PHYSICAL REVIEW D VOLUME 4, NUMBER 4 15 AUGUST 1971

Comments and Addenda

The Comments and Addenda section is for short communications which are not of such urgency as to justify publication in **Physical Review Letters** *and are not appropriate for regular Articles. It includes only the following types of communications: (1) comments on papers previously published in* **The Physical Review** *or* **Physical Review Letters***; (2) addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters***, in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section should be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts will follow the same publication schedule as articles in this journal, and galleys will be sent to authors.*

Comments on a Dual Multiparticle Theory with Nonlinear Trajectories*

M. Baker
Department of Physics, University of Washington, Seattle, Washington 98105

and

D. D. Coon†
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
(Received 17 May 1971)

Some features of the nonlinear trajectories of a dual multiparticle theory are examined and the development of the theory is discussed.

Within the past year and a half a dual multiparticle theory with nonlinear trajectories has been developed[1-7] in a manner which closely parallels the development of the generalized Veneziano model.[8-12] The theory includes the Veneziano model as a limiting case. The following work has been done since the four-point function was originally proposed[2]: (1) explicit construction of N-point tree graphs,[1] (2) verification that Veneziano tree graphs are obtained in the limit of linear trajectories,[1,5] (3) investigation of conjectured rules for loop diagrams[4] analogous to the rules of Kikkawa, Sakita, and Virasoro,[9] (4) factorization[6] of the tree graphs analogous to the work of Fubini and Veneziano,[10] (5) reformulation of the tree graphs in terms of an operator formalism[7] analogous to that of Fubini, Gordon, and Veneziano,[11] (6) discovery of Ward-like identities,[6] and (7) application[2] of the Adler self-consistency condition[13] analogous to the applications of Lovelace[14] and of Ademollo, Veneziano, and Weinberg.[15] Three of these advances [(4)–(6)] are quite new.

In this note we will first answer some questions which have been raised concerning the four-point Born term, B_4. We will then review the rules for constructing the N-point Born term B_N, emphasizing the simple, symmetric way in which duality is incorporated, and finally, we will comment on the present state of the theory.

The model arose in an attempt to find the most general meromorphic, dual Born term and to thereby avoid specialization to the Veneziano model. We have not proved that this is the most general possible model. However, no more-general meromorphic, dual Born terms have yet been proposed.[16] The four-point function of the model is given by

$$B_4(s,t)=G(\sigma,\tau)/G(\sigma)G(\tau), \tag{1}$$

where

$$G(\sigma)=\prod_{l=0}^{\infty}(1-\sigma q^l), \tag{2}$$

q is a parameter, $0<q<1$, $\sigma=as+b$, and $\tau=at+b$. The poles of B_4 occur at $\sigma,\tau=q^{-j}$, $j=0,1,2,\ldots$, with polynomial residues of order j and the trajectory function is

$$\alpha(t)=-(\ln\tau)/(\ln q). \tag{3}$$

As $|s|\to\infty$ for fixed t, B_4 has the Regge behavior $B_4(s,t)\sim(-as)^{\alpha(t)}$. If the coefficients a and b are chosen so that the q dependence of τ near $q=1$ is of the form

$$\tau=1+(1-q)\tau'(t)+(1-q)^2\tau''(t)+\cdots,$$

then

$$\lim_{q\to 1}\alpha(t)=\tau'=\alpha_{\text{Veneziano}}\,. \tag{4}$$

Thus, we can have trajectories which are linear or experimentally indistinguishable from linear trajectories.

We now consider two questions which have been raised concerning the logarithmic trajectory functions:

(a) *Trajectories of the form (3) have a left-hand cut.* In general, trajectories are not complex below threshold.[17] The exception to the rule occurs when two or more trajectories intersect. This is in fact what happens in our model. If we denote the integer-spaced[18] daughter-trajectory functions of the model by $\alpha_n(t)=\alpha(t)-n$, we see that $1/\alpha_n=0$ for all n at $\tau=at+b=0$ and $t=\infty$. These are just the branch points of α_n. Thus, the branch points occur when the leading trajectory and all of the daughters collide at the point at infinity $(1/j=0)$ in the j plane. In the linear limit, Eq. (4), the branch point at $t=-b/a$ moves off towards $t=-\infty$, where it annihilates the other branch point, leaving a linear trajectory. Of course the analyticity properties of α had to be physically acceptable since the resultant B_4 does possess the necessary analytic properties to be a physically allowable Born term.

(b) *The real part of* $\alpha(t)$ *rises as* $t\to-\infty$. Since as $|s|\to\infty$, B_4 is Regge-behaved for any finite t, this rise of α on the left leads to a unitarity violation for t sufficiently large and negative. Thus, as $|s|\to\infty$ for fixed $z=\cos\theta_s$, we have[1,2]

$$|B_4(s,t)|\sim\exp[-(\ln q)^{-1}(\ln a|s|)\ln|\tfrac{1}{2}as(1-z)|] \tag{5}$$

which gets large because $\ln q$ is negative. This means that $B_4(s,t)$ cannot be a satisfactory approximation to the complete unitarized amplitude in the fixed-angle region. However, since fixed z with $s\to\infty$ involves large momentum transfer, there is no physical reason to expect that the Born term should be a good approximation in this region. In a unitarized theory with B_4 as its Born term, higher-order diagrams should be important in the large-momentum-transfer region and should change the effective $\alpha(t)$ for large negative t.

Since Eq. (5) holds for z near -1, $B_4(s,t)$ also increases as $s\to\infty$ for fixed u. This bad behavior was mentioned in the original paper[2] on B_4 and was elaborated upon by Capra.[19] Since the origin of the bad fixed-u behavior is the same as that of the bad fixed-z behavior, any higher-order effect which damps out one will damp out the other. Physically we expect higher-order corrections to the fixed-u behavior of $B_4(s,t)$ to be important because B_4 can be written as a sum of either s- or t-channel resonance contributions while u-channel exchange contributions are not contained in B_4. Thus, $B_4(s,t)$ only describes peripheral processes for fixed t or fixed s and is not expected to give a good representation of the full scattering amplitude at fixed u and large s. A completely crossing-symmetric Born term, $B_4(s,t)+B_4(s,u)+B_4(t,u)$, has u-channel exchange contributions only in $B_4(s,u)+B_4(t,u)$. Only these terms can be expected to give a phenomenologically acceptable representation of the scattering amplitude for $|s|\to\infty$ with u fixed. It is a special property of the Veneziano limit that $B_4(s,t)$ also remains bounded in this limit. However, even in the Veneziano case, trouble can occur for fixed u when satellites or lower-lying trajectories are introduced.[20]

We thus see that $B_4(s,t)$ is a physically acceptable Born term since it violates unitarity only in that region where one expects higher-order corrections to be important. There remains the problem of constructing a consistent unitary dual theory with B_4 as a Born term. Consequently, we have embarked on a program which is analogous to that advocated for the Veneziano model by Kikkawa, Sakita, and Virasoro.[9] This is of course a large undertaking which is only at its early stages of development. Some reason for optimism can be found in the fact that the development of the nonlinear theory has so closely paralleled the development of the generalized Veneziano model.[8-12]

The essential first step in the above-mentioned program is the construction of the N-point Born term B_N with spin-zero external lines. We will now review the reasoning which led to its construction. For this purpose we first note that the infinite-product representation (1) of $B_4(s,t)$ can be expanded[1] in the following double power series in σ and τ:

$$B_4(s,t)=\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\sum_{n,m=0}^{\infty}\frac{\sigma^n}{f_n}q^{nm}\frac{\tau^m}{f_m}, \tag{6}$$

where $f_n=(1-q)\cdots(1-q^n)$ for $n>1$ and $f_0=1$. Equation (6) converges for $|\sigma|<1$, $|\tau|<1$. If we set $\tau=0$ in Eq. (6), we obtain the expansion

$$\frac{1}{G(\sigma)}=\sum_{n=0}^{\infty}\frac{\sigma^n}{f_n}. \tag{7}$$

Without the q^{nm} factor in Eq. (6), the n and m sums would have decoupled, and by Eq. (7) we would have obtained $[G(\sigma)G(\tau)]^{-1}$, which is just the product of the denominator factors of Eq. (1). The residues of the poles in σ would then have had poles in τ, in contrast with Eq. (6) where the residues of the poles in σ are polynomials in τ. Thus, the coupling factor q^{nm} in Eq. (6) prevents simultaneous poles in σ and τ, but allows poles in either σ or τ alone. We thus call q^{nm} a simultaneous-pole eliminator or "duality factor." With the above guide we can write down almost immediately the generalization of Eq. (6) to the N-point function B_N corresponding to a given set of Feynman diagrams such as all the planar tree graphs.

The positions of poles in B_N are determined as follows: If p_i is the momentum of the particle associated with an internal line L_i of one of the diagrams of the given set, then B_N must possess poles in the variable p_i^2 at values determined from

$$\sigma_i \equiv a_i p_i^2 + b_i = q^{-j}, \quad j=0,1,2,\ldots \tag{8}$$

where a_i and b_i are constants and q is the same parameter that appears in B_4. This guarantees that the mass spectrum associated with B_N is consistent with that deduced from B_4. We first construct a function with all the above poles by forming the product

$$B_N^{\text{sing}} = 1\Big/\prod_i G(\sigma_i), \tag{9}$$

where $G(\sigma)$ is defined by Eq. (2). The product (9) is taken over all lines L_i which carry distinct momentum p_i. B_N^{sing} is not a satisfactory candidate for B_N because it contains simultaneous poles in all the variables p_i^2. If we insert the expansion (7) for $1/G$ into Eq. (9), we obtain a multiple power series in the variables σ_i:

$$B_N^{\text{sing}} = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\sigma_1^{n_1}}{f_{n_1}} \frac{\sigma_2^{n_2}}{f_{n_2}} \cdots \frac{\sigma_r^{n_r}}{f_{n_r}}. \tag{10}$$

We want to construct a function B_N which has no simultaneous poles in any pair of dual variables p_i^2 and p_j^2 (p_i^2 and p_j^2 are dual variables if there is no Feynman diagram in the given set of tree graphs which contains both lines L_i and L_j). Using the analogy with Eq. (6) for B_4, we can construct such a function B_N by the following simple rule:

Rule: For each pair of dual variables p_i^2 and p_j^2 introduce a "pole-eliminating" factor $q^{n_i n_j}$ under the multiple sum in Eq. (10).

Using Eqs. (6) and (7), it is easy to verify[1] that insertion of these $q^{n_i n_j}$ factors into Eq. (10) gives a B_N which has the pole and residue structure of the totality of Feynman diagrams of the given set.

In order for the resulting dual Born term B_N to form the basis for construction of a dual multiparticle theory, B_N must possess the basic factorization property.[10] The proof of factorization for B_N has recently been carried out by Yu, Baker, and Coon.[6] Although their work on factorization is in principle analogous to that of Fubini and Veneziano[10] on the Veneziano N-point function, it is in detail completely different, as are the results. The degeneracy of the pole in B_N at $\sigma_i = q^{-j}$ is found[6] to be $\sim 6^j$. The number 6 enters through the use of $O(4,2)$ scalar products in the factorization. Since the poles are spaced exponentially, the degeneracy grows only as some power of the mass. Thus, even though the degeneracy in the nonlinear theory is greater than the degeneracy of the Veneziano model, the exponential mass spectrum compensates for the increased degeneracy so that the resulting asymptotic density of states is less than in the Veneziano model.

From the factorized amplitude, we obtain the N-point Born term with two external lines having arbitrary spin. Using this we have obtained expressions for the N-point planar loop diagrams. However, we have not yet been able to evaluate the integrands in the general case, because we have not yet found a convenient technique to facilitate evaluation of the traces which define these integrands.

As in the Veneziano model, the structure of the theory is more clearly understood by introducing an operator formalism. This formalism[7] differs fundamentally from that used in the linear theory.[11] In our case, the totality of spin states is generated by a *single* set of noncommuting creation operators[21] acting on a vacuum state. The commutation relations of these operators with their adjoints differ from the usual harmonic-oscillator commutation relations. Because the analysis of this operator structure has not been fully carried out, the development of the nonlinear theory has not yet proceeded as far as the development of the Veneziano model. When more detailed properties of the nonlinear theory are understood, we will have a better idea of whether the nonlinear B_N can form the basis for a complete dual theory, if indeed such a theory can be constructed at all. At present we are unaware of any physical or aesthetic reason for preferring either the linear or the nonlinear theory.

---

*Work supported in part by the U. S. Atomic Energy Commission, under Contracts No. AT(45-1)-1388 and No. AT(11-1)-1764.

†Address after 1 September 1971: Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213.

[1]M. Baker and D. D. Coon, Phys. Rev. D 2, 2349 (1970).

[2]D. D. Coon, Phys. Letters 29B, 669 (1969).

[3]D. D. Coon, Phys. Rev. 186, 1422 (1969).

[4]M. Baker and D. D. Coon, Phys. Rev. D 3, 2478 (1971).

[5]M. Baker and D. D. Coon (unpublished).

[6]S. Yu, M. Baker, and D. D. Coon (unpublished).

[7]M. Baker, D. D. Coon, and S. Yu (unpublished).

[8]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[9]K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969).

[10]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969).

[11]S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 697 (1969); Y. Nambu, in *Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970).

[12]V. Alessandrini, D. Amati, M. LeBellac, and D. Olive, Phys. Reports 1C, 269 (1971), and references contained

therein.
[13]S. Adler, Phys. Rev. 137, B1022 (1965).
[14]C. Lovelace, Phys. Letters 28B, 264 (1968).
[15]M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters 22, 83 (1969).
[16]Trajectories of the form of those in Refs. 1–7 have also been found by E. Cremmer and J. Nuyts, Nucl. Phys. B26, 151 (1971); J. Golden, Lett. Nuovo Cimento 1, 893 (1971).
[17]W. R. Frazer, in *Selected Topics in Particle Physics, Proceedings of the International School of Physics "Enrico Fermi," Course XLI*, edited by J. Steinberger (Academic, New York, 1968), p. 94; A. O. Barut and D. E. Zwanziger, Phys. Rev. 127, 974 (1962); N. F. Bali, Shu-Yuan Chu, R. N. Haymaker, and Chung-I Tan, *ibid.* 161, 1450 (1967).
[18]Integer-spaced daughters are obviously present since the polynomial residues of B_4 are not Legendre polynomials.
[19]F. Capra, Phys. Letters 30B, 53 (1969).
[20]As $|s|\to\infty$ for fixed u, a Veneziano term of the form

$$V_4(s,t)=\Gamma(n-\alpha(s))\Gamma(m-\alpha(t))/\Gamma(l-\alpha(s)-\alpha(t))$$

has the behavior

$$V_4(s,t)\sim s^{u-4M^2-2\alpha(0)-1+n+m}/\sin\pi\alpha(s).$$

Thus, for large s in the neighborhood of the positive, real axis and u fixed, $V_4(s,t)$ behaves as a power of s which becomes appreciable if $\alpha(0)$ is too small or n and m are too large. For example, with negligible external masses ($M=0$), if we set $u=0$, $n=m=1$, and $\alpha(0)=0$, $V_4\sim s$ as $s\to\infty$ near the real axis.
[21]This is in contrast with the infinite set of commuting creation operators of Ref. 11.

PHYSICAL REVIEW D VOLUME 4, NUMBER 4 15 AUGUST 1971

Scaling, Light-Cone Expansion, and the Van Hove Model*

Myron Bander
Department of Physics, University of California at Irvine, Irvine, California 92664
(Received 5 May 1970)

With certain assumptions on the coupling of two currents to particles of increasing spin, it is shown that the Van Hove model results in Bjorken scaling and Regge asymptotic behavior. The fields corresponding to these particles are related to the products appearing in the operator-product expansion near the light cone.

The Bjorken scaling limit[1] for deep-inelastic electron scattering, or more generally for the scattering of any current in the appropriate kinematic region, may be accounted for by the behavior of products of currents close to each other's light cone.[2–5] This scaling limit can be made consistent with Regge asymptotic[6] behavior; such a behavior may be suggested by the data on inelastic electron scattering.[7] In this note we shall point out how these results may be achieved in the context of the Van Hove model.[8] It may likewise shed some light on the nature of the bilocal operators appearing on the right side of the operator-product expansions.[3] It should be emphasized that none of the results will be derived; they will all be inserted into the model from the start. Our purpose is to show the consistency of these assumptions within a dynamical scheme, and as mentioned previously, to discuss their connection with the operator-product expansion.

For brevity we shall consider the scattering of a current by a spinless particle and study only the even-charge-conjugation amplitude analogous to W_2 of electroproduction. Let q_1 and p_1 (q_2 and p_2) be the four-momenta of the incoming (outgoing) current and particle; the amplitude under discussion is

$$T_{\mu\nu}=(2\pi)^3(4p_1^0p_2^0)^{1/2}\int e^{iq\cdot x}d^4x\,\langle p_1|[J_\mu^\alpha(x),\,J_\nu^\beta(0)]|p_2\rangle$$
$$=P_\mu P_\nu A(\nu,t,Q^2,\delta)+\cdots,\qquad(1)$$

with

$$P=\tfrac{1}{2}(p_1+p_2),\quad Q=\tfrac{1}{2}(q_1+q_2),\quad (P^2)^{1/2}\nu=P\cdot Q,\quad t=(p_1-p_2)^2,\ \text{and}\ \delta=q_2{}^2-q_1{}^2.$$

The conjectured Bjorken scaling limit for the A amplitude is

$$\lim_{\nu,Q^2\to\infty;\ Q^2/2\nu=\omega}\nu A(\nu,t,Q^2,\delta)=F(\omega,t).\qquad(2)$$