$$\langle m|U^2|n\rangle < \infty . \tag{6}$$

Consider in particular $\langle 0|U^2|0\rangle$. The Bessel inequality tells us that

$$\sum_{n=0}^{N}\langle 0|U|n\rangle\langle n|U|0\rangle \leq \langle 0|U^2|0\rangle . \tag{7}$$

On the other hand, we know that the sum

$$\sum_{n=0}^{N}\langle 0|H_I|n\rangle\langle n|H_I|0\rangle$$

diverges as $N\to\infty$, which shows that

$$\langle 0|U|n\rangle \neq \langle 0|H_I|n\rangle . \tag{8}$$

We summarize our arguments by pointing out once again that there is nothing intrinsically wrong with the U proposed in Ref. 2; it is a perfectly fine potential. It is *not*, however, the potential which reproduces the tree graphs of the theory from which it comes, but rather some other set. Thus, for example, if we were to check unitarity by using the original tree graphs and this U, it would fail. Finally, we would have no reason to believe that we can exclude the graphs involving contractions at the same point discussed in Ref. 3.

*Supported in part by the National Science Foundation.

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[4]It is easy to see that we can express U in terms of error functions. The resemblance between U expressed this way and the U of Eq. (5) is misleading, in that the two behave very differently at ∞.

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Further Comments on the Relations between Dual Models and Chiral Symmetry Breaking

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Certain peculiar aspects present in dual-resonance models (collectively called "algebraic duality") led to the construction of amplitudes coincident with those coming from the nonlinear σ model in the SU(2) case, and from a $U(3)\otimes U(3)$ nonlinear Lagrangian with a breaking which preserved the complete pseudoscalar nonet degeneracy in the more general case. In the present work we manage to split the singlet from the octet, breaking algebraic duality in a simple way.

Some work has been done with the intention of imposing some peculiar properties found in dual amplitudes as a way to select effective Lagrangian models.[1-4]

The present note is a continuation of Ref. 4, always within the spirit put forward in Ref. 1.

Let us begin by recalling the postulates of the so-called algebraic duality, introduced by Frye and Susskind. Within this framework the n-point pseudoscalar-meson amplitude is given by a sum of terms, each one corresponding to a distinct permutation[5] of the external legs. The term for a given permutation contains only the Mandelstam variables associated with that order and an appropriate factor to guarantee invariance[6] under SU(2) or SU(3) (trace of Pauli matrices or symmetrized trace of Gell-Mann matrices, both guessed by means of quark diagrams). Such an expression automatically rules out dependence on Mandelstam variables corresponding to channels with exotic quantum numbers. Furthermore, it is required that Adler zeros be satisfied within each permutation separately.[7]

These conditions, plus factorization, were basically enough to produce amplitudes equivalent to the ones obtained from well-defined nonlinear Lagrangians (in particular the nonlinear σ model[1,8]), when one is interested only in pions – and from a $U(3)\otimes U(3)$ model with the chiral symmetry broken in a way that preserves the complete nonet degeneracy, in the general case of the pseudoscalar mesons.[4]

The experimental fact of the gross mass difference between the singlet and the octet members makes it worthwhile to study the way in which such a splitting can be introduced.[2] This is, of course, nonunique, but we shall adopt the criterion of looking for a simple breaking pattern of the algebraic-duality postulates rather than simplicity in the group transformation properties of the chiral-symmetry-breaking term in the Lagrangian.

Proceeding as in Refs. 9 and 4, we define the meson matrix M which is a series expansion in the fields:

$$M=\sum_n a_n(if\phi)^n \tag{1}$$

where

$$\phi=\frac{1}{\sqrt{2}}\sum_{i=0}^{8}\lambda_i\phi_i\,. \tag{2}$$

We are free to choose $a_0=1$, $a_1=2$, and $a_2=2$. The chiral-invariant entity $M^\dagger M$ is taken to be 1. This provides relations of the coefficients among themselves.

We shall be interested only up to sixth order in the fields. The relevant relations are then

$$a_4=2(a_3-1)\,,\quad a_6=2a_5-4(a_3-1)+\tfrac{1}{2}a_3{}^2\,. \tag{3}$$

The Lagrangian

$$\mathcal{L}=-\frac{1}{8f^2}\mathrm{tr}(\partial_\mu M^\dagger\partial_\mu M)+\frac{m^2}{8f^2}\mathrm{tr}(M+M^\dagger) \tag{4}$$

proved to be consistent with algebraic duality if one sets $a_3=a_5=0$. We are going to study now a more complex Lagrangian with the addition of a term which will provide a shift for the mass of the SU(3) singlet. It must also give the necessary contributions to the meson interactions in order to achieve Adler's condition for the octet members and a minimal departure from algebraic duality.

We express the total chiral-symmetry-breaking term in the Lagrangian as

$$\mathcal{L}_{\mathrm{SB}}=-\frac{m^2}{2f^2}\mathrm{tr}[\mu(f\phi)]-\frac{\Delta m^2}{6f^2}[\mathrm{tr}\eta(f\phi)]^2\,, \tag{5}$$

where $\Delta m=m_0-m$ is the difference between the mass of the singlet and that of the octet. The restriction that only even powers of the fields appear in the Lagrangian implies

$$\mu(f\phi)=(f\phi)^2+b_4(f\phi)^4+b_6(f\phi)^6 \tag{6}$$

and

$$\eta(f\phi)=f\phi+c_3(f\phi)^3+c_5(f\phi)^5\,. \tag{7}$$

The vector currents are conserved, as can be easily checked using the expression for the infinitesimal transformation of the field:

$$\delta_V\phi=i\alpha_k[\tfrac{1}{2}\lambda_k,\phi]\,. \tag{8}$$

The situation for the axial-vector current is more delicate. We want to implement partial conservation of axial-vector current (PCAC) for the octet of axial-vector currents – namely, we want the breaking to be such that the relations

$$\partial_\mu A_\mu^k=\frac{m^2}{f\sqrt{2}}\phi_k\,,\quad k=1,\ldots,8 \tag{9}$$

hold up to the order in which we are interested. This can be implemented only after knowing the infinitesimal transformation of the fields.

M has simple transformation properties under chiral transformations, but we cannot expect the same thing to be true for the fields which form the basis of a nonlinear realization of the group. From

$$M[f\phi+\delta_A(f\phi)]-M(f\phi)=i\alpha_k\{\tfrac{1}{2}\lambda_k,M(f\phi)\}_+ \tag{10}$$

we get

$$\delta_A(f\phi)=\alpha_k(\tfrac{1}{2}\lambda_k+\tfrac{1}{4}f^2[(a_3-2)\{\phi^2,\lambda_k\}_++a_3\phi\lambda_k\phi]+\tfrac{1}{8}f^4[(a_3{}^2+2a_3-2a_5-4)\{\lambda_k,\phi^4\}_+$$
$$+2(a_3{}^2-a_3-a_5)(\phi\lambda_k\phi^3+\phi^3\lambda_k\phi)+(3a_3{}^2-4a_3-2a_5)(\phi^2\lambda_k\phi^2)])+O((f\phi)^6)\,. \tag{11}$$

The general expression for the current associated with an arbitrary transformation $\delta\phi$,

$$\alpha_p j_\mu^p(x)\equiv-\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_k)}\delta\phi_k\,, \tag{12}$$

implies that the divergence of the axial-vector current is given by

$$-\alpha_k\partial_\mu A_\mu^k=\mathcal{L}(f\phi+f\delta_A\phi)-\mathcal{L}(f\phi)\,. \tag{13}$$

For the Lagrangian (5), the right-hand side of (13) is

$$\mathcal{L}(f\phi+f\delta_A\phi)-\mathcal{L}(f\phi)=-\frac{m^2}{2f^2}\{2f^2\mathrm{tr}(\phi\delta_A\phi)+4b_4f^4\mathrm{tr}(\phi^3\delta_A\phi)+6b_6f^6\mathrm{tr}(\phi^5\delta_A\phi)\}$$
$$-\frac{\Delta m^2}{6f^2}\{2(f\,\mathrm{tr}\phi+c_3f^3\mathrm{tr}\phi^3+c_5f^5\mathrm{tr}\phi^5)[f\,\mathrm{tr}\,\delta_A\phi+3c_3f^3\mathrm{tr}(\phi^2\delta_A\phi)+5c_5f^5\mathrm{tr}(\phi^4\delta_A\phi)]\}\,. \tag{14}$$

Substituting (14) in (13) and using the expression (11) we get the following terms:

First order:

$$-\frac{m^2}{2f}\times 2\mathrm{tr}\,\{\phi[\delta_A(f\phi)]^{(0)}\}-\frac{\Delta m^2}{6f}\times 2\mathrm{tr}\phi\,\mathrm{tr}\,\{[\delta_A(f\phi)]^{(0)}\}=-\frac{m^2}{f}\frac{\alpha_k}{\sqrt{2}}\phi_k-\frac{\Delta m^2}{f}\frac{\alpha_0}{\sqrt{2}}\phi_0\,. \tag{15}$$

Second order: identically zero.

Third order:

$$\begin{aligned}&-(m^2/2f^2)\{2f\,\mathrm{tr}[\phi(f\delta_A\phi)^{(2)}]+4b_4f^3\mathrm{tr}[\phi^3(f\delta_A\phi)^{(0)}]\}\\&\quad-(\Delta m^2/6f^2)\{2f\,\mathrm{tr}\phi\,\mathrm{tr}(f\delta_A\phi)^{(2)}+6c_3f^3\mathrm{tr}\phi\,\mathrm{tr}[\phi^2(f\delta_A\phi)^{(0)}]+2c_3f^3\mathrm{tr}\phi^3\mathrm{tr}(f\delta_A\phi)^{(0)}\}\\&=-\tfrac{1}{2}m^2f\alpha_k[\tfrac{1}{4}(3a_3-4)\mathrm{tr}(\phi^3\lambda_k)+2b_4\mathrm{tr}(\phi^3\lambda_k)]\\&\quad-\tfrac{1}{6}\Delta m^2f\alpha_k[\tfrac{1}{2}(3a_3-4)\mathrm{tr}(\phi^2\lambda_k)\mathrm{tr}\phi+3c_3\mathrm{tr}\phi\,\mathrm{tr}(\lambda_k\phi^2)+c_3\mathrm{tr}\lambda_k\mathrm{tr}\phi^3]\,.\end{aligned} \tag{16}$$

Fourth order: identically zero.

Fifth order:

$$\begin{aligned}&-(m^2/2f^2)\{2f\,\mathrm{tr}[\phi(f\delta_A\phi)^{(4)}]+4b_4f^3\mathrm{tr}[\phi^3(f\delta_A\phi)^{(2)}]+6b_6f^5\mathrm{tr}[\phi^5(f\delta_A\phi)^{(0)}]\}\\&\quad-(\Delta m^2/6f^2)\{6c_3f^3\mathrm{tr}\phi\,\mathrm{tr}[\phi^2(f\delta_A\phi)^{(2)}]+2c_3f^3\mathrm{tr}(f\delta_A\phi)^{(2)}\mathrm{tr}\phi^3+2f\,\mathrm{tr}\phi\,\mathrm{tr}(f\delta_A\phi)^{(4)}\\&\qquad+10c_5f^5\mathrm{tr}\phi\,\mathrm{tr}[\phi^4(f\delta_A\phi)^{(0)}]+2c_5f^5\mathrm{tr}(f\delta\phi)^{(0)}\mathrm{tr}\phi^5+6c_3{}^2f^5\mathrm{tr}\phi^3\mathrm{tr}[\phi^2(f\delta_A\phi)^{(0)}]\}\\&=-\tfrac{1}{8}m^2f^3\alpha_k\{9a_3{}^2-4a_3-10a_5-8+4b_4(3a_3-4)+12b_6\}\mathrm{tr}(\phi^5\lambda_k)\\&\quad-\tfrac{1}{24}\Delta m^2\{[9a_3{}^2-4a_3-10a_5-8+6(3a_3-4)c_3+20c_5]\mathrm{tr}\phi\,\mathrm{tr}(\phi^4\lambda_k)\\&\qquad+[2c_3(3a_3-4)+12c_3{}^2]\mathrm{tr}\phi^3\mathrm{tr}\phi^2\lambda_k+20c_5\mathrm{tr}\lambda_k\mathrm{tr}\phi^5\}\,.\end{aligned} \tag{17}$$

The next contribution is of seventh order, higher than what we are interested in.

The first-order term, for a given k ($k=1,\ldots,8$), gives the right-hand side in Eq. (9). Thus all other terms must vanish order by order. From equating to zero the coefficients of $\phi_i\,\phi_j\,\phi_l$, with $i,\ j,\ l=1,\ldots,8$, we get the condition

$$b_4=1-\tfrac{3}{4}a_3\,. \tag{18}$$

Similarly, for $\phi_0\phi_i\,\phi_j$ we get, using (18),

$$c_3=\tfrac{1}{6}(4-3a_3)\,. \tag{19}$$

In the next order (17), the coefficient of $\mathrm{tr}\lambda_k\mathrm{tr}\phi^5$ does not contribute for the octet currents. The coefficient of $\mathrm{tr}\phi^3\mathrm{tr}\phi^2\lambda_k$ is identically zero by condition (19). Now equating to zero the coefficient of $\phi_i\,\phi_j\,\phi_l\,\phi_m\phi_n$ ($i,j,l,m,n=1,\ldots,8$), we get

$$b_6=\tfrac{1}{6}(-10a_3+5a_5+12)\,. \tag{20}$$

The vanishing of the remaining term gives

$$c_5=\tfrac{3}{5}b_6\,. \tag{21}$$

We will now look at the explicit expression for the amplitudes. Because of the relations (18)–(21) it is possible to express the amplitudes in terms of only two parameters, namely a_3 and a_5. They will be fixed keeping in mind the purpose of retaining as much as possible the spirit of algebraic duality.

We treat all particles as incoming and use the metric $g_{11}=g_{22}=g_{33}=-1$, $g_{44}=1$. $\mathfrak{T}\mathrm{r}$ is the symmetrized trace introduced in (4).

$$\begin{aligned}A_4=&\sum_{\text{distinct permutations}}\tfrac{1}{8}f^2\mathfrak{T}\mathrm{r}(\lambda_a\lambda_b\lambda_c\lambda_d)[2(s_{ab}+s_{bc})-a_3(s_{ab}+s_{bc}+s_{ac})-m^2(4-3a_3)]\\&-\tfrac{1}{24}\Delta m^2f^2(4-3a_3)[\mathrm{tr}\lambda_a\,\mathfrak{T}\mathrm{r}(\lambda_b\lambda_c\lambda_d)+\mathrm{tr}\lambda_b\,\mathfrak{T}\mathrm{r}(\lambda_a\lambda_c\lambda_d)+\mathrm{tr}\lambda_c\,\mathfrak{T}\mathrm{r}(\lambda_a\lambda_b\lambda_d)+\mathrm{tr}\lambda_d\,\mathfrak{T}\mathrm{r}(\lambda_a\lambda_b\lambda_c)]\,.\end{aligned} \tag{22}$$

The six-point amplitude has one contact term plus the contribution from tree diagrams. For the computation of this last part we refer to the techniques and notation in Ref. 4, particularly to relation (A6).

Here extra care is needed because the propagator for the exchange of a singlet is different from that of an octet member.

Allowing only octet members in the external legs, we have

$$A_6(\text{tree}) = -\tfrac{1}{32}f^4 \sum_{\text{distinct permutations}} \mathfrak{T}\text{r}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\lambda_f)\{(s_{abc}-m^2)^{-1}[2(s_{ab}+s_{bc})-a_3(s_{ab}+s_{bc}+s_{ac})-m^2(4-3a_3)]$$
$$\times[2(s_{de}+s_{ef})-a_3(s_{de}+s_{ef}+s_{df})-m^2(4-3a_3)]$$
$$+(s_{bcd}-m^2)^{-1}[2(s_{bc}+s_{cd})-a_3(s_{bc}+s_{cd}+s_{bd})-m^2(4-3a_3)]$$
$$\times[2(s_{ef}+s_{fa})-a_3(s_{ef}+s_{fa}+s_{ea})-m^2(4-3a_3)]$$
$$+(s_{cde}-m^2)^{-1}[2(s_{cd}+s_{de})-a_3(s_{cd}+s_{de}+s_{ce})-m^2(4-3a_3)]$$
$$\times[2(s_{fa}+s_{ab})-a_3(s_{fa}+s_{ab}+s_{fb})-m^2(4-3a_3)]\}$$
$$+\tfrac{1}{256}f^4\sum_{\{X\}}\sum_{pX,pY}\tfrac{2}{3}\mathfrak{T}\text{r}(\lambda_a\lambda_b\lambda_c)\mathfrak{T}\text{r}(\lambda_d\lambda_e\lambda_f)(s_{abc}-m^2)^{-1}$$
$$\times[2(s_{ab}+s_{bc})-a_3(s_{ab}+s_{bc}+s_{ac})-m^2(4-3a_3)][2(s_{de}+s_{ef})-a_3(s_{de}+s_{ef}+s_{df})-m^2(4-3a_3)]$$
$$-\tfrac{1}{96}f^4\sum_{\{X\}}\mathfrak{T}\text{r}(\lambda_a\lambda_b\lambda_c)\mathfrak{T}\text{r}(\lambda_d\lambda_e\lambda_f)(s_{abc}-m_0^2)^{-1}(4-3a_3)^2$$
$$\times(s_{ab}+s_{bc}+s_{ac}-2m^2-m_0^2)(s_{de}+s_{ef}+s_{df}-2m^2-m_0^2), \tag{23}$$

where X indicates a given set of three indices (for example a, b, c) and Y is the complement (d, e, f). The symbol $\sum_{pX,pY}$ means sum over all possible permutations of a, b, c and d, e, f within each set. Finally, $\sum_{\{X\}}$ indicates the sum over different sets of three indices.

$$A_6(\text{contact}) = -\tfrac{1}{64}f^4\sum_{\text{distinct permutations}}\mathfrak{T}\text{r}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\lambda_f)$$
$$\times\{-2a_3^2(s_{abc}+s_{bcd}+s_{cde})+8(a_3-1)(s_{ab}+s_{bc}+s_{cd}+s_{de}+s_{ef}+s_{fa})$$
$$-\tfrac{4}{3}a_5(s_{abc}+s_{abd}+s_{abe}+s_{abf}+s_{acd}+s_{ace}+s_{acf}+s_{ade}+s_{adf}+s_{aef})+4m^2(12+5a_5-10a_3)\}$$
$$-\tfrac{1}{96}\Delta m^2 f^4(4-3a_3)^2\sum_{\{X\}}\mathfrak{T}\text{r}(\lambda_a\lambda_b\lambda_c)\mathfrak{T}\text{r}(\lambda_d\lambda_e\lambda_f). \tag{24}$$

We find that the best choice is $a_3 = a_5 = 0$. First and most important, this is the only way to eliminate dependence on Mandelstam variables associated with exotic quantum numbers. Note that the function multiplying one symmetrized trace contains only Mandelstam variables corresponding to that permutation. Furthermore, the coefficient of the product of two traces does not contain terms mixing indices of the two sets.

In the amplitudes considered we still have a part which is a sum of terms, each one associated with a distinct permutation, and the factor multiplying the symmetrized trace in each one satisfies Adler's condition independently. The whole part that breaks algebraic duality exhibits Adler's zeros because the rest of the amplitude does and we have imposed PCAC; it does not have exotic dependence, but the terms in it cannot be put in correspondence to distinct permutations of the external legs. Looking at the resulting expression for the Lagrangian we can see that

$$-\frac{m^2}{2f^2}\text{tr}[\mu(\phi)] = \frac{m^2}{8f^2}\text{tr}(M+M^\dagger) \tag{25}$$

up to sixth order, as was true in the degenerate case.[4]

As a final comment we want to explain why we did not use a model with only the octet of pseudoscalar mesons as a way to bypass the problem of the degeneracy of the whole nonet. This is inconsistent in general, because even if one restricts the transformations to SU(3) rather than the full U(3), the trace of $\delta_A\phi$ is, in general, different from zero. The condition $\delta_A\phi = 0$ implies $M = e^{2if\phi}$,[9] which produces a strong dependence on exotic-channel variables.

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Errata

Brans-Dicke Theory Under a Transformation of Units and the Three Tests, R. E. Morganstern [Phys. Rev. D 3, 2946 (1971)]. The third term on the right-hand side of Eq. (7a) should contain a factor of $\frac{1}{4}$.

General Properties of q-Number Schwinger Terms, Susumu Okubo [Phys. Rev. D 3, 409 (1971)]. Equation (1.6) should read

$$\sigma^{ba}(x)-\sigma^{ab}(x)=\frac{\partial}{\partial x_\mu}Q_\mu{}^{ab}(x)+\frac{\partial^2}{\partial x_l\partial x_k}\sum{}_{lk}{}^{ab}(x)\,. \qquad (1.6)$$

Note that the change is a *plus* sign instead of a *minus*.

Three-Body Problems in the Amado Model, Y. Avishai [Phys. Rev. D 3, 3232 (1971)]. The following paper is to be added to Ref. 1 in the original paper: J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935 (1965).