[13]E. P. Tryon, Columbia University Report No. NYO-1932(2)-153, 1969 (unpublished); Columbia University Report No. NYO-1932(2)-196, 1971 (to be published in Phys. Letters B).

[14]D. Morgan and G. Shaw, Phys. Rev. D 2, 520 (1970).

[15]M. G. Olsson, Phys. Rev. 162, 1338 (1967); T. Akiba and K. Kang, Phys. Letters 25B, 35 (1967).

[16]This explains why the model with $L = 1$ generates the "universal curve" relating a_0 to a_2, since the derivation by Morgan and Shaw (Ref. 14) is primarily based on the sum rule for $2a_0 - 5a_2$.

[17]The only difference is that with $L = 2$ the D waves are unitarized, whereas with $L = 1$ they are not. However, the D-wave absorptive parts are negligible below 1.5 GeV except for the f_0 (1250), so the $\mathrm{Im}[\Delta A^{(2)I}]$ are negligible up to 1.5 GeV or more. Thus unitarizing the D waves for $L = 2$ would not have an appreciable effect on the S waves or P wave below 1 GeV.

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Eikonal Approximation to the Vertex Function on the Mass Shell*

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The eikonal approximation for graphs representing the exchange of an arbitrary connected amplitude l times in the vertex function is developed for a ϕ^3 theory and quantum electrodynamics when the two external particles are on the mass shell and the momentum transfer is large and spacelike. The simplest case of elementary-particle exchange is calculated. Summing over l, we obtain

$$\Gamma^{\text{eikonal}}_{\text{scalar}}(q^2) = g\exp\left[\frac{g^2}{32\pi^2}\frac{1}{-q^2}\ln^2\left(\frac{-q^2}{m^2}\right)\right] \text{ and } \Gamma^{\mu\,\text{eikonal}}_{\text{QED}}(q^2) = e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\exp\left[-\frac{e^2}{16\pi^2}\ln^2\left(\frac{-q^2}{\mu^2}\right)\right],$$

where μ is a photon mass introduced to eliminate infrared divergence problems.

I. INTRODUCTION

The off-mass-shell vertex function at high energy transfer was studied originally by Sudakov,[1] who considered radiative corrections in quantum electrodynamics (QED) by calculating the asymptotic behavior of Feynman integrals. More recent calculations for both on and off the mass shell were done by Jackiw[2] using an infinite-momentum technique similar to Weinberg's.[3] The vertex function, obtained in QED for crossed-ladder radiative corrections by Jackiw, is given by[4]

$$\Gamma^\mu = e\bar{u}\gamma^\mu u\exp\left[\frac{-e^2}{16\pi^2}\ln^2\left(\frac{-q^2}{\mu^2}\right)\right].$$

This result was conjectured from low-order calculations.

Recently the eikonal approximation has aroused much interest as a useful tool in calculating high-energy elastic scattering amplitudes. Abarbanel and Itzykson[5] have clearly demonstrated for QED that in the high-energy limit the eikonal method gives the correct behavior of those graphs considered. (The exact high-energy behavior was calculated for QED by Cheng and Wu[6] in their extensive work.) Later much work was done in applying the eikonal approximation in ϕ^3 theory[7] as well as QED[8] to obtain high-energy amplitudes.

It is natural, therefore, to apply the eikonal method to the graphs of the vertex function.[9,10] In the asymptotic region there are two distinct types of contributions to the Feynman integral for a vertex graph: eikonal contributions and noneikonal contributions. An eikonal contribution corresponds to a region of integration where the large momentum of the incoming particle is carried essentially unchanged by a line of propagators to the vertex and another line of propagators carries the large momentum from the vertex to the outgoing particle. In such a region we can picture the incoming particle as moving through the interaction region, emitting only soft virtual particles, until it reaches the vertex which is a hard interaction. At this point, large momentum is carried away, and then the particle continues through the interaction region, absorbing soft virtual particles, and finally emerging as the outgoing particle.

The distinguishing mark of a noneikonal contribution is that there are hard interactions at places other than the vertex. That is, either a propagator carries both the large incoming momentum and the large outgoing momentum or large momentum is split between two propagators.

Figure 1 illustrates the two types of contributions

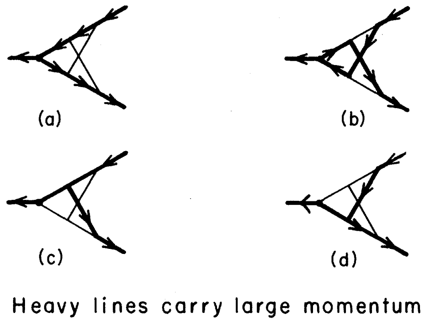

FIG. 1. A graph of the vertex function in ϕ^3 theory showing the regions of integration that contribute to the asymptotic behavior. (a) and (b) represent eikonal regions; (c) and (d) represent noneikonal regions.

to the asymptotic behavior. In this graph there are two eikonal regions of integrations [Figs. 1(a) and 1(b)] and two noneikonal regions of integration [Figs. 1(c) and 1(d)] contributing to the asymptotic value.[11]

The eikonal approximation may fail to give the correct asymptotic behavior for two reasons – one serious and one minor. The serious failing of the eikonal approximation occurs when noneikonal regions contribute asymptotically.[11]

The second problem concerns counting and has two aspects. First, in the perturbation expansion of the S matrix, topologically different graphs may have different relative weights; e.g., the two-rung cross ladder has a factor $\frac{1}{2}$ and the straight ladder has a factor 1. [Fig. 2(a)] The eikonal technique, in giving the amplitude for the appropriate sum of graphs, assigns to each graph the same weight. Consequently, this is one error. Secondly, the crossed graphs may have eikonal regions other than those considered. For example, Fig. 1(b) shows a possible routing of the large momentum which is ignored by a naive application of the eikonal method, since the eikonal routing for the crossed graph corresponding to the two-rung straight-ladder graph is shown in Fig. 1(a). This is another error. However, since this extra region is in fact an eikonal region, it can be calculated by the eikonal method. In the eikonal approximation for the fifth-order graphs shown in Fig. 2(a) the crossed ladder has two regions [Figs. 1(a) and 1(b)] in which it contributes and we can represent the sum of the eikonal regions as in Fig. 2(b). The graphs of Figs. 1(a) and 1(b) are topologically the same so that they have the same contribution. Thus the first pair of graphs in Fig. 2(b) and the second pair have the same value. Therefore we obtain the naive eikonal results in fifth order. The error from the symmetry factor in the crossed graph and the error from neglecting the other eikonal region in the crossed graph have canceled. We conjecture that this cancellation occurs in all higher orders also; we have, however, not yet shown this conjecture.

Bearing in mind that the eikonal approximation fails in a ϕ^3 theory[11] and is valid in QED[10] for the vertex function, we proceed to compute the eikonal approximation to the vertex graphs representing the exchange of the arbitrary connected amplitude D l times in a boson ϕ^3 theory and QED. We consider the eikonal region where all the momenta of D are small, i.e., the large momenta flow along world lines 1 and 2 shown in Fig. 3 for a ϕ^3 theory and Fig. 4 for QED.

In Sec. II we consider the boson case. We first find the eikonal approximation to the exact wave function of a spinless boson particle in an external potential v, $\varphi(x, p_i; v)$. Then by introducing auxiliary external potentials v and v', we may write the eikonal approximation to the vertex graphs considered when $(p_1+\frac{1}{2}q)^2=(p_2-\frac{1}{2}q)^2=m^2$ and $-q^2 \gg m^2$ as

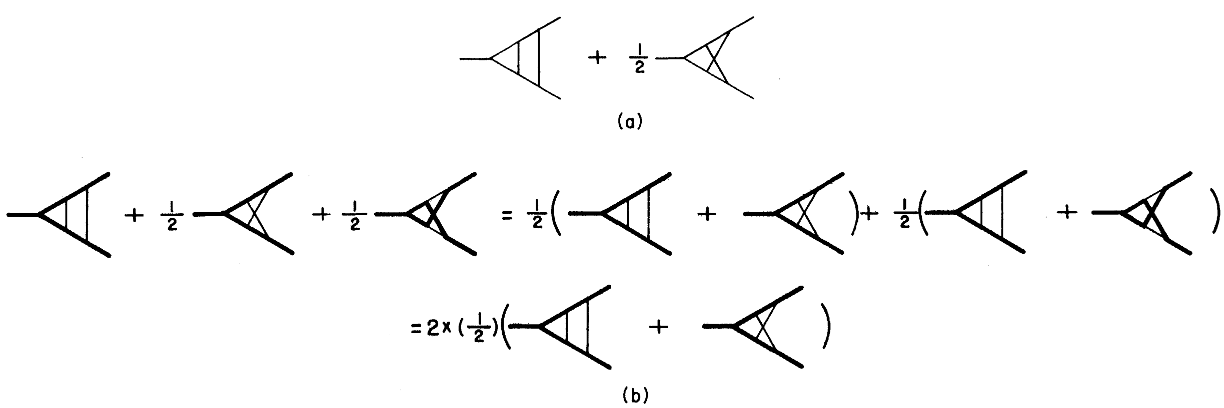

FIG. 2. (a) shows the straight and crossed two-rung ladder graphs in ϕ^3 theory. (b) shows the eikonal regions of these two graphs contributing to the high-energy behavior.

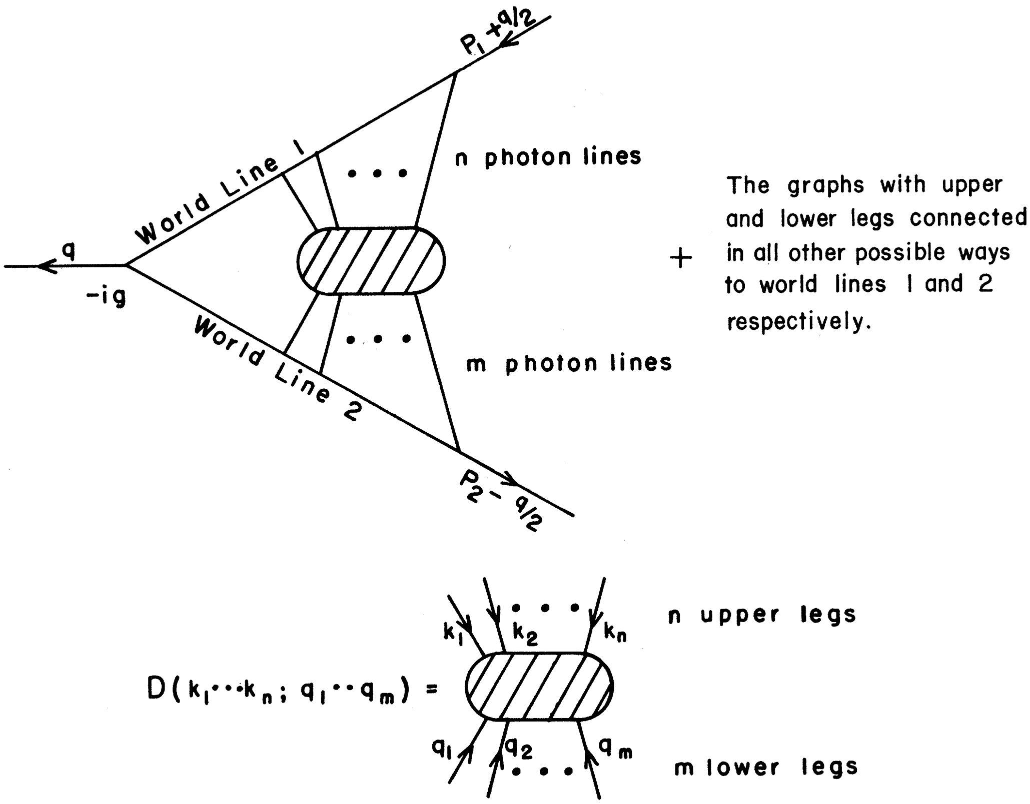

FIG. 3. $\Gamma_l\,(q^2)$ in a ϕ^3 theory.

$$-i(2\pi)^4\delta^4(p_1-p_2)\Gamma_l^{\text{eikonal}}(q^2)=\int d^4x\, K_l\varphi_2^{\text{eikonal}\,\dagger}(x,p_2-\tfrac{1}{2}q;v')e^{-iq\cdot x}(-ig)\varphi_1^{\text{eikonal}}(x,p_1+\tfrac{1}{2}q;v)\big|_{v=v'=0},$$

where

$$K_l=\frac{1}{l!}\left[\prod_{i=1}^{n}\left(\int\frac{d^4k_i}{(2\pi)^4}\frac{\delta}{\delta v(k_i)}\right)\prod_{j=1}^{m}\left(\int\frac{d^4q_j}{(2\pi)^4}\frac{\delta}{\delta v'(q_j)}\right)D(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l.$$

Using the brick-wall frame, these integrals can be evaluated to obtain finally

$$\Gamma_l^{\text{eikonal}}(q^2)=g\frac{1}{l!}\left[\left(\frac{i}{Q}\right)^{n+m}\prod_{i=1}^{n}\left(\int_{-\infty}^{0}d\alpha_i\int\frac{d^4k_i}{(2\pi)^4}e^{ik_i\cdot\eta_-\alpha_i}\right)\prod_{j=1}^{m}\left(\int_0^{\infty}d\alpha_j'\int\frac{d^4q_j}{(2\pi)^4}e^{iq_j\cdot\eta_+\alpha_j'}\right)D(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l$$

$$(Q^2=-q^2>0).$$

In Sec. III we consider the QED case. The eikonal approximation to the exact wave function for a fermion in an external vector potential $v^\mu(x)$ is obtained by relating it to the wave function for a spinless boson found in Sec. II. The computation of the vertex function for the exchange of the amplitude $D^{\alpha_1\cdots\beta_m}(k_1,\ldots,q_m)$ l times is then carried out as in Sec. II. We obtain

$$\Gamma_l^{\mu\,\text{eikonal}}(q^2)=e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\frac{1}{l!}\left[i^{n+m}\prod_{i=1}^{n}\left(\int_{-\infty}^{0}d\sigma_i\int\frac{d^4k_i}{(2\pi)^4}e^{ik_i\eta_-\sigma_i}(\eta_-)_{\alpha_i}\right)\right.$$

$$\left.\times\prod_{j=1}^{m}\left(\int_0^{\infty}d\sigma_j'\int\frac{d^4q_j}{(2\pi)^4}e^{iq_j\cdot\eta_+\sigma_j'}(\eta_+)_{\beta_j}\right)D^{\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_m}(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l.$$

Finally in Sec. IV we explicitly calculate for both ϕ^3 theory and QED the case of elementary-particle-exchange graphs. These results,

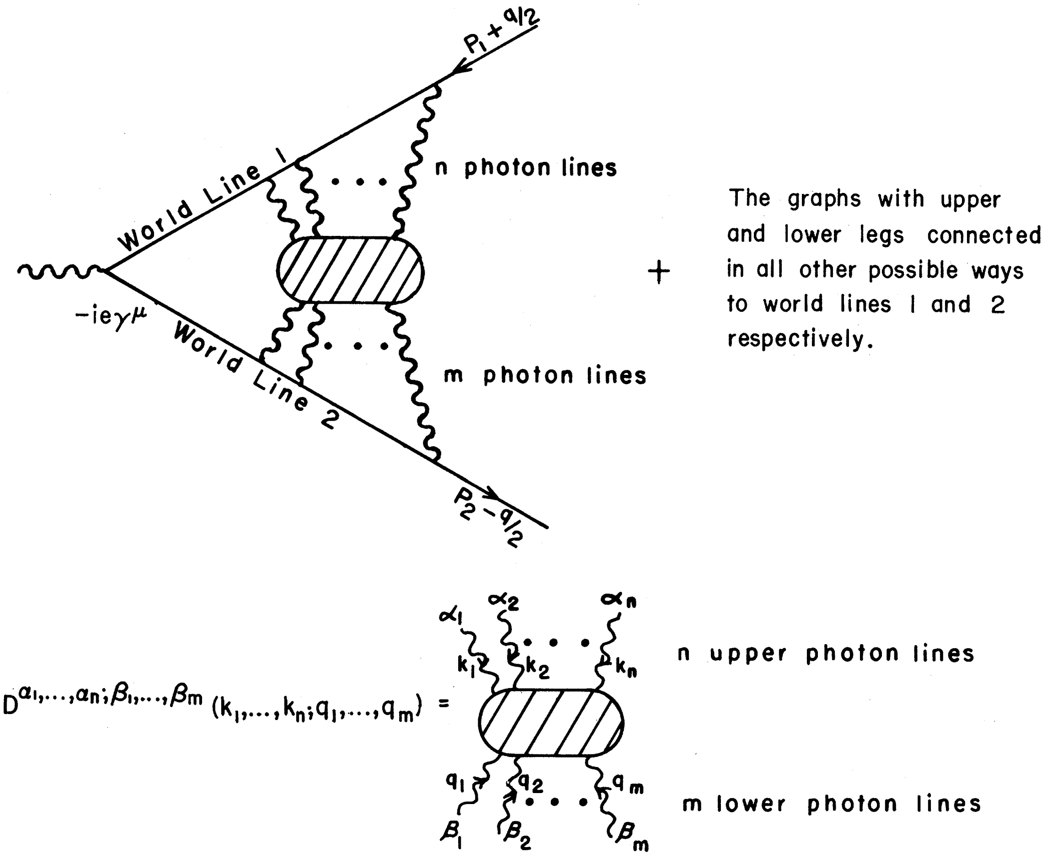

FIG. 4. $\Gamma_l^\mu(q^2)$ in QED.

$$\Gamma^{\text{eikonal}}_{\text{scalar}}(q^2)=g\frac{1}{l!}\left[\frac{g^2}{16\pi^2}\frac{1}{2Q^2}\ln^2\left(\frac{Q^2}{m^2}\right)\right]^l$$

and

$$\Gamma^{\mu\,\text{eikonal}}_{l\,\text{QED}}(q^2)=e\,\frac{\bar u_{\lambda_f}\gamma^\mu u_{\lambda_i}}{l!}\left[\frac{-e^2}{16\pi^2}\ln^2\left(\frac{Q^2}{\mu^2}\right)\right]^l,$$

are then compared with the exact low-order calculations cited previously in this section.

II. THE BOSON CASE

First we consider the wave function $\varphi(x)$ for a spinless particle of mass m in an external potential $v(x)$. $\varphi_1(x,p_i;v)$ satisfies the differential equation

$$\left[\Box^2+m^2-v(x)\right]\varphi_1(x,p_i;v)=0,\quad \Box^2=-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}$$

where the boundary condition is given by $\varphi_1\to e^{ip_i\cdot x}$ as $t\to-\infty$ and $p_i^2=m^2$. We can make the eikonal approximation by the method used by Johnson.[12] We assume that in the region $v(x)\neq 0$,

$$\varphi_1(x,p_i;v)=e^{ip_i\cdot x}\tilde\varphi(x,p_i;v),$$

where $\tilde\varphi$ is a slowly varying function of x and $\tilde\varphi(x)\to 1$ as $t\to-\infty$. The eikonal approximation is made by dropping $\Box^2$ compared to $p_i\cdot\partial/\partial x$ in the differential equation for $\tilde\varphi(x)$. (In momentum space, this approximation becomes $|p_i\cdot q_j|>|q_j^2|$ for every j, where q_j is the momentum carried away by the jth interaction with the external potential v.) We thus have

$$\left[\frac{2}{i}p_i\cdot\frac{\partial}{\partial x}-v(x)\right]\tilde{\varphi}^{\text{eikonal}}(x,p_i;v)=0\,. \tag{1}$$

The solution of the above differential equation is facilitated by using the coordinate system defined by

$$x^\mu \to x'^\mu+\frac{p_i^\mu\tau}{m}\ \text{and}\ \ x'^\mu\cdot p_{i\mu}=0\,.$$

In this "proper-time" system,

$$\tau=\frac{1}{m}p_i\cdot x\ \ \text{and}\ \ p_i\cdot\frac{\partial}{\partial x}=m\frac{\partial}{\partial\tau}\,.$$

Thus Eq. (1) becomes

$$\left[\frac{2}{i}m\frac{\partial}{\partial\tau}-v(x)\right]\tilde{\varphi}^{\text{eikonal}}(x)=0\,,$$

where

$$\tilde{\varphi}^{\text{eikonal}}(x)\to 1\ \text{as}\ \tau\to-\infty\,.$$

Solving for $\varphi^{\text{eikonal}}(x)$, we obtain for $\varphi^{\text{eikonal}}(x)=e^{ip_i\cdot x}\tilde{\varphi}^{\text{eikonal}}(x)$,

$$\varphi_1^{\text{eikonal}}(x,p_i;v)=e^{ip_i\cdot x}\exp\left[\frac{i}{2m}\int_{-\infty}^{0}d\sigma\,v\left(x^\mu+\frac{p_i^\mu\sigma}{m}\right)\right]. \tag{2}$$

The eikonal wave function for a state that becomes the free-particle state $e^{ip_f\cdot x}$ as $t\to+\infty$ can be found by a similar analysis or directly from Eq. (2) by applying the time-reversal operator:

$$\varphi_2^{\text{eikonal}}(x,p_f;v)=e^{ip_f\cdot x}\exp\left[\frac{-i}{2m}\int_0^{\infty}d\sigma' v\left(x^\mu+\frac{p_f^\mu\sigma'}{m}\right)\right]. \tag{3}$$

We may now construct the contribution to the vertex graphs shown in Fig. 3 from the eikonal region of the exact expressions where the initial particle carries large momentum and continues essentially undeflected by each interaction along world line 1 until it reaches the vertex, which is a "hard" interaction; i.e., the emitted meson carries away large momentum. The remaining large momentum then continues essentially undeflected along world line 2 until it is carried away by the final particle.

By introducing auxiliary external potentials[13] v and v' and using the results derived above, Eqs. (2) and (3), we may write the eikonal approximation to the vertex graphs shown in Fig. 3 when $(p_1+\frac{1}{2}q)^2=(p_2-\frac{1}{2}q)^2=m^2$ and $-q^2\gg m^2$:

$$-i(2\pi)^4\delta^4(p_1-p_2)\Gamma_1^{\text{eikonal}}(q^2)=\int d^4x\,K_1\varphi_2^{\text{eikonal}\,\dagger}(x,p_2-\tfrac{1}{2}q;v')e^{-iq\cdot x}(-ig)\varphi_1^{\text{eikonal}}(x,p_1+\tfrac{1}{2}q;v)\Big|_{v=v'=0}\,. \tag{4}$$

Here, $\varphi_1^{\text{eikonal}}(x,p_1+\frac{1}{2}q;v)$ is the wave function of a particle in an external potential $v(x)$ that represented a free particle of momentum $p_1+\frac{1}{2}q$ before the scattering (i.e., $t\to-\infty$). This wave function represents world line 1 and is given explicitly in the eikonal approximation by Eq. (2). Similarly, $\varphi_2^{\text{eikonal}}(x,p_2-\frac{1}{2}q;v')$ is the wave function of a particle in an external potential $v'(x)$ that will represent a free particle of momentum $p_2-\frac{1}{2}q$ after the scattering (i.e., $t\to+\infty$). This wave function represents world line 2 and is given explicitly in the eikonal approximation by Eq. (3). The emission of a meson of momentum q at the vertex gives a factor $(-ig)e^{-iq\cdot x}$. Finally, K_1 is an operator containing the amplitude D that is exchanged between world lines 1 and 2,

$$K_1=\prod_{i=1}^{n}\left[\int\frac{d^4k_i}{(2\pi)^4}\frac{\delta}{\delta v(k_i)}\right]\prod_{j=1}^{m}\left[\int\frac{d^4q_j}{(2\pi)^4}\frac{\delta}{\delta v'(q_j)}\right]D(k_1,\ldots,k_n;q_1,\ldots,q_m)\,. \tag{5}$$

Because of the symmetry in $(k_1,\ldots,k_n)$ and $(q_1,\ldots,q_m)$ in the eikonal approximation, we could not expect the eikonal approximation to be valid for any individual Feynman graph, since it is not symmetric, and so we are considering the sum of the Feynman graphs with the upper n legs of D attached in all possible ways to world line 1 and the m lower legs of D attached in all possible ways to world line 2.[14]

The eikonal approximation for the vertex graphs corresponding to the connected amplitude D exchanged l times with the nl upper legs and the ml lower legs connected in all possible ways to world lines 1 and 2, respectively, is

$$-i(2\pi)^4\Gamma_l^{\text{eikonal}}(q^2)\delta^4(p_1-p_2)=\int d^4x\,K_l\,\varphi_2^{\text{eikonal}\,\dagger}(x,p_2-\tfrac{1}{2}q;v')e^{-iq\cdot x}(-ig)\varphi_1^{\text{eikonal}}(x,p_1+\tfrac{1}{2}q;v)\Big|_{v=v'=0}, \tag{6}$$

where

$$K_l=\frac{1}{l!}(K_1)^l\,.$$

Equations (2), (3), and (5) are substituted into Eq. (6), and the functional derivatives are evaluated. We then let $v=v'=0$, to obtain

$$(2\pi)^4\Gamma_l^{\text{eikonal}}(q^2)\delta^4(p_1-p_2)=g\int d^4x\,e^{i(p_1-p_2)\cdot x}\frac{1}{l!}\left[\left(\frac{i}{2m}\right)^{n+m}\prod_{i=1}^{n}\left\{\int_{-\infty}^{0}d\sigma_i\int\frac{d^4k_i}{(2\pi)^4}\exp\left[ik_i\left(x+\frac{p_1+\frac{1}{2}q}{m}\sigma_i\right)\right]\right\}\right.$$
$$\left.\times\prod_{j=1}^{m}\left\{\int_0^{\infty}d\sigma_j'\int\frac{d^4q_j}{(2\pi)^4}\exp\left[iq_j\left(x+\frac{p_2-\frac{1}{2}q}{m}\sigma_j'\right)\right]\right\}D(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l. \tag{7}$$

We choose to evaluate the integral in the brick-wall frame:

$$q^\mu=(0,0,Q,0)\,,\qquad p_1^\mu=(0,0,p^3,p^4)\,.$$

Since $(p_1-\frac{1}{2}q)^2=(p_1+\frac{1}{2}q)^2=m^2$, $p^3=0$ and $(p^4)^2=m^2+\frac{1}{4}Q^2$. Because $Q^2\gg m^2$, $p^4\simeq\frac{1}{2}Q+m^2/Q$. Thus to order $1/Q$,

$$p_1+\tfrac{1}{2}q=\left(\tfrac{1}{2}Q+\frac{m^2}{2Q}\right)\eta_-+\frac{m^2}{2Q}\eta_+\,,$$
$$p_1-\tfrac{1}{2}q=\left(\tfrac{1}{2}Q+\frac{m^2}{2Q}\right)\eta_++\frac{m^2}{2Q}\eta_-\,, \tag{8}$$

where

$$\eta_+=(0,0,-1,1)\,,\qquad \eta_-=(0,0,1,1)\,.$$

Within the eikonal approximation we keep only the terms of order Q in the expression for $p_1+\frac{1}{2}q$ and $p_1-\frac{1}{2}q$ given in Eq. (8) and substitute these expressions for $p_1+\frac{1}{2}q$ and $p_1-\frac{1}{2}q$ into Eq. (7). We make the change of variables

$$\sigma_i,\sigma_j'\to\alpha_i=\frac{Q}{2m}\sigma_i\,,\quad \alpha_j'=\frac{Q}{2m}\sigma_j'\,.$$

We then notice that $D(k_1,\ldots,k_n;q_1,\ldots,q_m)$ must have an over-all δ function in it, so $\sum_{i=1}^n k_i+\sum_{j=1}^m q_j=0$, and therefore the x dependence of the expression in square brackets in (7) drops out. We can then perform the x^μ integration to get $(2\pi)^4\delta^4(p_1-p_2)$. We thus obtain, for the ϕ^3 theory, the general form of the eikonal approximation for the vertex graphs on the mass shell and $-q^2\gg m^2$ representing the exchange of the amplitude D l times,

$$\Gamma_l^{\text{eikonal}}(q^2)=g\frac{1}{l!}\left[\left(\frac{i}{Q}\right)^{n+m}\prod_{i=1}^{n}\left(\int_{-\infty}^{0}d\alpha_i\int\frac{d^4k_i}{(2\pi)^4}e^{ik_i\cdot\eta_-\alpha_i}\right)\prod_{j=1}^{m}\left(\int_0^{\infty}d\alpha_j'\int\frac{d^4q_j}{(2\pi)^4}e^{iq_j\cdot\eta_+\alpha_j'}\right)D(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l. \tag{9}$$

III. THE FERMION CASE

We now wish to construct the eikonal approximation to the vertex function in a fermion–vector-boson theory, i.e., QED with a massive photon. First we must find the wave function for a fermion in an external vector potential $v^\mu(x)$. Let $\Psi_{\rm I}(x,p_i,\lambda_i;v^\mu)$ represent the wave function of a fermion in an external vector potential, where the fermion was a free particle of momentum p_i and spin λ_i before the scattering (i.e., $t\to-\infty$). We write $\Psi_{\rm I}$ as the sum of its perturbation series in $v^\mu(x)$:

$$\Psi_{\rm I}(x,p_i,\lambda_i;v^\mu)=\sum_{n=0}^{\infty}\Psi_{{\rm I}n}(x,p_i,\lambda_i;v^\mu)\,.$$

We can relate $\Psi_{{\rm I}n}(x,p_i,\lambda_i;v^\mu)$ to $\varphi_{1n}(x,p_i;v)$ in the eikonal approximation. We write $\Psi_{{\rm I}n}^{\text{eikonal}}$ in momentum space using Eq. (2):

$$\varphi_{1n}^{\text{eikonal}}(p,p_1+\tfrac{1}{2}q;v)=\left(\frac{i}{2m}\right)^n\frac{1}{n!}\prod_{i=1}^{n}\left[\int_{-\infty}^{0}d\sigma_i e^{iq_i(p_1+\frac{1}{2}q)\sigma_i}\int\frac{d^4q_i}{(2\pi)^4}v(q_i)\right](2\pi)^4\delta^4\left(p-p_1-\tfrac{1}{2}q+\sum_{i=1}^{n}q_i\right).$$

Since $S_F(p)=(\not p+m)\Delta_F(p)$ and $p_1+\frac{1}{2}q=\frac{1}{2}Q\eta_-$,

$$\Psi_{\text{I}n}^{\text{eikonal}}(p,\tfrac{1}{2}Q\eta_-,\lambda_i;v^\mu)=\frac{1}{n!}\left(\frac{i}{Q}\right)^n\prod_{i=1}^{n}\left[\left(\not p_1+\tfrac{1}{2}\not q-\sum_{j=1}^{i}\not q_j+m\right)\gamma_{\mu_i}\right]$$

$$\times\prod_{i=1}^{n}\left[\int\frac{d^4q_i}{(2\pi)^4}\left(\frac{-i}{q_i\cdot\eta_-}\right)v^{\mu_i}(q_i)\right](2\pi)^4\delta^4\left(p-\tfrac{1}{2}Q\eta_-+\sum_{i=1}^{n}q_i\right)u_{\lambda_i}(\tfrac{1}{2}Q\eta_-).$$

The eikonal approximation allows us to drop the $\sum_{j=1}^{i}\not q_j+m$ terms relative to $\frac{1}{2}Q\eta_-\cdot\gamma=\frac{1}{2}Q(\gamma^0-\gamma^z)$ terms. Thus

$$\Psi_{\text{I}n}^{\text{eikonal}}(p,\tfrac{1}{2}Q\eta_-,\lambda_i;v^\mu)=\frac{i^n}{n!}\prod_{i=1}^{n}[\tfrac{1}{2}(1+\gamma^0\gamma^z)\gamma^0\gamma_{\mu_i}]\prod_{i=1}^{n}\left[\int\frac{d^4q_i}{(2\pi)^4}\left(\frac{-i}{q_i\cdot\eta_-}\right)v^{\mu_i}(q_i)\right](2\pi)^4\delta^4\left(p-\tfrac{1}{2}Q\eta_-+\sum_{i=1}^{n}q_i\right)u_{\lambda_i}(\tfrac{1}{2}Q\eta_-).$$

In the limit $Q\to\infty$ the Dirac equation becomes $\gamma^0\gamma^z u_{\lambda_i}(\frac{1}{2}Q\eta_-)=u_{\lambda_i}(\frac{1}{2}Q\eta_-)$. Therefore the largest terms as $Q\to\infty$ involve the scattering by Dirac matrices which do not vanish in the $\gamma^0\gamma^z$ subspace.[12] This projection is 1 for $\gamma^0\gamma^0$ and $\gamma^0\gamma^z$ and 0 for $\gamma^0\gamma^x$ and $\gamma^0\gamma^y$. Thus, in effect, we may replace each vertex by $(\eta_-)_{\mu_i}$. So we can write

$$\Psi_{\text{I}n}^{\text{eikonal}}(p,\tfrac{1}{2}Q\eta_-,\lambda_i;v^\mu)=(2\pi)^4\frac{i^n}{n!}u_{\lambda_i}(\tfrac{1}{2}Q\eta_-)\prod_{i=1}^{n}\left[\int\frac{d^4q_i}{(2\pi)^4}\left(\frac{-i}{q_i\cdot\eta_-}\right)(\eta_-)_{\mu_i}v^{\mu_i}(q_i)\right]\delta^4\left(p-\tfrac{1}{2}Q\eta_-+\sum_{i=1}^{n}q_i\right).$$

Thus, summing the perturbation series,

$$\Psi_{\text{I}}^{\text{eikonal}}(x,\tfrac{1}{2}Q\eta_-,\lambda_i;v^\mu)=e^{i\frac{1}{2}Q\eta_-\cdot x}u_{\lambda_i}(\tfrac{1}{2}Q\eta_-)\exp\left[i\int_{-\infty}^{0}d\sigma_i\eta_-\cdot v(x+\sigma_i\eta_-)\right].\qquad(10)$$

Similarly, the wave function $\Psi_{\text{II}}^{\text{eikonal}}(x,p_f,\lambda_f;v^\mu)$ of a fermion that will be a free particle of momentum $p_f=\frac{1}{2}Q\eta_+$ and spin λ_f after the scattering (i.e., $t\to+\infty$) is written

$$\Psi_{\text{II}}^{\text{eikonal}}(x,\tfrac{1}{2}Q\eta_+,\lambda_f;v^\mu)=u_{\lambda_f}(\tfrac{1}{2}Q\eta_+)e^{i\frac{1}{2}Q\eta_+\cdot x}\exp\left[-i\int_0^\infty d\sigma_j'\eta_+\cdot v(x+\sigma_j'\eta_+)\right].\qquad(11)$$

Now the operator K_1 for the vertex graphs shown in Fig. 4 for QED is given by

$$K_1=\prod_{i=1}^{n}\left(\int\frac{d^4k_i}{(2\pi)^4}\frac{\delta}{\delta v(k_i)}\right)\prod_{j=1}^{m}\left(\int\frac{d^4q_j}{(2\pi)^4}\frac{\delta}{\delta v'(q_j)}\right)D^{\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_m}(k_1,\ldots,k_n;q_1,\ldots,q_m).\qquad(12)$$

The analysis of the eikonal contribution to the vertex function in QED on the mass shell and $-q^2\gg m^2$ representing the exchange of the amplitude $D^{\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_m}(k_1,\ldots,k_n;q_1,\ldots,q_m)$ l times is now exactly the same as in the scalar case. We obtain[15]

$$\Gamma_l^{\mu\,\text{eikonal}}(q^2)=e\bar u_{\lambda_f}\gamma^\mu u_{\lambda_i}\frac{1}{l!}\left[i^{n+m}\prod_{i=1}^{n}\left(\int\frac{d^4k_i}{(2\pi)^4}e^{ik_i\cdot\eta_-\sigma_i}(\eta_-)_{\alpha_i}\right)\right.$$

$$\left.\times\prod_{j=1}^{m}\left(\int_0^\infty d\sigma_j'\int\frac{d^4q_j}{(2\pi)^4}e^{iq_j\cdot\eta_+\sigma_j'}(\eta_+)_{\beta_j}\right)D^{\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_m}(k_1,\ldots,k_n;q_1,\ldots,q_m)\right]^l.\qquad(13)$$

IV. AN EXPLICIT CALCULATION

Explicit calculations can now be done for the particularly simple case of elementary-particle-exchange graphs in ϕ^3 and QED.

In the scalar case,

$$D(k_1,q_1)=ig^2(2\pi)^4\delta^4(k_1-q_1)(k_1^2-m^2+i\epsilon)^{-1}.$$

In QED,

$$D^{\alpha_1\beta_1}(k_1,q_1)=-ie^2(2\pi)^4\delta^4(k_1-q_1)g^{\alpha_1\beta_1}(k_1^2-\mu^2+i\epsilon)^{-1},$$

where μ is the photon mass.

Using these expressions in the vertex functions (9) and (13), respectively,

$$\Gamma_{l\,\text{scalar}}^{\text{eikonal}}(q^2)=g\frac{1}{l!}\left[\frac{-ig^2}{Q^2}I(q^2,m^2)\right]^l,$$

$$\Gamma_{l\,\text{QED}}^{\mu\,\text{eikonal}}(q^2)=e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\frac{1}{l!}[2ie^2I(q^2,\mu^2)]^l,$$

where

$$I(q^2,m^2)=\int_{-\infty}^0 d\sigma\int_0^\infty d\sigma'\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot\eta_-\sigma}e^{-ik\cdot\eta_+\sigma'}(k^2-m^2+i\epsilon)^{-1}.$$

We can perform the $k\cdot\eta_-$ integration in I by contour integration and then perform the σ and σ' integrations:

$$I(q^2,m^2)=\frac{i}{2(2\pi)^3}\int d^2\vec{k}\int\frac{dk_+}{k_+}\frac{\theta(k_+)}{\vec{k}^2+m^2}\qquad(k_+=k\cdot\eta_+).$$

The eikonal approximation gives us the limits on the integrations remaining since it requires $|(p_1+\frac{1}{2}q)\cdot k|\geqslant|k^2|$ and $|(p_1-\frac{1}{2}q)\cdot k|\geqslant|k^2|$.

These limits are $Q\geqslant|k_+|\geqslant m^2/Q$ and $Q|k_+|\geqslant\vec{k}^2$. The remaining integrations can now be performed to obtain

$$I(q^2,m^2)=\frac{i}{32\pi^2}\ln^2\left(\frac{Q^2}{m^2}\right).$$

Thus

$$\Gamma_{l\,\text{scalar}}^{\text{eikonal}}(q^2)=g\frac{1}{l!}\left[\frac{g^2}{32\pi^2Q^2}\ln^2\left(\frac{Q^2}{m^2}\right)\right]^l \tag{14}$$

and

$$\Gamma_{l\,\text{QED}}^{\mu\,\text{eikonal}}(q^2)=e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\frac{1}{l!}\left[\frac{-e^2}{16\pi^2}\ln^2\left(\frac{Q^2}{\mu^2}\right)\right]^l. \tag{15}$$

Assuming that it is meaningful to sum over l, we obtain the vertex functions representing the sum of all elementary exchange graphs in the limit $-q^2=Q^2\gg\{^{\mu^2}_{m^2}\}$ on the mass shell:

$$\Gamma_{\text{scalar}}^{\text{eikonal}}(q^2)=\sum_{l=0}^\infty\Gamma_l=g\exp\left[\frac{g^2}{32\pi^2}\frac{1}{Q^2}\ln^2\left(\frac{Q^2}{m^2}\right)\right], \tag{16}$$

$$\Gamma_{\text{QED}}^{\mu\,\text{eikonal}}(q^2)=\sum_{l=0}^\infty\Gamma_l^\mu=e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\exp\left[\frac{-e^2}{16\pi^2}\ln^2\left(\frac{Q^2}{\mu^2}\right)\right]. \tag{17}$$

In the QED case the results obtained in the eikonal approximation [(15) and (17)] agree with the exact low-order calculations mentioned previously and also satisfy the general form obtained by Mack.[4]

In the ϕ^3 theory the eikonal result (14) does not give the correct high-energy behavior of the elementary-particle-exchange graphs. An analysis of the exact high-energy behavior of particle exchange graphs in fifth order ($l=2$) shows that this failure is due to the existence at high energy of "noneikonal" regions of integration in the crossed graph.[11]

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[1]V. V. Sudakov, Zh. Eksperim. i Teor. Fiz. 30, 87 (1954) [Soviet Phys. JETP 3, 65 (1956)].

[2]R. Jackiw, Ann. Phys. (N.Y.) 48, 292 (1968). This problem was also studied by Cassandro and Cini; how-

ever their numerical results are not in agreement with subsequent work: M. Cassandro and M. Cini, Nuovo Cimento 34, 1719 (1964).

[3]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[4]Mack has also considered the vertex function from a very different viewpoint, viz., a statistical approach, and he has obtained a general form for the vertex function of the type $\Gamma \sim \exp[-A \ln^2(aq^2)]$: G. Mack, Phys. Rev. 154, 1617 (1967).

[5]H. D. J. Abarbanel and C. Itzykson, Phys. Rev. Letters 23, 53 (1969). S. J. Chang and S. K. Ma, *ibid.* 22, 1334 (1969), also showed that the eikonal method for the scattering amplitude gave the correct asymptotic behavior in QED for the ladder diagrams.

[6]There are many articles on this subject by Cheng and Wu; see, for example, H. Cheng and T. Wu, Phys. Rev. Letters 22, 666 (1969).

[7]In a ϕ^3 theory, see, for example, S. J. Chang and T. M. Yan. Phys. Rev. Letters 25, 1586 (1970); B. Hasslacher, D. K. Sinclair, G. M. Cicuta, and R. L. Sugar, *ibid.* 25, 1591 (1970).

[8]In QED, see, for example, S. J. Chang and P. M. Fishbane, Phys. Rev. D 2, 1104 (1970).

[9]M. Lévy and J. Sucher, Phys. Rev. 186, 1656 (1969), developed the eikonal approximation in the vertex function in a manner somewhat similar to the method presented in this paper, although they considered only elementary-particle-exchange diagrams. Using the techniques of Lévy and Sucher for elementary particle exchange, the vertex calculation in QED has been done by L. F. Li, Phys. Rev. D 2, 614 (1970) and in the scalar theory by J. L. Cardy, Nucl. Phys. B28, 477 (1971).

[10]While this paper was being prepared I became aware of two other papers in preparation on the eikonal method for the vertex function. P. M. Fishbane and J. D. Sullivan, University of Illinois report (unpublished), and T. Appelquist and J. Primack, Harvard University report (unpublished).

[11]The asymptotic behavior was computed exactly, showing the failure of the eikonal approximation, by E. Eichten and R. Jackiw, Phys. Rev. D 4, 439 (1971).

[12]K. A. Johnson presents these arguments in his lectures at the Latin-American School of Physics, La Plata, Argentina, 1970 (unpublished). A similar approximation occurs also in R. Blankenbecler and R. Sugar, Phys. Rev. D 2, 3024 (1970).

[13]Abarbanel and Itzykson introduced auxiliary external potentials to facilitate the calculation of the eikonal approximation in the two-particle elastic scattering amplitude. See Ref. 5.

[14]The connected amplitude D contains an appropriate combinatorial factor to avoid overcounting the vertex graphs when the legs of D are attached in all possible ways.

[15]Although renormalization counterterms are required to cancel the ultraviolet divergences in QED, these results will be essentially unmodified since they have no ultraviolet divergences.