7}$$

which violates the continuity theorem[24] if $\alpha_L(t)=\alpha_L^{\mathrm{II}}(t)$ and if there are no shielding cuts. In order to see the difficulty, we note that in our example the limit $\lambda\to\alpha_L(t)$ and the continuation $F\leftrightarrow F^{\mathrm{II}}$ can be interchanged. So we find from Eq. (4.7)

$$F(t,\lambda=\alpha_L(t))=-1/2i\rho(t),$$

which is in contradiction with Eq. (4.6).

As an indication of the way a shielding cut works, we consider an oversimplified case. We give an explicit function which has a logarithmic singularity of the type (4.1) in sheet I, but which is bounded at the same point in sheet II. We write

$$g(t,\lambda)=\{[t_S(\lambda)-t]^{1/2}+(t_0-t)^{1/2}\}\ln[\lambda-\alpha_L(t)],$$

with

$$t_s(\lambda)=\lambda+t_0-\alpha_L(t), \quad\text{or}\quad \alpha_S(t)=t-t_0+\alpha_L(t),$$

so that

$$t_s(\lambda=\alpha_L(t))=t_0.$$

Then

$$g(t,\lambda\to\alpha_L(t))\sim2(t_0-t)^{1/2}\ln[\lambda-\alpha_L(t)],$$

but

$$g^{\mathrm{II}}(t,\lambda\to\alpha_L(t))\to0$$

since

$$g^{\mathrm{II}}(t,\lambda)=\{[t_s(\lambda)-t]^{1/2}-(t_0-t)^{1/2}\}\ln[\lambda-\alpha_L(t)].$$

Hence $g(t,\lambda)$ is an example of a function which is compatible with the continuity theorem. More complicated forms are required in order to comply explicitly with the unitarity condition (4.3).

In Sec. III we have written the partial-wave amplitudes with complex trajectories $\alpha_{1,2}(t)=1\pm\text{const}\sqrt{t}$ as superpositions of terms like

$$\{[\lambda-\alpha_1(t)][\lambda-\alpha_2(t)]\}^{-1/2}. \tag{4.8}$$

There are several ways in which these expressions can be made compatible with t-channel unitarity. Most simply, we could choose weight functions $\rho_\pm$ such that $F(t,\lambda)$ remains finite for $\lambda\to\alpha_{1,2}(t)$. Then there is no problem. In the following section we give some explicit models of this type.

A more interesting possibility is the following[25]: As it stands, the form (4.8) may be considered as a superposition of poles and branch points which only coincide near $t=0$ if higher-order terms are neglected. In the place of more elaborate constructions, we give again a very simple example. Consider the function

$$\frac{\beta(t,\lambda)+\{[\lambda-\alpha_{c1}(t)][\lambda-\alpha_{c2}(t)]\}^{1/2}}{[\lambda-\alpha_1(t)][\lambda-\alpha_2(t)]}, \tag{4.9}$$

with

$$\begin{aligned}\alpha_{1,2}(t)&=1\pm(at+b^2t^2)^{1/2}+\cdots,\\ \alpha_{c1,2}(t)&=1\pm(at)^{1/2}+\cdots,\end{aligned} \tag{4.10}$$

and $\beta(t,\lambda)=bt+\cdots$. This expression coincides with Eq. (4.8) for $|bt|\ll1$. We know that Regge-pole trajectories generally have a branch point at the two-particle threshold $t=t_0$, and we assume that $\alpha_{1,2}(t)$ has one, but that it may be approximated near $t=0$ by the expression in Eq. (4.10). Then there is no difficulty with t-channel unitarity. Branch-point trajectories like $\alpha_c(t)$ in Eq. (4.9) generally are associated with inelastic thresholds in the t channel and remain regular at $t=t_0$. We see that our *Ansatz* (3.1) for $F(t,\lambda)$ can be considered as the limiting case of a pole-cut relationship where poles and branch-point trajectories coincide near $t=0$, and where the Regge poles acquire the necessary thresholds in the t channel.

The pole-cut relations described above have an important additional feature.[25] With an appropriate choice of functions, like in our example, one branch of the pole surface $\alpha(t)$ can be removed from the physical sheet of the λ plane for $t>0$. It remains in a secondary sheet with respect to the branch points $\alpha_{c1,2}(t)$ and has no direct physical consequence if $t>0$.

Another possibility for making the *Ansatz* (3.1) compatible with t-channel unitarity would be the introduction of shielding cuts. One may also try to introduce singular surface with a t-dependent character. Here we do not consider these possibilities.

V. AN EXPLICIT EXAMPLE

In this section we want to explore an explicit example for the case of an amplitude with constant asymptotic cross sections. We denote these asymptotic cross sections by σ and $\bar{\sigma}$, and we know that the leading terms for the forward amplitudes are given by[3,7,26]

$$F_-(s,0)\sim -\frac{2}{\pi}s(\ln s)\frac{\sigma-\bar{\sigma}}{16\pi}+is\frac{\sigma-\bar{\sigma}}{16\pi}, \tag{5.1}$$

$$F_+(s,0)\sim is\frac{\sigma-\bar{\sigma}}{16\pi}, \tag{5.2}$$

with $F_\pm=F\pm\bar{F}$. For the continued partial-wave amplitudes we consider expressions of the form (3.1). In particular, we make for $\rho_-(\xi)$ the special *Ansatz*

$$\rho_-(\xi)=\frac{\bar{\sigma}-\sigma}{8\pi}\times\begin{cases}\dfrac{\xi}{1-\gamma} & \text{for } 0\leq\xi<1-\gamma\\[2ex] -\dfrac{\xi}{\gamma} & \text{for } 1-\gamma<\xi\leq 1.\end{cases} \tag{5.3}$$

We do not need to specify $\rho_+(\xi)$ for the moment. From the *Ansatz* (5.3) we obtain the important function

$$\int_0^\xi dx\,\frac{\rho_-(x)}{x}=\frac{\bar{\sigma}-\sigma}{8\pi}\times\begin{cases}\dfrac{\xi}{1-\gamma} & \text{for } 0\leq\xi\leq 1-\gamma\\[2ex] \dfrac{1-\xi}{\gamma} & \text{for } 1-\gamma\leq\xi\leq 1,\end{cases} \tag{5.4}$$

which determines the dispersive part of the amplitude. It is seen to satisfy the condition

$$\int_0^1 d\xi\,\frac{\rho_-(\xi)}{\xi}=0, \tag{5.5}$$

which has been discussed in Sec. III. *A priori*, the parameter γ can be varied between $\gamma=0$ and $\gamma=1$, but in the boundary cases the weight functions (5.3) and (5.4) become generalized functions.

We have shown in Sec. III that the high-energy limit of $F_-(s,t)$ is dominated by the real $s\ln s$ terms in Eq. (3.15), not only for $t=0$ but also for fixed values of $t<0$. Hence we can calculate the asymptotic elastic cross section in terms of the function (5.4). We write

$$\sigma_{el}=\tfrac{1}{4}\sigma^{(-)}, \tag{5.6}$$

where

$$\sigma^{(-)}(s)=\frac{16\pi}{s^2}\int_{-s}^0 dt\,|F_-(s,t)|^2 \sim\frac{32\pi}{a}\frac{4}{\pi^2}\int_0^1 d\xi\,\xi^{-1}\left(\int_0^\xi dx\,\frac{\rho_-(x)}{x}\right)^2. \tag{5.7}$$

The cross section $\sigma^{(-)}$ by itself is, up to factors, the charge-exchange (e.g., $\pi^-p\to\pi^0n$) or regeneration ($K_S^0p\to K_L^0p$) cross section. It follows from Eqs. (2.6) and (5.1) that this cross section has the lower bound[27]

$$\sigma^{(-)}(s)\geq\frac{(\sigma-\bar{\sigma})^2}{\pi^3a}. \tag{5.8}$$

In addition, we have the upper bound

$$\sigma^{(-)}(s)\leq 4\max(\sigma,\bar{\sigma}). \tag{5.9}$$

Hence the limit of the ratio[18]

$$R=\frac{\pi^3a\sigma^{(-)}}{(\sigma-\bar{\sigma})^2} \tag{5.10}$$

for $s\to\infty$ varies between

$$1\leq R\leq\frac{4\pi^3a\max(\sigma,\bar{\sigma})}{(\sigma-\bar{\sigma})^2}. \tag{5.11}$$

We can express R in terms of the weight function $\rho_-(\xi)$:

$$R=\frac{1}{2}\int_0^1 d\xi\,\xi^{-1}\frac{\left(\int_0^\xi dx\,\rho_-(x)/x\right)^2}{\left(\int_0^1 d\xi\int_0^\xi dx\,\rho_-(x)/x\right)^2}, \tag{5.12}$$

and for our special case (5.4) this becomes

$$R(\gamma)=\frac{2}{\gamma^2}\left(\ln\frac{1}{1-\gamma}-\gamma\right) \tag{5.13}$$

with $R(0)=1$ and $R(1)=\infty$. We see that $\gamma=0$ corresponds to a maximal violation of the Pomeranchuk theorem, and that there is a finite value $\gamma_{max}>1$ so that $R(\gamma_{max})$ equals the upper bound in (5.11).

The simplicity of our *Ansatz* (5.3) makes it possible to give explicit expressions for $F_-(t,\lambda)$ and for $F_-(s,t)$. For the partial-wave amplitude we find

$$F_-(t,\lambda)\propto\frac{\bar{\sigma}-\sigma}{8\pi}\frac{-1}{at}\left(\frac{1}{\gamma(1-\gamma)}[(\lambda-1)^2-at(1-\gamma)^2]^{1/2}-\frac{1}{\gamma}[(\lambda-1)^2-at]^{1/2}-\frac{\lambda-1}{1-\gamma}\right). \tag{5.14}$$

Except for the extreme boundary case $\gamma=0$, we have only square-root branch points in the complex λ plane. Hence there is no formal difficulty with the continued unitarity condition, even if we wanted to continue the expression (5.14) as it is to $t \geq t_0$. By construction, F_- is well behaved at $t=0$, where it has a simple pole at $\lambda=1$, and for $\lambda=1$, it is regular at $t=0$ because of (5.5).

The high-energy limit of the amplitude in the s channel is obtained from Eq. (3.15). It is given by

$$F_-(s,t) \sim -\frac{2}{\pi} s(\ln s)\frac{\sigma-\bar{\sigma}}{16\pi}\left(\frac{2}{\gamma\sqrt{-at}\,\ln s}[J_1((1-\gamma)\sqrt{-at}\,\ln s)-J_1(\sqrt{-at}\,\ln s)] + \frac{2}{\gamma}\int_{1-\gamma}^{1} dx\, J_0(x\sqrt{-at}\,\ln s)\right)$$
$$+\, is\frac{\sigma-\bar{\sigma}}{16\pi}\frac{2}{\gamma\sqrt{-at}\,\ln s}[J_1(\sqrt{-at}\,\ln s)-J_1((1-\gamma)\sqrt{-at}\,\ln s)]. \quad (5.15)$$

The physical properties of the differential cross section can be obtained from this expression. But it is more transparent to look at the mathematical limits $\gamma \to 0$ and $\gamma \to 1$. In particular, for $\gamma \to 0$ we have

$$F_-(t,\lambda) \propto \frac{\sigma-\bar{\sigma}}{8\pi}\left(\frac{-1}{at}\{[(\lambda-1)^2-at]^{1/2}-(\lambda-1)\}-[(\lambda-1)^2-at]^{-1/2}\right) \quad (5.16)$$

and

$$F_-(s,t) \sim -\frac{2}{\pi}s(\ln s)\frac{\sigma-\bar{\sigma}}{16\pi}\frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}} + is\frac{\sigma-\bar{\sigma}}{16\pi}2\left(\frac{J_1(\sqrt{\tau})}{\sqrt{\tau}}-J_0(\sqrt{\tau})\right), \quad (5.17)$$

where we have used $\tau=-at(\ln s)^2$. The characteristic form of the leading real term in Eq. (5.17) is actually a general consequence of the fact that $R(0)=1$ corresponding to a maximal violation of the Pomeranchuk theorem.[18] This can be easily seen from Eqs. (5.1) and (5.12), which, for $R=1$, imply

$$\int_0^{\xi} dx\, \frac{\rho_-(x)}{x} = \frac{\bar{\sigma}-\sigma}{8\pi}\xi\theta(1-\epsilon-\xi), \quad (5.18)$$

and hence

$$\rho_-(\xi) = \frac{\bar{\sigma}-\sigma}{8\pi}\xi[\theta(1-\xi)-\delta(1-\xi)]. \quad (5.19)$$

So we see that this result is independent of our special *Ansatz* (5.3).

We know that the limit $\gamma \to 1$ is unphysical in the sense that $\gamma \leq \gamma_{max} < 1$ with $R(\gamma_{max}) < \infty$ as long as $|\sigma-\bar{\sigma}|$ is nonzero. Nevertheless the limit is of interest as a boundary case of the model. For $\gamma \to 1-\epsilon$, we find

$$F_-(t,\lambda) \propto \frac{\bar{\sigma}-\sigma}{8\pi}\frac{1}{at}\left([(\lambda-1)^2-at]^{1/2}-(\lambda-1)-\left[\frac{1}{\epsilon^2}(\lambda-1)^2-at\right]^{1/2}+\frac{1}{\epsilon}(\lambda-1)\right). \quad (5.20)$$

In this relation the limits $\lambda \to 1$ and $\epsilon \to 0$ are not interchangeable. For the high-energy limit, we obtain

$$F_-(s,t) \sim -\frac{2}{\pi}s(\ln s)\frac{\sigma-\bar{\sigma}}{16\pi}2\left(\int_0^1 dx\, J_0(x\sqrt{\tau})-\frac{J_1(\sqrt{\tau})}{\sqrt{\tau}}\right) + is\frac{\sigma-\bar{\sigma}}{16\pi}\frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}}, \quad (5.21)$$

where we have taken the limit $\epsilon \to 0$.

The most interesting part of this limiting case is the feature that the dominating real part has no zeros on the real axis as a function of $\sqrt{\tau}=\sqrt{-at}\,\ln s$.[28] This is in contrast to the opposite boundary case $\gamma=0$, where we have rapid oscillations as seen in Eq. (5.17). For $0 \leq \gamma < 1$, our model provides an example for the transition between the two extreme cases.

By using the general formulas of Sec. III, we can write down further examples which illuminate specific aspects of the problem and which comply with s- and t-channel constraints.

Note added in proof. Recent Serpukhov experiments (up to about 60 GeV/c) show a decrease of $\sigma_{tot}(K^-p)-\sigma_{tot}(K^+p)$ corresponding to a power law. It remains to be seen whether this is the actual asymptotic behavior. On the other hand, these same experiments indicate that the sum of the cross sections increases with increasing energy. If this increase is asymptotic and faster than $\sim \ln s$, it follows from our discussions in Sec. II that complex trajectories are required for the positive-signature amplitude. Explicit examples for such amplitudes can be constructed in direct analogy to the considerations of this section. For further details, see the paper quoted in Ref. 13. Limiting cases of amplitudes with rising cross section have been considered in Sec. III.

VI. CONCLUSIONS

We have seen that leading Regge trajectories $\alpha(t)$ with square-root branch points at $t=0$ are generally expected to be present in amplitudes which require faster-than-logarithmic shrinkage of the diffraction peak.[29] The more detailed conditions have been summarized in Sec. III. Note that we are talking here about shrinkage which is required by unitarity because of the growth of the amplitude. It is not an extra feature as in the case of ordinary Regge poles. In our arguments, we use s- and t-channel properties of the amplitudes, and we assume that the relevant singularities in the complex angular momentum plane are nonessential. But isolated essential singularities and natural boundaries are also considerably restricted by the analytic properties of $F(t,\lambda)$ as a function of two complex variables. In certain cases they can be excluded, but in this paper we have not discussed these problems.[30]

We note that trajectories of the form $\alpha(t)=\alpha(0)+\text{const}\sqrt{t}+O(t)$ can, of course, also be present in cases where they are not required by general principles.[31]

Continued partial-wave amplitudes with complex singular surfaces of the type (1.1) are most naturally related to representations of scattering amplitudes in terms of superpositions of Bessel functions of the argument $\xi\sqrt{-at}\,\ln s$, $0\leq\xi\leq 1$. In this sense, the picture of high-energy scattering involving these complex trajectories is actually rather intuitive from the point of view of the s channel. It is possible to relate it to quasiclassical pictures.

Using a rather general *Ansatz* for the partial-wave amplitude near $(t,\lambda)=(0,1)$, we have evaluated the high-energy limits of the amplitude in terms of Bessel-function representations. These representations are very useful for the construction of specific models.

The character of the singular surfaces (1.1) of $F_\pm(t,\lambda)$ is dependent upon the specific asymptotic properties of the amplitude $F(s,t)$, in particular for $t=0$. Although this high-energy limit is physical for $t\leq 0$, the general notions of dispersion theory require also that we take into account the t-channel properties of the amplitude, in particular, the unitarity constraints. To do this requires some analytic continuation, and it is most simply done by using the continued partial-wave amplitudes. We have shown that, unless we want to introduce very special shielding cuts, or singular surfaces with t-dependent character, the actual character of the trajectories (1.1) should be either such that they are by themselves compatible with t-channel unitarity, or that they represent the degenerate limit of a pole-cut relationship. In a pole-cut relationship of this type, the Regge poles and branch points are of the form (1.1) near $t=0$, but they are different near the threshold $t=t_0$, where the poles develop a branch point and the branch-point trajectories are weak and do not disturb unitarity.

Incorporating the constraints mentioned above, we have constructed a one-parameter family of explicit examples for amplitudes with complex trajectories (1.2). We have chosen the particular case of constant asymptotic total cross sections σ and $\bar\sigma$, with $\sigma\neq\bar\sigma$, but the same approach can be used in order to construct models for amplitudes with rising cross sections. In the complex angular momentum plane, the model amplitudes have square-root branch points which are compatible with t-channel unitarity requirements. In the s channel, they consist of a few Bessel functions. Some important limiting cases for amplitudes with increasing cross sections have been presented in Sec. III.

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[29]The singular surfaces considered in this paper are actual singularities of the partial-wave amplitudes $F_\pm(t,\lambda)$, and they are *not* trajectories which appear only in intermediate steps of iteration schemes or graph summations.

[30]An example of a function $f(s,t)$, which gives shrinkage as in Eq. (2.20), is

$$f(s,t)\propto \exp[ct(\ln s)^\beta]=s^{ct(\ln s)^{\beta-1}}$$

with $\beta\geq 1+\epsilon$, $\epsilon>0$. This function is excluded because it corresponds to an amplitude which is not bounded by a polynomial for $\mathrm{Re}\,t>0$. It would violate the basic assumptions of relativistic field theory, from which one can derive polynomial boundedness in one variable.

[31]Complex Regge trajectories have also been considered for situations with $\alpha(0)<1$. See, for example, G. F. Chew and D. Snider, Phys. Letters 31B, 75 (1970); F. Zachariasen, in *Proceedings of the Coral Gables Conference on Fundamental Interactions at High Energies II*, edited by A. Perlmutter *et al.* (Gordon and Breach, New York, 1970), pp. 103–122. This paper contains further references: F. Zachariasen, CERN Report No. CERN-TH-1290 (unpublished).