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# Application of a Separable-Potential Model to $p$-$p$ Final-State Interactions in $\gamma + d \to \pi^- + p + p$

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We use wave functions calculated by using a new separable potential to which has been added a Coulomb potential to give an impulse-approximation treatment of $p$-$p$ final-state interactions, which we compare with $n$-$n$ and $n$-$p$ final-state interactions. Our treatment involves the choice of a potential which gives exact analytic solutions to the Schrödinger equation, so that no approximations are made which might destroy the completeness and orthogonality of the wave functions used. We carry out computations for $\gamma + d \to \pi^- + p + p$ with our model, and conclude that no Coulomb effects should be noticeable when a bremsstrahlung beam is used as the $\gamma$ source.

## INTRODUCTION

Separable potentials have been used by many authors[1] to give a description of low-energy nucleon-nucleon scattering, and Harrington[1] has pointed out that the addition of a Coulomb potential (which is not separable) to a separable potential in configuration space gives rise to an equation which is separable. Harrington has used the usual Yamaguchi[1] potential as an example, and has shown how the integrals which then occur can be evaluated by a perturbation expansion. We point out that by choosing a different potential, which depends on confluent hypergeometric functions, we can carry out all integrations explicitly, and give answers in closed form. Moreover, the scattering amplitude calculated by this method turns out to be the lowest-order term in Harrington's perturbation expansion. A disadvantage is that our potential contains an explicit charge dependence.

In Sec. II of this paper, we apply the wave functions obtained from our potential to a calculation of the effect of the $p$-$p$ final-state interactions in $\gamma + d \to \pi^- + p + p$, and compare this with $\gamma + d \to \pi^+ + n + n$.

## I. SOLUTION OF THE SCHRÖDINGER EQUATION

If the Schrödinger equation for the $p$-$p$ system is

$$\left(-\frac{\Delta}{M} + \frac{a}{Mr}\right)\psi(\vec{r}) + \int d^3r' V(\vec{r}\,')V(\vec{r})\psi(\vec{r}\,') = \frac{k^2}{M}\psi(\vec{r}), \tag{1.1}$$

where $a = Me^2$ in natural units of $c = \hbar = 1$, it can be solved explicitly in a manner similar to that used to solve the Schrödinger equation for a pure separable potential. We shall write down here such results as are essential to the main purpose of this paper. Fuller derivations may be found in Harrington's paper.

Equation (1.1) can be written in spherical coordinates, and the wave functions of definite orbital angular momentum, $l$, calculated. For nonzero $l$ the solution is given by the regular Coulomb wave function

$$\psi_l(r)=\frac{[N_l(\kappa)]^{1/2}(-)^l M_{\kappa,l+1/2}(-2ikr)}{-2ikr}$$
$$=i^l F_l(kr)/kr, \tag{1.2}$$

where

$$N_l(\kappa)=\frac{\Gamma(l+1+\kappa)\Gamma(l+1-\kappa)e^{+i\pi\kappa}}{[(2l+1)!]^2}, \tag{1.3}$$

$$\kappa=-ia/2k, \tag{1.4}$$

and $F_l(kr)$ is the regular Coulomb wave function as defined by Hull and Breit.[2] (The function $M_{\kappa,l+1/2}$ is a Whittaker function defined in Ref. 2.) For $l=0$ we omit the angular momentum index, and set

$$\psi(r)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty p^2dp\,\frac{F_0(pr)}{pr}\phi(p), \tag{1.5}$$

so that

$$\phi(p)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty r^2dr\,\frac{F_0(pr)}{pr}\psi(r). \tag{1.6}$$

In this case, it is found that the scattering solution is given by

$$\phi(p)=\frac{\delta(p-k)}{4\pi k^2}+A\,\frac{4\pi M v(p)}{p^2-k^2-i\epsilon}. \tag{1.7}$$

Here

$$v(p)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty r^2dr\,\frac{F_0(pr)}{pr}V(r) \tag{1.8}$$

and

$$A=\frac{v(k)}{4\pi[1-4\pi M\int p'^2dp'v(p')^2/(p'^2-k^2-i\epsilon)]}. \tag{1.9}$$

In the case that the potential is the usual Yamaguchi potential, given by

$$V_Y(r)=g(\tfrac{1}{2}\pi)^{1/2}e^{-\beta r}/r, \tag{1.10}$$

one may calculate that

$$v_Y(p)=g\sqrt{N(\kappa)}\,(\beta+ip)^{-1+ia/2p}(\beta-ip)^{-1-ia/2p} \tag{1.11}$$

and the integrations may be performed analytically in the perturbation expansion used by Harrington. It is important to note that the zero-order term in this expansion is equivalent to setting equal to one the factor $[(\beta+ip)/(\beta-ip)]^{ia/2p}$, which varies by about 5% over the entire range of integration. We may therefore look upon this zero-order approximation as the exact solution for the potential.

$$v(p)=g\sqrt{N(\kappa)}\,(\beta^2+p^2)^{-1}, \tag{1.12}$$

whose configuration-space form is

$$V(r)=g\left(\frac{\pi}{2}\right)^{1/2}\Gamma\left(1+\frac{a}{2\beta}\right)\frac{W_{-a/2\beta,1/2}(2\beta r)}{r}$$
$$\sim g\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-\beta r}}{r}\Gamma\left(1+\frac{a}{2\beta}\right)(2\beta r)^{-a/2\beta}$$
$$\times\,{}_2F_0\left(1+\frac{a}{2\beta},\frac{a}{2\beta},-\frac{1}{2\beta r}\right). \tag{1.13}$$

In the limit $a\to 0$, $V(r)$ of course reduces to $V_Y(r)$. Because of the simplicity of the form of $v(p)$, we shall use this potential for all the calculations in this paper. This gives rise to a great simplification, in that all our calculations can be carried out exactly.

By the use of Harrington's methods, the integration in (1.9) can be done, and $A$ evaluated as

$$A=\frac{v(k)}{4\pi\{1-[(\alpha+\beta)/(\beta-ik)]^2[1+C(k)]\}}, \tag{1.14}$$

where $\alpha$ is given by

$$\pi^2Mg^2=\beta(\alpha+\beta)^2, \tag{1.15}$$

and

$$C(k)=\frac{1}{(\beta+ik)^2}\left\{2k\beta\frac{a}{k}\left[\psi\left(1+\frac{ia}{2k}\right)-\ln\left(\frac{ia}{2k}\right)\right]+(\beta^2+k^2)\left[\frac{a}{\beta}-\frac{a^2}{\beta^2}\psi'\left(1+\frac{a}{2\beta}\right)\right]-2a\beta\left[\psi\left(1+\frac{a}{2\beta}\right)-\ln\left(\frac{a}{2\beta}\right)\right]\right\}. \tag{1.16}$$

The configuration-space wave function has not been calculated before, and can be done as follows. We must evaluate the right-hand side of Eq. (1.6). The first part, involving the $\delta$ function, is trivial, and the second part involves the integral

$$\int_0^\infty p^2dp\,\frac{[N(\tau)]^{1/2}}{(p^2-k^2-i\epsilon)(\beta^2+p^2)}\,\frac{[N(\tau)]^{1/2}M_{\tau,1/2}(-2irp)}{-2irp}, \tag{1.17}$$

where

$$\tau=-ia/2p. \tag{1.18}$$

Using the explicit expression (1.3) for $N(\tau)$, and the formula for Whittaker functions

$$-\Gamma(1+\tau)\Gamma(1-\tau)M_{\tau,1/2}(-2irp)e^{i\pi\tau}=\Gamma(1-\tau)W_{\tau,1/2}(-2irp)-\Gamma(1+\tau)W_{-\tau,1/2}(2irp), \tag{1.19}$$

we may rewrite the integral (1.17) as

$$\int_{-\infty}^\infty p^2dp\,\frac{\Gamma(1-\tau)W_{\tau,1/2}(-2ipr)}{2irp(p^2-k^2-i\epsilon)(\beta^2+p^2)}. \tag{1.20}$$

Since for $r>0$, the $W$ function decays exponentially in the upper-half $p$ plane, we may close the contour there and evaluate the integral from the residues of the two poles inside this contour. Doing this, we obtain

$$\left(\frac{\pi}{2}\right)^{1/2}\psi(r)=\frac{F_0(kr)}{4\pi kr}+\frac{2\pi^2 MgA}{r}\,\frac{1}{\beta^2+k^2}\times[\Gamma(1-\kappa)W_{\kappa,1/2}(-2irk)-\Gamma(1-\kappa')W_{\kappa',1/2}(2r\beta)],\quad(1.21)$$

where $\kappa=-ia/2k$ and $\kappa'=-a/2\beta$.

Defining

$$N(\kappa)=\frac{a\pi/k}{e^{a\pi/k}-1}\equiv C_0(a/k),\quad(1.22)$$

by comparison with p.65 of Ref. 3, or from Harrington's paper, it is found that the phase shift is given by

$$e^{2i\delta}-1=\frac{4ik\beta(\alpha+\beta)^2C_0(a/k)}{(\beta+ik)^2\{(\beta-ik)^2-(\alpha+\beta)^2[1+C(k)]\}}.\quad(1.23)$$

Since we have shown that the approximation we are using comes from a Schrödinger equation with a real potential, δ as given by (1.23) must be real, as can be checked by using the standard properties of the $\psi$ function as given in Ref. 4.

From (2.21) an effective-range expansion can be derived, in which the scattering length and effective range are given, respectively, by

$$a_0=\frac{-2(\alpha+\beta)^2}{\beta[\beta^2-(\alpha+\beta)^2(1-2X+Y)]},\qquad r_0=\frac{2\beta^2+(\alpha+\beta)^2(1-Y)}{\beta(\alpha+\beta)^2}.\quad(1.24)$$

Here $X$ and $Y$ are given by

$$X=\frac{a}{\beta}\left[\psi\left(1+\frac{a}{2\beta}\right)-\ln\left(\frac{a}{2\beta}\right)\right]$$

and

$$Y=\frac{a}{\beta}-\frac{a^2}{2\beta^2}\psi'\left(1+\frac{a}{2\beta}\right).$$

## II. APPLICATION TO $p$-$p$ FINAL-STATE INTERACTIONS

Since we have deduced wave functions for the $p$-$p$ system, we now apply these to a study of $p$-$p$ final-state interactions and compare these with $n$-$n$ final-state interactions. The problem has been treated before by Phillips[5] in considerable detail, and the numerical calculations based on our formulas agree with his. We carry out these calculations for two reasons.

(i) The wave functions we have calculated give rise to analytic formulas for the energy spectrum in the final state.

(ii) Since our wave functions are exact solutions of a Schrödinger equation, they have two important properties: They are a complete orthogonal set of functions, and they give rise to real $p$-$p$ phase shifts. Phillips has taken care of the reality of phase shifts, in his calculations, but not the completeness of wave functions. The result of using an incomplete or nonorthogonal set of wave functions which give correct phase shifts is to give correctly the general shape of the energy spectrum of the final state, but the integrated spectrum can be in error. We show in Eqs. (2.15) the consequence of completeness, which gives an approximate formula for the integrated spectrum.

At the end of this section we give the results of some computations for the final-state interaction in $\gamma+d\to\pi^-+p+p$, and compare this with $\gamma+d\to\pi^+ +n+n$.

### A. Derivation of Formulas

In a reaction of the form

$$A+d\to B+N_1+N_2,\quad(2.1)$$

where $d$, $N_1$, $N_2$ are, respectively, a deuteron, a nucleon, and another nucleon, and in which it is believed that the fundamental reaction occurring is

$$A+N_1\to B+N_2,\quad(2.2)$$

one uses the impulse approximation. The formulation that is used in this article will be found in Ref. 3. We do all our calculations in the rest frame of the deuteron. We define our kinematics in Fig. 1. Let

$$\vec{\Delta}=\text{three-momentum transfer}=\tfrac{1}{2}(\vec{P}_A-\vec{P}_B),\quad(2.3)$$

so that

$$\vec{P}_{N_1}+\vec{P}_{N_2}=2\vec{\Delta},\quad(2.4)$$

and let

$$\vec{P}_{N_1}-\vec{P}_{N_2}=2\vec{p}.\quad(2.5)$$

Then the differential cross section to a state in which the total spin of the $N_1$-$N_2$ system is $s'$ will

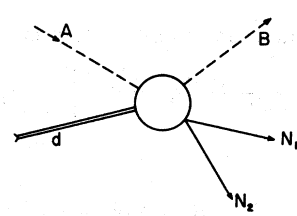


FIG. 1. Kinematics for the reaction $A+d\to B+N_1+N_2$.

be

$$\frac{d\sigma^{s'}}{d\Omega_p d^3\vec{p}_B} = |T|^2 |S_{s'}(\vec{p}, \vec{\Delta})|^2 \times (\text{kinematical factors}), \tag{2.6}$$

in which

$$S_{s'}(\vec{p}, \vec{\Delta}) = \int d^3r\, \psi^*_{s'}(\vec{p}, \vec{r}) \psi_d(r) e^{i\vec{\Delta}\cdot\vec{r}}, \tag{2.7}$$

$T$ is the $T$ matrix for $A + N_1 \to B + N_2$, and the kinematic factors are slowly varying functions of $\vec{p}$.

In order to avoid vagueness, we shall concentrate our attention on the reaction

$$\gamma + d \to \pi^- + p + p. \tag{2.8}$$

Schilling[6] has given a treatment of this reaction, in which the problems of the Coulomb interaction were neglected. (Our treatment will show that the neglect was quite justifiable.) In this specific case, Eq. (2.6) may be replaced by

$$\frac{d\sigma^{s'}}{d\Omega_p d^3Q} = \frac{1}{2s+1} \frac{(2\pi)^2}{M^2 E} \frac{4p}{Q_0} \times \sum_{M,M'} |\langle s'M'|T|sM\rangle|^2 |S_{s'}(\vec{p}, \vec{\Delta})|^2, \tag{2.9}$$

in which $s'$, $M'$ are spin parameters of the final two-nucleon system, and $s$, $M$ are spin parameters of the initial deuteron.

One normally sums over a certain range of values $Q_0$, and over the solid angle $d\Omega_p$, since in general the final protons are unobserved. Thus we finally need

$$\frac{d\sigma^{s'}}{d\Omega_Q} = \int_{\min}^{\max} QQ_0 dQ_0 \int d\Omega_p \frac{d\sigma^{s'}}{d\Omega_p d^3Q}. \tag{2.10}$$

The $T$-matrix element is assumed largely constant over the ranges of these integrations, so that we are led (after change of variables) to the equations

$$\frac{d\sigma^{s'}}{dQ_0 d\Omega_Q} = \frac{2\pi^2}{E} \frac{pQ}{M} \bar{S}_{s'}(p, \Delta) |T^{s'}|^2, \tag{2.11}$$

$$\frac{d\sigma^{s'}}{d\Omega_Q} = \frac{2\pi^2}{M} I_{s'}(p, \Delta) |T^{s'}|^2, \tag{2.12}$$

where

$$\bar{S}_{s'}(p, \Delta) = \int d\Omega_p |S_{s'}(\vec{p}, \vec{\Delta})|^2 \tag{2.13}$$

and

$$I_{s'}(p, \Delta) = \int_0^W p^2 dp \frac{Q}{E} \bar{S}_{s'}(p, \Delta), \tag{2.14}$$

where $W$ is that value of $p$ corresponding to the minimum value of $Q_0$ detected.

Most authors make the approximation that $W$ is essentially infinite and $Q/E \approx 1$, so that one may use the completeness formula

$$I_s(p, \Delta) = 1 + (-)^s S(\tfrac{1}{2}\Delta), \tag{2.15}$$

where

$$S(p) = \int d^3r\, e^{i\vec{p}\cdot\vec{r}} |\psi_d(r)|^2, \tag{2.16}$$

where $\psi_d$ is the deuteron wave function.

We can calculate analytically the quantities $\bar{S}_{s'}(p, \Delta)$ where the final state consists of two protons or two neutrons. We use the wave functions (2.5) for the $s$ wave, and pure Coulomb wave functions for higher angular momentum, and the deuteron wave function

$$\psi_d(r) = \sqrt{2\pi} \frac{[\mu\beta(\mu+\beta)]^{1/2}}{\beta - \mu} \frac{e^{-\mu r} - e^{-\beta r}}{r}, \tag{2.17}$$

which can be derived as a separable-potential wave function.

For simplicity, we have chosen an average value of $\beta$ for all two-nucleon systems of $\beta = 260$ MeV, and we have chosen a value of $\alpha$ for each case which reproduces the correct scattering lengths. These values are given in Table I.

### B. Analytic Formulas for the $n$-$n$ System

In this section $s'$ takes on the values "$t$" for triplet and "$s$" for singlet. We use a separable potential for the $n$-$n$ system. The results are

$$S_t(\vec{p}, \vec{\Delta}) = -\frac{1}{\pi\sqrt{2}} \frac{[\mu\beta(\mu+\beta)]^{1/2}}{\beta-\mu} \left( \frac{1}{(\vec{p}-\vec{\Delta})^2+\mu^2} - \frac{1}{(\vec{p}-\vec{\Delta})^2+\beta^2} - \frac{1}{(\vec{p}+\vec{\Delta})^2+\mu^2} + \frac{1}{(\vec{p}+\vec{\Delta})^2+\beta^2} \right), \tag{2.18}$$

which gives the transition to a triplet final state,

$$S_s(\vec{p}, \vec{\Delta}) = \frac{1}{\pi\sqrt{2}} \frac{[\mu\beta(\mu+\beta)]^{1/2}}{\beta-\mu} \left[ \frac{1}{(\vec{p}-\vec{\Delta})^2+\mu^2} - \frac{1}{(\vec{p}-\vec{\Delta})^2+\beta^2} + \frac{1}{(\vec{p}+\vec{\Delta})^2+\mu^2} - \frac{1}{(\vec{p}+\vec{\Delta})^2+\beta^2} + \frac{e^{2i\delta}-1}{2p\Delta} A(p, \Delta) \right], \tag{2.19}$$

where

$$A(p, \Delta) = \ln\left( \frac{[\mu + i(p+\Delta)][\beta + i(p-\Delta)][(\mu+\beta) - i\Delta][2\beta + i\Delta]}{[\mu + i(p-\Delta)][\beta + i(p+\Delta)][(\mu+\beta) + i\Delta][2\beta - i\Delta]} \right), \tag{2.20}$$

TABLE I. Values of separable-potential parameters used in computations (experimental data from Ref. 7).

| Units | | $p$-$p$ | $n$-$n$ | $n$-$p$ (singlet) | $n$-$p$ (triplet) |
|---|---|---|---|---|---|
| GeV | $\alpha$ | −0.0028 | −0.0101 | −0.007 92 | 0.0464 |
| GeV | $\beta$ | 0.260 | 0.260 | 0.260 | 0.260 |
| GeV | $\alpha+\beta$ | 0.2572 | 0.2499 | 0.252 08 | 0.3064 |
| fm | $a_0$(th) | −7.79 | −18.41 | −23.78 | 5.42 |
| fm | $r_0$(th) | 2.23 | 2.40 | 2.37 | 1.85 |
| fm | $a_0$(exp) | −7.786 ± 0.008 | −18.42 ± 1.53 | −23.714 ± 0.0013 | 5.425 ± 0.004 |
| fm | $r_0$(exp) | 2.840 ± 0.009 | | 2.704 ± 0.087 | 1.749 ± 0.008 |

and the phase shift, $e^{2i\delta}$, is given by the separable potential and has the analytic form

$$e^{2i\delta}=\frac{(\beta-ik)^2[(\beta+ik)^2-(\alpha+\beta)^2]}{(\beta+ik)^2[(\beta-ik)^2-(\alpha+\beta)^2]}\,. \tag{2.21}$$

Elementary integration techniques now give

$$\overline{S}_t(p,\Delta)=\frac{\mu\beta(\mu+\beta)}{2\pi p^2\Delta^2(\beta-\mu)^2}\left[\frac{2}{\xi^2-1}+\frac{2}{\eta^2-1}-\frac{2}{\eta-\xi}\ln\left(\frac{\eta-1}{\eta+1}\,\frac{\xi+1}{\xi-1}\right)\right.$$
$$\left.+\frac{2}{\eta+\xi}\ln\left(\frac{\eta+1}{\eta-1}\,\frac{\xi+1}{\xi-1}\right)-\frac{1}{\xi}\ln\left(\frac{\xi+1}{\xi-1}\right)-\frac{1}{\eta}\ln\left(\frac{\eta+1}{\eta-1}\right)\right], \tag{2.22}$$

$$\overline{S}_s(p,\Delta)=\frac{\mu\beta(\mu+\beta)}{2\pi p^2\Delta^2(\beta-\mu)^2}\left\{\frac{2}{\xi^2-1}+\frac{2}{\eta^2-1}-\frac{2}{\eta-\xi}\ln\left(\frac{\eta-1}{\eta+1}\,\frac{\xi+1}{\xi-1}\right)-\frac{2}{\eta+\xi}\ln\left(\frac{\eta+1}{\eta-1}\,\frac{\xi+1}{\xi-1}\right)\right.$$
$$+\frac{1}{\xi}\ln\left(\frac{\xi+1}{\xi-1}\right)+\frac{1}{\eta}\ln\left(\frac{\eta+1}{\eta-1}\right)+2\,\mathrm{Re}[(e^{2i\delta}-1)A(p,\Delta)]$$
$$\left.\times\ln\left(\frac{\xi+1}{\xi-1}\,\frac{\eta-1}{\eta+1}\right)+4(\sin^2\delta)|A(p,\Delta)|^2\right\}, \tag{2.23}$$

where in these expressions

$$\xi=\frac{p^2+\Delta^2+\mu^2}{2p\Delta}$$
$$\eta=\frac{p^2+\Delta^2+\beta^2}{2p\Delta}\,.$$

### C. $p$-$p$ Final State

The initial state is still a deuteron, so we define

$$Z_{s'}(\vec{p},\vec{\Delta})=\int d^3r\,\psi^*_{s'}(\vec{r})e^{i\vec{\Delta}\cdot\vec{r}}\,\psi_d(r), \tag{2.24}$$

in which

$$\psi_{s'}(\vec{r})=[\psi(\vec{p},\vec{r})+(-)^{s'}\psi(-\vec{p},\vec{r})]/\sqrt{2} \tag{2.25}$$

and

$$\psi(\vec{p},\vec{r})=\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\sigma_l}\psi_l(r)\,.$$

$\psi_l(r)$ is the separable-potential wave function in the presence of a Coulomb potential as defined in Eqs. (1.2) and (1.21), $\vec{p}\cdot\vec{r}=pr\cos\theta$, and $\sigma_l$ is the Coulomb phase shift as defined in Ref. 2.

We do the necessary integrations in the Appendix, and find that

$$Z_t(\vec{p},\vec{\Delta})=\frac{1}{\pi\sqrt{2}}\,\frac{[\mu\beta(\mu+\beta)]^{1/2}}{\beta-\mu}\,e^{-i\pi\kappa/2}\Gamma(1+\kappa)\big([(\mu+ip)^2+\Delta^2]^{\kappa}\{[\mu^2+(\vec{p}-\vec{\Delta})^2]^{-(1+\kappa)}-[\mu^2+(\vec{p}+\vec{\Delta})^2]^{-(1+\kappa)}\}$$
$$-[(\beta+ip)^2+\Delta^2]^{\kappa}\{[\beta^2+(\vec{p}-\vec{\Delta})^2]^{-(1+\kappa)}-[\beta^2+(\vec{p}+\vec{\Delta})^2]^{-(1+\kappa)}\}\big), \quad (2.26)$$

$$Z_s(\vec{p},\vec{\Delta})=\frac{1}{\pi\sqrt{2}}\,\frac{[\mu\beta(\mu+\beta)]^{1/2}}{\beta-\mu}\Big(e^{-i\pi\kappa/2}\Gamma(1+\kappa)\big([(\mu+ip)^2+\Delta^2]^{\kappa}\{[\mu^2+(\vec{p}-\vec{\Delta})^2]^{-(1+\kappa)}+[\mu^2+(\vec{p}+\vec{\Delta})^2]^{-(1+\kappa)}\}$$
$$-[(\beta+ip)^2+\Delta^2]^{\kappa}\{[\beta^2+(\vec{p}-\vec{\Delta})^2]^{-(1+\kappa)}+[\beta^2+(\vec{p}+\vec{\Delta})^2]^{-(1+\kappa)}\}\big)+\frac{e^{-2i\sigma_0}(e^{-2i\delta}-1)}{2p\Delta}B(p,\Delta)\Big), \quad (2.27)$$

where

$$B(p,\Delta)=\frac{e^{i\pi\kappa/2}}{\Gamma(1+\kappa)}\,[\phi(\mu;p,\Delta)-\phi(\mu;-i\beta,\Delta)-\phi(\mu;p,-\Delta)+\phi(\mu;-i\beta,-\Delta)$$
$$-\phi(\beta;p,\Delta)+\phi(\beta;-i\beta,\Delta)+\phi(\beta;p,-\Delta)-\phi(\beta;-i\beta,-\Delta)] \quad (2.28)$$

and

$$\kappa=-ia/p, \quad (2.29)$$

$$a=e^2M \text{ ($e$ is the charge of the proton)}, \quad (2.30)$$

and

$$\phi(\alpha;k,\Delta)=\frac{i[\alpha-i(\Delta+k)]}{2k(1+\kappa)}F\left(1,1,2+\kappa,\frac{\alpha-i(\Delta+k)}{-2ik}\right). \quad (2.31)$$

We may also calculate the integrals

$$\int d\Omega_p|Z_{s'}(\vec{p},\vec{\Delta})|^2\equiv\overline{Z}_{s'}(p,\Delta) \quad (2.32)$$

by elementary methods, to obtain

$$\overline{Z}_t(p,\Delta)=\frac{2\mu\beta(\mu+\beta)}{\pi(\mu-\beta)^2}|D|^2\left[|F|^2\left\{\frac{4}{\xi^2-1}-\frac{1}{\kappa\xi}\left[\left(\frac{\xi+1}{\xi-1}\right)^{\kappa}-\left(\frac{\xi-1}{\xi+1}\right)^{\kappa}\right]\right\}+|G|^2\left\{\frac{4}{\eta^2-1}-\frac{1}{\kappa\eta}\left[\left(\frac{\eta+1}{\eta-1}\right)^{\kappa}-\left(\frac{\eta-1}{\eta+1}\right)^{\kappa}\right]\right\}\right.$$
$$\left.-2\,\mathrm{Re}\left(F^*G\left\{\frac{2}{\kappa(\eta-\xi)}\left[\left(\frac{\xi+1}{\eta+1}\right)^{\kappa}-\left(\frac{\xi-1}{\eta-1}\right)^{\kappa}\right]-\frac{2}{\kappa(\eta+\xi)}\left[\left(\frac{\xi+1}{\eta-1}\right)^{\kappa}-\left(\frac{\xi-1}{\eta+1}\right)^{\kappa}\right]\right\}\right)\right] \quad (2.33)$$

and

$$\overline{Z}_s(p,\Delta)=\frac{2\mu\beta(\mu+\beta)}{\pi(\beta-\mu)^2}\left\{|D|^2\left[|F|^2\left\{\frac{4}{\xi^2-1}+\frac{1}{\kappa\xi}\left[\left(\frac{\xi+1}{\xi-1}\right)^{\kappa}-\left(\frac{\xi-1}{\xi+1}\right)^{\kappa}\right]\right\}+|G|^2\left\{\frac{4}{\xi^2-1}+\frac{1}{\kappa\eta}\left[\left(\frac{\eta+1}{\eta-1}\right)^{\kappa}-\left(\frac{\eta-1}{\eta+1}\right)^{\kappa}\right]\right\}\right.\right.$$
$$\left.-2\,\mathrm{Re}\left(F^*G\left\{\frac{2}{\kappa(\eta-\xi)}\left[\left(\frac{\xi+1}{\eta+1}\right)^{\kappa}-\left(\frac{\xi-1}{\eta-1}\right)^{\kappa}\right]+\frac{2}{\kappa(\eta+\xi)}\left[\left(\frac{\xi+1}{\eta-1}\right)^{\kappa}-\left(\frac{\xi-1}{\eta+1}\right)^{\kappa}\right]\right\}\right)\right]$$
$$+4\,\mathrm{Re}\left(\frac{e^{2i\sigma_0}(e^{2i\delta}-1)}{2p\Delta}\,\frac{DB^*(p,\Delta)}{\kappa}\{F[(\xi-1)^{-\kappa}-(\xi+1)^{-\kappa}]-G[(\eta-1)^{-\kappa}-(\eta+1)^{-\kappa}]\}\right)$$
$$\left.+\frac{2|B(p,\Delta)|^2\sin^2\delta}{p^2\Delta^2}\right\}, \quad (2.34)$$

where

$$D=e^{-i\pi\kappa/2}\Gamma(1+\kappa)(2p\Delta)^{-1-\kappa}, \quad (2.35)$$

$$F=[(\mu+ip)^2+\Delta^2]^{\kappa}, \quad (2.36)$$

$$G=[(\beta+ip)^2+\Delta^2]^{\kappa}. \quad (2.37)$$

These formulas are not particularly illuminating to look at, but have the advantage of relative ease of computation for numerical purposes. They depend, of course, on the validity of the model potential, but by choice of $\alpha$ and $\beta$ the scattering length and effective range can always be fitted.

The fundamental requirements of reality of phase shift and orthogonality and completeness of wave functions have also been satisfied. Thus, due to the well-known insensitivity of low-energy scattering to much more than effective range and scattering length, we can place quite a large degree of reliance on these calculations.

### D. Results of Computations

We have done calculations for

$$\gamma + d \to \pi^+ + p + p$$

and (2.38)

$$\gamma + d \to \pi^- + n + n$$

at an incident $\gamma$ energy of 3.4 GeV, and momentum transfer

$$t = (p_\gamma - p_\pi)^2 = -0.03\ \text{GeV}^2\,. \qquad (2.39)$$

This means that the number $\Delta = |\vec{\Delta}|$ is given by

$$\Delta = \tfrac{1}{2}(-t + t^2/4M_N^2)^{1/2} \qquad (2.40)$$

$$= 0.08697\ \text{GeV}\,. \qquad (2.41)$$

In Fig. 2, which gives the energy spectrum of the final pion if the $n$-$n$ or $p$-$p$ systems are in singlet states, we observe the same qualitative features as Phillips,[5] that the Coulomb interaction reduces the peak of high energies, but causes a small amount of enhancement of lower energies.

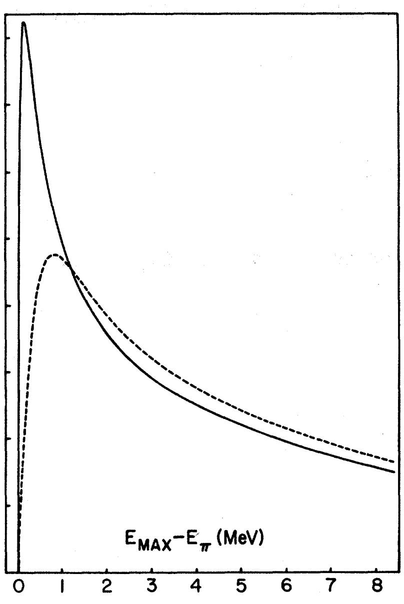


FIG. 2. Energy spectrum for the final pion from photoproduction on deuterium in the case where the two final nucleons are in a singlet state. The solid line is $\gamma + d \to \pi^+ + n + n$, the dashed line is $\gamma + d \to \pi^- + p + p$. The incident $\gamma$ energy is 3.4 GeV, and the momentum transfer squared, $t$, is $-0.03$ GeV$^2$. The vertical scale is in arbitrary units, each of which equals 5.5 of the arbitrary units in Fig. 3.

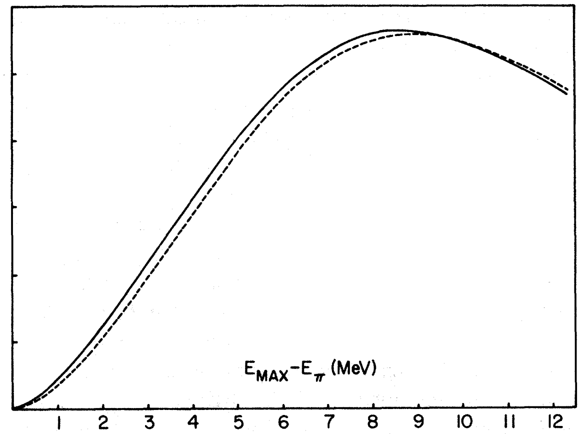


FIG. 3. Energy spectrum of the pion from photoproduction on deuterium in the case where the final two nucleons are in a triplet state. The solid line is $\gamma + d \to \pi^+ + n + n$, and the dashed line is $\gamma + d \to \pi^- + p + p$. The incident $\gamma$ energy is 3.4 GeV, and the momentum transfer squared, $t$, is $-0.03$ GeV$^2$. The vertical scale is in arbitrary units, each of which is 1/5.5 of the arbitrary units in Fig. 2.

In Fig. 3, which gives the energy spectrum of the final pion if the $n$-$n$ or $p$-$p$ systems are in triplet states and there is no final-state interaction, we see that the Coulomb interaction tends to displace the spectrum without changing its shape.

In Fig. 4 we show the integrated spectra for the two singlet cases. Notice that at low final pion energy, the value of the integrated spectrum becomes very close to the value given by completeness arguments. The slight difference is accounted for by kinematical factors omitted to give Eq.

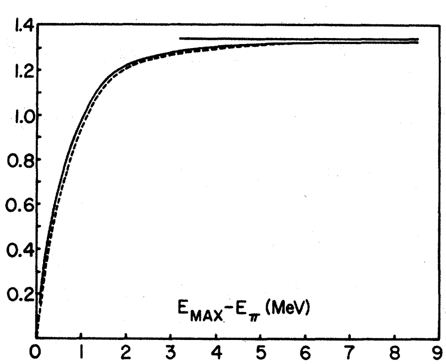


FIG. 4. Integrated spectrum in the case of singlet nucleon final states. The horizontal scale represents the upper limit of integration, while the lower limit occurs when $E = E_{max}$. The solid horizontal line is the value of the total integrated spectrum as predicted by completeness arguments in Eq. (2.15).

(2.15), rather than the approximation that $W$ is infinite. When these kinematical factors are omitted, the difference between the completeness prediction and the value calculated is an order of magnitude less than that in Fig. 4. Equation (2.15) gives, in fact, only an upper bound on the integrated spectrum. During the process of developing this formalism, it was found by the author that approximations in which completeness and orthogonality of the wave functions are not guaranteed can quite easily violate this bound, and it is for this reason that such stress has been laid on this point.

At present, experiments done with bremsstrahlung beams cannot detect the difference between $p$-$p$ and other two-nucleon final states. However, with improved techniques, a $\gamma$ beam with sharply defined energy may be developed. With such a beam the effects calculated may be measurable.

## CONCLUSIONS

The separable-potential description has been found to give a manageable treatment of $p$-$p$ final-state interactions in the impulse approximation, in which all fundamental requirements of completeness and orthogonality of the wave functions and reality of $p$-$p$ phase shifts are retained.

We have found that with the present resolutions available from bremsstrahlung photon beams, no effects will be noticed, since the energy range in which $p$-$p$ final-state interactions differ from $n$-$p$ and $n$-$n$ final-state interactions is only of the order of a few MeV. Furthermore, the closure approximation (2.15) has been shown to be quite good, although it is possible that the slight discrepancy noticed (see Fig. 4) might be measurable in experiments with good statistics.

## APPENDIX: EVALUATION OF INTEGRALS REQUIRED IN SEC. II

We wish to evaluate the integral given in Eq. (2.24). Now

$$\psi(\vec{p},\vec{r}) = (2\pi)^{-3/2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\sigma_l}\psi_l(r) \tag{A1}$$

$$= (2\pi)^{-3/2}\Bigg[\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\sigma_l}i^l\frac{F_l(pr)}{pr} + \frac{\exp[\pi a/4p+2i\sigma_0](e^{2i\delta}-1)}{2ipr\,\Gamma(1-\kappa)} \times\{[\Gamma(1-\kappa)W_{\kappa,1/2}(-2ipr)+\Gamma(1+a/2\beta)W_{-a/2\beta,1/2}(2\beta r)]\}\Bigg]. \tag{A2}$$

We wish to evaluate

$$\int d^3r\,\psi^*(\vec{p},\vec{r})e^{i\vec{\Delta}\cdot\vec{r}}e^{-\alpha r}/r \equiv I(\vec{p},\vec{\Delta},\alpha). \tag{A3}$$

First we treat the part of the integral coming from the first part of Eq. (A2). We write

$$e^{i\vec{\Delta}\cdot\vec{r}} = \sum(2l+1)i^l\vec{C}_l(\hat{\Delta})\cdot\vec{C}(\hat{r})j_l(\Delta r), \tag{A4}$$

where we have used the notation of Brink and Satchler[8] for the spherical harmonics $C_{lm}$. Using the orthogonality of spherical harmonics, the first part may be written as

$$J = \left(\frac{2}{\pi}\right)^{1/2}\sum_l\int r^2dr(2l+1)P_l(\hat{\Delta}\cdot\hat{p})e^{-i\sigma_l}\frac{F_l^*(pr)}{pr}\,\frac{j_l(\Delta r)e^{-\alpha r}}{r}. \tag{A5}$$

To evaluate $J$ we need to compute the integrals

$$I_l = \int r\,dr\,\frac{F_l^*(pr)}{pr}j_l(\Delta r)e^{-\alpha r} \tag{A6}$$

using the fact that

$$\frac{F_l(pr)}{pr} = [N_l(\kappa)]^{1/2}(2pr)^l F(l+1-\kappa,\,2l+2,\,-2ipr)e^{ipr} \tag{A7}$$

and

$$j_l(\Delta r) = \frac{\Gamma(l+1)}{\Gamma(2l+2)}(2\Delta r)^l F(l+1,\,2l+2,\,-2i\Delta r)e^{i\Delta r}, \tag{A8}$$

where $F(a,b,z)$ is the confluent hypergeometric function as defined by Landau and Lifshitz.[9] Equations (A7) and (A8) may be obtained from Refs. 4 and 2. Making the substitutions (A7) and (A8), the integrals can be evaluated by the methods of Landau and Lifshitz to give

$$I_l=\sqrt{N_l(\kappa)}\,\Gamma(l+1)\left(\frac{\alpha+i(p-\Delta)}{\alpha-i(p+\Delta)}\right)^{\kappa}[\alpha^2+(p+\Delta)^2]^{-1}\left(\frac{4p\Delta}{\alpha^2+(p+\Delta)^2}\right)^{l}F\left(l+1+\kappa,\,l+1,\,2l+2,\,\frac{4p\Delta}{\alpha^2+(p+\Delta)^2}\right). \tag{A9}$$

Substituting the explicit values for $N_l(\kappa)$ and $\exp(i\sigma_l)$ as given in Eqs. (1.3) and Ref. 2, we find that

$$J=4\pi\sum_{l=0}^{\infty}(2l+1)P_l(\hat{\Delta}\cdot\hat{p})e^{-i\pi\kappa/2}\frac{\Gamma(l+1+\kappa)\Gamma(l+1)}{\Gamma(2l+2)}\left(\frac{\alpha+i(p-\Delta)}{\alpha-i(p+\Delta)}\right)^{\kappa}[\alpha^2+(p+\Delta)^2]^{-1}$$
$$\times\left(\frac{4p\Delta}{\alpha^2+(p+\Delta)^2}\right)^{l}F\left(l+1+\kappa,\,l+1,\,2l+2,\,\frac{4p\Delta}{\alpha^2+(p+\Delta)^2}\right). \tag{A10}$$

We can sum Eq. (A10) if we know the sum

$$\sum_{l=0}^{\infty}(2l+1)z^l\,\frac{\Gamma(l+1+\kappa)\Gamma(l+1)}{\Gamma(2l+2)}F(l+1+\kappa,\,l+1,\,2l+2,\,z)P_l(\cos\omega). \tag{A11}$$

From Landau and Lifshitz, p. 503, and Eq. (A8), we know that

$$\int_0^{\infty}dt\,e^{t/2}j_l(\tfrac{1}{2}it)(it)^{-l}e^{-t/z}t^{l+\kappa}=\frac{\Gamma(l+1+\kappa)\Gamma(l+1)}{\Gamma(2l+2)}z^{l+1+\kappa}F(l+1+\kappa,\,l+1,\,2l+2,\,z). \tag{A12}$$

Substituting the left-hand side of (A12) into (A11), we find that (A11) is equal to

$$z^{-(1+\kappa)}\int_0^{\infty}\exp\left[\tfrac{1}{2}t\left(1-\frac{2}{z}\right)\right]t^{\kappa}\sum_{l=0}^{\infty}j_l(\tfrac{1}{2}it)(-i)^lP_l(\cos\omega)(2l+1)dt,$$

and using the Rayleigh expansion, Eq. (A4), (A11) equals

$$z^{-(1+\kappa)}\int_0^{\infty}\exp\left[t\left(\frac{1+\cos\omega}{2}-\frac{1}{z}\right)\right]t^{\kappa}dt=z^{-(1+\kappa)}\Gamma(1+\kappa)\left(\frac{1+\cos\omega}{2}-\frac{1}{z}\right)^{-(1+\kappa)}. \tag{A13}$$

Now substitute

$$z=4p\Delta/[\alpha^2+(p+\Delta)^2]$$

and

$$\cos\omega=\hat{p}\cdot\hat{\Delta}, \tag{A14}$$

and use the resulting form of Eq. (A13) and substitute into Eq. (A10), to find that

$$J=(2/\pi)^{1/2}e^{-i\pi\kappa/2}\Gamma(1+\kappa)[(\alpha+ip)^2-\Delta^2]^{\kappa}[\alpha^2+(\vec{p}-\vec{\Delta})^2]^{-(1+\kappa)}. \tag{A15}$$

This completes the integration coming from the first part of Eq. (A

To evaluate the other integrals coming from the second part of Eq. (A2), we must consider

$$\int d^3r\,\frac{W_{-\kappa,1/2}(2ipr)}{-2ipr}e^{i\vec{\Delta}\cdot\vec{r}}\,e^{-\alpha r}/r \tag{A16}$$

which is equal to

$$\frac{\pi}{p\Delta}\int dr\,r^{-1}e^{-\alpha r}(e^{i\Delta r}-e^{-i\Delta r})W_{-\kappa,1/2}(2ipr). \tag{A17}$$

Now the separate integrals containing $e^{i\Delta r}$ and $e^{-i\Delta r}$ do not converge, but their difference does. Consider then

$$I(\Delta,\epsilon)=\int_0^{\infty}dr\,r^{-1+\epsilon}e^{-(\alpha-i\Delta)r}W_{-\kappa,1/2}(2ipr), \tag{A18}$$

which does converge for $\epsilon>0$, but not for $\epsilon=0$. If we calculate this, and then take

$$\lim_{\epsilon\to 0}[I(\Delta,\epsilon)-I(-\Delta,\epsilon)],$$

we obtain a value for (A17). We proceed to do this.

First substitute

$$W_{-\kappa,1/2}(2ipr)=e^{-ipr}(2ipr)U(1+\kappa,\,2,\,2ipr), \tag{A19}$$

as given in the chapter on the confluent hypergeometric function in Ref. 4, which also gives the integral representation

$$\Gamma(a)U(a, b, z) = \int_0^\infty e^{-zt}(1+t)^{b-a-1}t^{a-1}dt\ , \tag{A20}$$

which is valid for Re$z>0$. Since we are using $z = 2ipr$, we shall let $p$ have a small negative imaginary part, which is set equal to zero at the end of the calculation. We shall not write this in explicitly. Hence

$$\begin{aligned} I(\Delta, \epsilon) &= \int_0^\infty dt \int_0^\infty dr\, r^{-1+\epsilon} e^{-(\alpha - i\Delta)r} e^{-ipr}(2ipr)\frac{1}{\Gamma(1+\kappa)} e^{-2iprt}(1+t)^{-\kappa}t^\kappa \\ &= \frac{2ip\Gamma(1+\epsilon)}{\Gamma(1+\kappa)}\int_0^\infty dt\left(\frac{1+t}{t}\right)^{-\kappa}\{[\alpha - i(\Delta - p)] + 2ipt\}^{-(1+\epsilon)}\ . \end{aligned} \tag{A21}$$

Let $u = t/(1+t)$; then the integral can be transformed to

$$I(\Delta, \epsilon) = \frac{2ip\Gamma(1+\epsilon)}{\Gamma(1+\kappa)}[\alpha - i(\Delta - p)]^{-(1+\epsilon)}\int_0^1 du\, u^\kappa(1-u)^{-(1-\epsilon)}\left(1 - u\frac{\alpha - i(\Delta+p)}{\alpha - i(\Delta - p)}\right)^{-(1+\epsilon)}$$

which may be recognized from the chapter on hypergeometric functions in Ref. 4 as a hypergeometric function: explicitly,

$$I(\Delta, \epsilon) = 2ip\frac{\Gamma(1+\epsilon)\Gamma(\epsilon)}{\Gamma(1+\epsilon-\kappa)}[\alpha - i(\Delta - p)]^{-(1+\epsilon)}F\left(1+\epsilon, 1+\kappa, 1+\epsilon+\kappa, \frac{\alpha - i(\Delta+p)}{\alpha - i(\Delta - p)}\right). \tag{A22}$$

This integral is continuous at Im$p=0$, so we now let Im$p=0$. The divergence as $\epsilon \to 0$ is plain. We now separate Eq. 22 into convergent and divergent parts at $\epsilon = 0$, using the equation

$$(1-z)^a F(a, b, c, z) = F\left(a, b, c, \frac{z}{z-1}\right) \tag{A23}$$

which is also obtainable from Ref. 4. Thus

$$\begin{aligned} I(\Delta, \epsilon) &= (2ip)^{-\epsilon}\frac{\Gamma(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+\epsilon-\kappa)}F\left(1+\epsilon, \epsilon, 1+\epsilon+\kappa, \frac{\alpha - i(\Delta+p)}{-2ip}\right) \\ &= (2ip)^{-\epsilon}\left[\frac{\Gamma(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+\epsilon+\kappa)} + \sum_{n=1}^\infty \frac{\Gamma(1+\epsilon+n)\Gamma(\epsilon+n)}{\Gamma(1+\epsilon+\kappa+n)n!}\left(\frac{\alpha - i(\Delta+p)}{-2ip}\right)^n\right]. \end{aligned} \tag{A24}$$

The part which diverges at $\epsilon = 0$ is the first part in the curly brackets, which is independent of $\Delta$. Hence $I(\Delta, \epsilon) - I(\Delta, \epsilon)$ depends on the second part in the curly brackets only, which we call $I'(\Delta, \epsilon)$, and when $\epsilon \to 0$, $I'(\Delta, 0)$ is finite, and is given by

$$\begin{aligned} I'(\Delta, 0) &= \frac{\alpha - i(\Delta+p)}{-2ip}\sum_{n=0}^\infty \frac{\Gamma(1+n)}{\Gamma(2+\kappa+n)}\left(\frac{\alpha - i(\Delta+p)}{-2ip}\right)^n \\ &= \frac{1}{\Gamma(2+\kappa)}\frac{\alpha - i(\Delta+p)}{-2ip}F\left(1, 1, 2+\kappa, \frac{\alpha - i(\Delta+p)}{-2ip}\right). \end{aligned} \tag{A25}$$

Define the function $\phi(\alpha; p, \Delta)$ by

$$\phi(\alpha; p, \Delta) = i[\alpha - i(\Delta+p)][(1+\kappa)(2p)]^{-1}F\left(1, 1, 2+\kappa, \frac{\alpha - i(\Delta+p)}{-2ip}\right), \tag{A26}$$

so that

$$I'(\Delta, 0) = \phi(\alpha; p, \Delta)/\Gamma(1+\kappa)\ . \tag{A27}$$

This now completes the steps necessary to evaluate Eq. (A3). Collecting all together,

$$\begin{aligned} I(\vec{p}, \vec{\Delta}, \alpha) &= (2/\pi)^{1/2}e^{-i\pi\kappa/2}\Gamma(1+\kappa)[(\alpha+ip)^2 + \Delta^2]^\kappa[\alpha^2 + (\vec{p} - \vec{\Delta})^2]^{-(1+\kappa)} \\ &\quad + \frac{1}{\sqrt{2\pi}}\frac{e^{-2i\sigma_0}(e^{-2i\delta}-1)}{2p\Delta}\frac{e^{i\pi\kappa/2}}{\Gamma(1+\kappa)}[\phi(\alpha; p, \Delta) - \phi(\alpha; p, -\Delta) - \phi(\alpha; -i\beta, \Delta) + \phi(\alpha; -i\beta, -\Delta)]\ . \end{aligned} \tag{A28}$$

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# Phenomenology of $K \to 2\pi$ Decays

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The theoretical expressions for experimentally determined quantities in $K \to 2\pi$ decays and leptonic kaon decays are used to determine solutions for various theoretical parameters. Two solutions are found: one for which the $|\Delta I| = \frac{1}{2}$ dominance rule is valid, and another for which it is not. Both solutions yield $\epsilon \cong \eta_{+-} \cong \eta_{00}$. Even precise measurements of $\phi_{00}$ may not allow one to distinguish between the two solutions.

## I. INTRODUCTION

With the recent measurement[1] of the phase angle of the ratio of $K_L \to \pi^0\pi^0$ to $K_S \to \pi^0\pi^0$ decay, $\phi_{00}$, it has become of interest to recalculate the phenomenological parameters of $K \to 2\pi$ decays previously calculated by Roper.[2] In the calculation of Ref. 2 no approximations were made concerning the relative sizes of the $|\Delta I| = \frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ decay amplitudes. We have expanded the previous calculations by including the $\Delta S = \pm \Delta Q$ amplitudes of leptonic kaon decays, which various authors[3] have used to find an approximate value for $\mathrm{Re}\epsilon$. Since it is impossible at this time to ascertain which of the available numerical values to use for the phase-shift difference $\delta_2 - \delta_0$ between $I = 2$ and $I = 0$ $s$-wave scattering for $\pi$-$\pi$ interactions, the results are given with both Walker's[4] value and Marateck's[5] value for $\delta_2 - \delta_0$ as inputs.

We begin with Roper's[2] formulation of $K \to 2\pi$ decays. By means of the experimental values of various quantities we are able to derive values of the ratio $\bar{b}_3$ of the complex $|\Delta I| = \frac{3}{2}$ reduced matrix element to the $|\Delta I| = \frac{1}{2}$ reduced matrix element, the ratio $\bar{b}_5$ of the complex $|\Delta I| = \frac{5}{2}$ reduced matrix element to the $|\Delta I| = \frac{1}{2}$ reduced matrix element, the complex $K^0$-$\bar{K}^0$ mixing parameter $\epsilon$, the complex ratio $x$ of the $\Delta S = -\Delta Q$ amplitude to the $\Delta S = +\Delta Q$ amplitude in leptonic kaon decays, and the $\pi$-$\pi$ phase-shift difference $\delta_2 - \delta_0$.

The above values were found by using the four solutions found by Roper[2] as inputs in a least-squares fit to the data by means of the exact equations. The four solutions reduced to two solutions, one of which satisfies the $|\Delta I| = \frac{1}{2}$ dominance rule and the other of which does not.

By assuming very precise values for $\phi_{00}$ we show that even great precision may not allow one to distinguish between the two solutions obtained here.

## II. THEORY

We write the isotopic-spin amplitudes as

$$\langle 0, 0|A|K^0\rangle = \beta_0, \quad \langle 0, 0|A|\bar{K}^0\rangle = \beta_0\,,$$

$$\langle 2, 0|A|K^0\rangle = \beta_2, \quad \langle 2, 0|A|\bar{K}^0\rangle = \beta_2^*\,,$$

$$\langle 2, 1|A|K^+\rangle = \beta_1, \quad \langle 2, -1|A|K^-\rangle = \beta_1^*\,,$$

where

$$\beta_0 = b_1/\sqrt{2}\,,$$

$$\beta_2 = (b_3 + b_5)/\sqrt{2}\,,$$

and

$$\beta_1 = (\tfrac{3}{4})^{1/2}(b_3 - \tfrac{2}{3}b_5).$$